CHAPTER-2: REGULAR Γ - SEMI GROUPS

Various concepts of regularity on semi groups have been investigated by R. Croisot. His studies have been presented in the book of Clifford.A.H. and G.B.Preston [6] as ‘R.Croisot theory’. One of the central places in this theory is held by the left regularity. The idea of generalization of a commutative semi group was first introduced by Kazim and Naseeruddin [19]. M.K.Sen [27, 28, 29] and N.K.Saha [29] defined the concepts of Γ-semi group and regular Γ-semi group. This chapter is concerned with the basic concepts and some results on regular and completely regular Γ-semi groups. We obtained some results on left regular Γ-semi groups using the cancellative property and studied the properties of regular Γ-semi groups satisfying some identities of three variables. It is also observed that every cancellative Γ-semi group is regular if and only if it is g-regular. By using the identity \(a\alpha b\beta c = c\alpha b\) for all \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\), it is proved that a regular \(\Gamma\)-semi group \(S\) is one of the following (i) commutative \(\Gamma\)-semi group (ii) left (right) regular \(\Gamma\)-semi group (iii) completely regular \(\Gamma\)-semi group and (iv) commuting regular \(\Gamma\)-semi group.

2.1. Some properties of regular \(\Gamma\)-semi groups:

In this section we discuss some properties of regular \(\Gamma\)-semi groups and left regular \(\Gamma\)-semi groups. Some definitions needed for the study of this section and the subsequent sections are also presented.

2.1.1 Definition: A \(\Gamma\)-semi group \(S\) is said to be a completely regular \(\Gamma\)-semi group if for all \(a \in S\) there exists \(x \in S\) and \(\alpha \in \Gamma\) such that \(a = a\alpha x\beta a\) and \(a\alpha x = x\alpha a\)

2.1.2. Definition: A \(\Gamma\)-semi group \(S\) is said to be a commuting regular \(\Gamma\)-semi group if for all \(a, b \in S\) there exists \(x \in S\) and \(\alpha, \beta, \gamma \in \Gamma\) such that \(a\alpha b = b\alpha a(\beta x\gamma)b\alpha a\).

2.1.3. Definition: A \(\Gamma\)-semi group \(S\) is called

(i) \(L\)-Commutative if \(x\alpha a\beta b = x\alpha b\beta a\)
(ii) \( R \)-Commutative if \( a\alpha b\beta x = b\alpha a\beta x \).

(iii) External commutative if \( a\alpha x\beta b = b\alpha x\beta a \)

(iv) Conditional commutative if \( a\alpha x\beta b = b\alpha x\beta a \Rightarrow a\alpha b = b\alpha a \) for all \( a, b, x \in S \)

and \( \alpha, \beta \in \Gamma \)

2.1.4. Definition: A \( \Gamma \)-semi group \( S \) is said to be left (right) singular \( \Gamma \)-semi group if 
\[ a\alpha b = a \) (a\alpha = b) \] for all \( a, b \in S \) and \( \alpha \in \Gamma \).

2.1.5. Definition: A \( \Gamma \)-semi group \( S \) is said to be a rectangular \( \Gamma \)-band if it satisfy the identity 
\[ a\alpha b\beta a = a \) for all \( a, b \in S \) and \( \alpha, \beta \in \Gamma \).

2.1.6. Definition: A \( \Gamma \)-semi group \( S \) is said to be a simple \( \Gamma \)-semi group if 
\[ a\alpha a \) = 1 \] for all \( a \in S \).

2.1.7. Definition: A \( \Gamma \)-semi group \( S \) is called an inverse \( \Gamma \)-semi group if for every \( a \in S \) there exists unique element \( x \) such that \( a = a\alpha x\beta a \) and \( x = x\alpha a\beta x \).

2.1.8. Definition: A \( \Gamma \)-semi group \( S \) is said to be g-regular if for every \( a \in S \) there exists \( x \) such that \( x = x\alpha a\beta x \)

2.1.9. Note: (i) Every regular \( \Gamma \)-semi group is g-regular.

(ii) Every inverse \( \Gamma \)-semi group is g-regular.

(iii) Every simple \( \Gamma \)-semi group with an identity element is g-regular.

2.1.10. Definition: An element ‘\( a \)' of a \( \Gamma \)-semi group \( S \) is called an \( E \) -inversive if there exists an element \( x \) such that \( (a\alpha x)\beta(a\alpha x) = (a\alpha x) \). i.e. \( a\alpha x \) \( E(S) \), where \( E(S) \) is a set of idempotent elements of \( S \).

2.1.11. Definition: A \( \Gamma \)-semi group \( S \) is called an \( E \)-inversive if every element of \( S \) is an \( E \)-inversive.

2.1.12. Result [30]: Every regular \( \Gamma \)-semi group is \( E \)-inversive. i.e. \( a = a\alpha x\beta a \Rightarrow a\alpha x = (a\alpha x)\beta(a\alpha x) \Rightarrow a\alpha x \in E(S) \)
2.1.13. Definition: A Γ-semi group $S$ is said to be a left (right) cancellative Γ-semi group if $a \alpha x = b \alpha x \Rightarrow a = b (x \alpha a = x \alpha b \Rightarrow a = b)$ for all $a, b \in S$ and $\alpha \in \Gamma$.

2.1.14. Definition: A Γ-semi group $S$ is said to be a cancellative Γ-semi group if it is left and right cancellative Γ-semi group.

2.1.15. Definition: A Γ-semi group $S$ is said to be a commutative Γ-semi group if $a \alpha b = b \alpha a$ for all $a, b \in S$ and $\alpha \in \Gamma$.

2.1.16. Definition: A Γ -semi group $S$ is said to be a weakly separative if $a \alpha a = a \alpha b = b \alpha a = b \alpha b \Rightarrow a = b$ for all $a, b \in S$ and $\alpha \in \Gamma$.

2.1.17. Definition: A Γ-semi group $S$ is said to be a quasi separative if $a \alpha a = a \alpha b = b \alpha b \Rightarrow a = b$ for all $a, b \in S$ and $\alpha \in \Gamma$.

2.1.18. Definition: A Γ-semi group $S$ is said to be separative if $a \alpha b \beta c = a \alpha c \beta b \alpha b \beta c$ for every $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

2.1.19. Definition: A Γ-semi group $S$ is said to be left (right) permutable if $a \alpha b \beta c = a \alpha \beta a \beta c$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

2.1.20. Definition: A Γ-semi group $S$ is said to be permutative if it is left and right permutative.

2.1.21. Definition: A Γ-semi group $S$ is said to be Γ-medial if $a \alpha b \gamma d = a \alpha c \beta b \gamma d$ for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

2.1.22. Definition: A Γ-semi group $S$ is said to be right (left) zero Γ-semi group if $a \alpha b = b$ for all $a, b \in S$ and $\alpha \in \Gamma$.

2.1.23. Definition: An element $a$ of a Γ-semi group $S$ is called an idempotent if $a \alpha a = a$. If every element of $S$ is an idempotent, then $S$ is called an idempotent Γ-semi group or Γ-band.

2.1.24. Definition: A commutative Γ-band is called a Γ-semi lattice.

2.1.25. Definition: A Γ-semi group $S$ is called right (left) simple Γ-semi group if it contains no
proper right (left) $\Gamma$-ideal of $S$.

2.1.26. **Definition:** A $\Gamma$-semi group $S$ is called right (left) simple $\Gamma$-semi group if $s\Gamma S = S$ ($\Gamma s = S$) for all $s \in S$.

2.1.27. **Definition:** A $\Gamma$-semi group $S$ is called right $\Gamma$-group if it is right simple and left cancellative $\Gamma$-semi group.

2.1.28. **Theorem:** A $\Gamma$-semi group $S$ is completely regular $\Gamma$-semi group if and only if it is left and right regular $\Gamma$-semi group.

**Proof:** Let $S$ be a $\Gamma$-semi group. Assume that $S$ is a completely regular $\Gamma$-semi group. Then for every $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x \beta a$ and $a\alpha x = x\alpha a$.

To show that $S$ is right regular $\Gamma$-semi group,

Consider $a = a\alpha x \beta a = a = (a\alpha x) \beta a = a\alpha (x \beta a) = a\alpha (a \beta x) = a \alpha a \beta x \Rightarrow a = a\alpha a \beta x$

$\therefore S$ is a right regular $\Gamma$-semi group.

Similarly, $a = a\alpha x \beta a = (a\alpha x) \beta a = (x \alpha a) \beta a = x \alpha (a \beta a) \Rightarrow a = x \alpha a \beta a$

$\therefore S$ is a left regular $\Gamma$-semi group.

Conversely, let $S$ be a left and right regular $\Gamma$-semi group. Then for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha a \beta x$ and $a = x \alpha a \beta a$.

Consider $a = a\alpha a \beta x = a\alpha (a) \beta x = a\alpha (x \alpha a \beta a) \beta x = a\alpha a \alpha (a \beta a \beta x) = a\alpha \alpha a \alpha \Rightarrow a = a\alpha \alpha a \alpha$

And $a\alpha x = (x \alpha a \beta a) \alpha x = x \alpha (a \beta a \alpha x) = x \alpha a$

$\therefore S$ is a completely regular $\Gamma$-semi group.

2.1.29. **Theorem:** Every regular $\Gamma$-semi group is an $E$-inversive $\Gamma$-semi group.

**Proof:** Let $S$ be a regular $\Gamma$-semi group. Then for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x \beta a$

Consider $a \alpha x \beta a = a \Rightarrow x \beta (a \alpha x \beta a) = x \beta a \Rightarrow (x \beta a) \alpha (x \beta a) = x \beta a$
Put \( x\beta a = c \Rightarrow c\alpha c = c \)

\[ \therefore c \text{ is an idempotent. i.e. } x\beta a \in E(S) \]

Similarly, \( a\alpha x\beta a = a \Rightarrow (a\alpha x\beta a)\alpha x = a\alpha x \Rightarrow (a\alpha x)\beta (a\alpha x) = a\alpha x \)

Put \( a\alpha x = d \Rightarrow d\beta d = d \)

\[ \therefore d \text{ is an idempotent. i.e. } a\alpha x \in E(S) \]

Hence \( S \) is an \( E \)-inversive \( \Gamma \)-semi group.

**2.1.30. Theorem:** If a regular \( \Gamma \)-semi group \( S \) has a unique idempotent, then it commutes with all the elements of \( S \).

**Proof:** Let \( S \) be a regular \( \Gamma \)-semi group. Then for any \( a \in S \) there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = a\alpha x\beta a \). This implies \( a\alpha x\beta a\alpha x = a\alpha x \Rightarrow (a\alpha x)\beta (a\alpha x) = a\alpha x \)

\[ \Rightarrow a\alpha x \text{ is an idempotent.} \]

Similarly, we can show that \( x\beta a \) is an idempotent in \( S \) for all \( a \in S \).

Since \( S \) has a unique idempotent, we have \( a\alpha x = x\beta a \) for all \( a \in S \)

Now let \( a\alpha x\beta a = (a\alpha x)\beta a \)

\[ \Rightarrow a\alpha x\beta a = (x\beta a)\beta a \Rightarrow a\alpha x\beta a = (x\beta a)\beta a \]

\[ \therefore \text{ Every element of } S \text{ commutes with an idempotent element of } S. \]

**2.1.31. Theorem:** Let \( S \) be a right (left) zero \( \Gamma \)-semi group. Then the following conditions are equivalent.

(i) \( S \) is a rectangular \( \Gamma \)-band.

(ii) \( a\alpha b = b\alpha a \Rightarrow a = b \).

(iii) \( S \) is a \( \Gamma \)-band and \( a\alpha b\beta c = a\alpha c \).

**Proof:** Let \( S \) be a right zero \( \Gamma \)-semi group. Then \( a\alpha b = b \) for all \( a, b \in S \) and \( \alpha \in \Gamma \).

(i) \( \Rightarrow \) (ii)
Assume that $S$ is a rectangular $\Gamma$-band. Then $a\alpha b\beta a = a$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$.

Consider $a\alpha b = b\alpha a$

$\Rightarrow a\beta(a\alpha b) = a\beta(b\alpha a)$

$\Rightarrow a\beta(a\alpha b) = a$

$\Rightarrow (a\beta\alpha)a\beta = a$

$\Rightarrow a\alpha b = a$ \hspace{1cm} (\because S$ is a right zero $\Gamma$-semi group)

$\Rightarrow b = a$

Hence $a\alpha b = b\alpha a \Rightarrow a = b$

(ii) $\Rightarrow$ (i)

Assume that $a\alpha b = b\alpha a \Rightarrow a = b$ for all $a, b \in S$ and $\alpha \in \Gamma$.

To prove that $S$ is a rectangular $\Gamma$-band, consider

$a\alpha b = b\alpha a$

$\Rightarrow (a\alpha b)\beta a = (b\alpha a)\beta a$

$\Rightarrow a\alpha b\beta a = b\alpha(a\beta a)$ \hspace{1cm} (\because S$ is a right zero $\Gamma$-semi group)

$\Rightarrow a\alpha b\beta a = b\alpha a$ \hspace{1cm} (\because S$ is a right zero $\Gamma$-semi group)

$\Rightarrow a\alpha b\beta a = a$ for all $a, b \in S$

Hence $S$ is a rectangular $\Gamma$-band.

(i) $\Rightarrow$ (iii)

Assume that $S$ is a rectangular $\Gamma$-band. Then $a\alpha b\beta a = a$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$. Since $S$ is right zero $\Gamma$-semi group, we have $a\alpha a = a$ for all $a \in S$ and $\alpha \in \Gamma \Rightarrow S$ is a $\Gamma$-band.

Again consider $a\alpha b\beta c = a\alpha b\beta(c)$

$= a\alpha b\beta(c\alpha a\beta c)$ \hspace{1cm} (\because S$ is a rectangular $\Gamma$-band)

$= a\alpha (b\beta c)a\alpha b\beta c$
\[= a\alpha c\alpha a\beta c \quad (\because b\beta c = c)\]

\[= a\alpha (c\alpha a\beta c)\]

\[\therefore a\alpha b\alpha c = a\alpha c\quad (\because S \text{ is a rectangular } \Gamma\text{-band})\]

(iii) \(\Rightarrow\) (i)

Assume that \(S\) is a \(\Gamma\)-band and \(a\alpha b\beta c = a\alpha c\).

Consider \(a\alpha b\beta c = a\alpha c\)

Put \(c = a\quad \Rightarrow a\alpha b\beta a = a\alpha a\)

\[\Rightarrow a\alpha b\beta a = a \text{ for all } a \in S\]

\[\therefore S \text{ is a rectangular } \Gamma\text{-band.} \]

2.1.3. Theorem: If \(S\) is a regular \(\Gamma\)-semi group, then the following statements are equivalent.

(i) Every \(\alpha\)-idempotent is a left identity of \(S\).

(ii) \(S\) is a left cancellative \(\Gamma\)-semi group.

(iii) \(S\) is a right \(\Gamma\)-group.

(iv) \(S\) is a right simple \(\Gamma\)-semi group.

(v) The set of all \(\alpha\)-idempotents of \(S\) is right zero \(\Gamma\)-semi group.

Proof: Let \(S\) be a regular \(\Gamma\)-semi group. Then for any \(a \in S\) there exists \(x \in S\) and \(\alpha, \beta \in \Gamma\) such that \(a = a\alpha x\beta a\)

(i) \(\Rightarrow\) (ii)

Assume that every \(\alpha\)-idempotent is a left identity of \(S\). Let \(e\) be an \(\alpha\)-idempotent of \(S\). Then \(e\alpha a = a\) for all \(a \in S\)

Since \(S\) is regular \(\Gamma\)-semi group, we have \(a = a\alpha x\beta a\quad \Rightarrow x\beta a = x\beta a\alpha x\beta a\quad \Rightarrow x\beta a = (x\beta a)\alpha (x\beta a)\quad \Rightarrow (x\beta a)\) is an idempotent \(\Rightarrow (x\beta a)\) is a left identity.

Now consider \(a\alpha b = a\alpha c\quad \Rightarrow (x\beta a)\alpha b = (x\beta a)\alpha c\).
\[ \therefore \ aab = aac \Rightarrow b = c \]  
\[ (\because x\beta a \text{ is left identity of } S ) \]

Hence \( S \) is a left cancellative \( \Gamma \)-semi group.

\((ii) \Rightarrow (iii)\)

Let \( S \) be a left cancellative \( \Gamma \)-semi group. Since \( S \) is a regular \( \Gamma \)-semi group and left cancellative \( \Gamma \)-semi group, so, \( S \) is a right \( \Gamma \)-group.

\((iii) \Rightarrow (iv)\)

Let \( S \) be a right \( \Gamma \)-group. Then \( S \) is right simple \( \Gamma \)-semi group.

\((iv) \Rightarrow (v)\)

Let \( S \) be a right simple \( \Gamma \)-semi group. Then we have \( s\Gamma S = S \) for all \( s \in S \). Let \( e \) and \( f \) be two idempotents of \( S \). Then

\[ e = f\beta x \Rightarrow e = f\alpha f\beta x \Rightarrow e = f\alpha (f\beta x) \Rightarrow e = f\alpha e \Rightarrow S \text{ is right zero } \Gamma \text{-semi group.} \]

\((v) \Rightarrow (i)\)

Assume that the set of all \( \alpha \)-idempotents of \( S \) is right zero \( \Gamma \)-semi group.

Let \( a \in S \) and \( e \) be an idempotent of \( S \). Then there exists \( x \in S \) such that \( e = a\alpha x \)

Consider \[a = a\alpha x\beta a \Rightarrow e\alpha a = (e)\alpha a\alpha x\beta a \Rightarrow e\alpha a = (a\alpha x)\alpha (a\alpha x)\beta a \]

\[ \Rightarrow e\alpha a = a\alpha x\beta a \Rightarrow e\alpha a = a \]

Therefore, every \( \alpha \)-idempotent is a left identity.

\[ \text{2.1.33. Theorem: Every singular } \Gamma \text{-semi group } S \text{ is (i) Rectangular } \Gamma \text{-band (ii) } \Gamma \text{-Semi lattice} \]

Proof: Let \( S \) be a singular \( \Gamma \)-semi group. Then for all \( a, b \in S \) and \( \alpha \in \Gamma \ aab = a \) and \( aab = b \)

Consider \[ a\alpha b\beta a = (a\alpha b)\beta a \Rightarrow a\alpha b\beta a = a\beta a \]  
\[ (\because S \text{ is singular}) \]

\[ \Rightarrow a\alpha b\beta a = a \]

\[ \therefore S \text{ is a rectangular } \Gamma \text{-band.} \]

\((ii) \) Since \( S \) is singular \( \Gamma \)-semi group, \( aab = a \) and \( aab = b \) for all \( a, b \in S \)
and \( \alpha \in \Gamma \). Therefore \( a\alpha a = a \) for all \( a \in S \) \( \Rightarrow \) \( S \) is band.

Also \( a\alpha b = a = b\alpha a \Rightarrow a\alpha b = b\alpha a \)

\[ \therefore S \text{ is commutative } \Gamma \text{-semi group.} \]

Hence \( S \) is a \( \Gamma \)-semi lattice.

2.1.34. **Theorem:** Let \( S \) be a rectangular \( \Gamma \)-band with an identity \( a\alpha a' = a = a'\alpha a \) for all \( a \in S \) and for some \( a' \in S \). Then \( S \) is a singular \( \Gamma \)-semi group.

**Proof:** Let \( S \) be a rectangular \( \Gamma \)-band. Then \( a\alpha b\beta a = a \) for all \( a, b \in S \) and \( \alpha \in \Gamma \).

Assume that \( S \) satisfies the identity \( a\alpha a' = a = a'\alpha a \) for all \( a \in S \) and for some \( a' \in S \) and \( \alpha \in \Gamma \). Consider \( a\alpha b = (a\alpha a')(\alpha(b\alpha a')) = a\alpha a'\alpha b\alpha a' = a\alpha (a'\alpha b\alpha a') = a\alpha a' = a \)

\[ \therefore a\alpha b = a \]

\( \Rightarrow S \) is a left singular \( \Gamma \)-semi group.

Similarly, consider \( a\alpha b = (a'\alpha a)(\alpha(a'\beta)) \Rightarrow a\alpha b = (a'\alpha a\alpha a') \alpha b) \]

\[ \Rightarrow a\alpha b = (a'\alpha b) \Rightarrow a\alpha b = b \Rightarrow S \text{ is a right singular } \Gamma \text{-semi group.} \]

Hence \( S \) is a singular \( \Gamma \)-semi group.

2.2. **Cancellative left regular \( \Gamma \)-semi groups:**

In this section we prove some results on the structures of left regular \( \Gamma \)-semi groups using the cancellative property.

2.2.1. **Theorem:** A cancellative \( \Gamma \)-semi group is regular if and only if it is g-regular.

**Proof:** Let \( S \) be a cancellative \( \Gamma \)-semi group. Assume that \( S \) is regular \( \Gamma \)-semi group. Then for all \( a \in S \) there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = a\alpha x\beta a \)

\[ \Rightarrow a\alpha x = a\alpha x\beta a\alpha x \Rightarrow x = \beta a\alpha x \]

\[ (\because S \text{ is a cancellative } \Gamma \text{-semi group}) \]

\[ \Rightarrow \text{for every } a \in S \text{ there exists } x \in S \text{ such that } x = \beta a\alpha x \]
\[ \therefore \, a \text{ is } g\text{-regular} \]

Hence \( S \) is a \( g \)-regular \( \Gamma \)-semi group.

Conversely, assume that \( S \) is a \( g \)-regular \( \Gamma \)-semi group. Then for any \( a \in S \) there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( x = x\alpha a\beta x \)

\[ \Rightarrow xaa = x\alpha a\beta xaa \Rightarrow a = a\beta x\alpha a \quad (\therefore S \text{ is cancellative}) \]

\[ \Rightarrow a \text{ is regular} \]

Hence \( S \) is a regular \( \Gamma \)-semi group.

**2.2.2. Theorem:** A cancellative \( \Gamma \)-semi group is left (right) regular \( \Gamma \)-semi group if and only if it is \( g \)-regular \( \Gamma \)-semi group.

**Proof:** Similar to the proof of Theorem 2.2.1

**2.2.3. Theorem:** A cancellative left regular \( \Gamma \)-semi group is commutative.

**Proof:** Let \( S \) be a cancellative left regular \( \Gamma \)-semi group. Then

Consider

\[ (a\alpha b)^2 = (a\alpha b)\alpha(a\alpha b) \]

\[ \Rightarrow a^2\alpha b^2 = a\alpha b\alpha a\alpha b \]

\[ \Rightarrow a\alpha a\alpha b^2 = a\alpha b\alpha a\alpha b \]

\[ \Rightarrow a\alpha b^2 = b\alpha a\alpha b \]

\[ \Rightarrow a\alpha b \alpha b = b\alpha a\alpha b \]

\[ \Rightarrow a\alpha b = b\alpha a \]

\[ \therefore S \text{ is a commutative } \Gamma \text{-semi group} \]

Hence a cancellative left regular \( \Gamma \)-semi group is commutative.

**2.2.4. Theorem:** A cancellative right regular \( \Gamma \)-semi group is commutative.

**Proof:** Similar to the proof of Theorem 2.2.3.

**2.2.5. Theorem:** A cancellative \( \Gamma \)-semi group \( S \) is left regular \( \Gamma \)-semi group if and only if \( S \) is
completely regular $\Gamma$-semi group.

**Proof:** Let $S$ be a cancellative $\Gamma$-semi group. Assume that $S$ is left regular $\Gamma$-semi group. Then for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha a \beta a$

Consider 

$$a\alpha x \beta a = a\alpha x (x\alpha a \beta a)$$

$$\Rightarrow (a\alpha x) \beta a = (a\alpha x \beta x \alpha a) \beta a$$

$$\Rightarrow a\alpha x = a\alpha x \beta x \alpha a$$

(\because S is cancellative)

$$\Rightarrow a\alpha x = a\alpha x (x \alpha a)$$

$$\Rightarrow a\alpha x = a\alpha x \beta (a \alpha x)$$

(\because S is commutative)

$$\Rightarrow a\alpha x = (a\alpha x \beta a) \alpha x$$

$$\Rightarrow a = (a\alpha x \beta a)$$

(\because S is cancellative)

$$\Rightarrow a$$ is regular for all $a \in S$

And also, by the Theorem 2.2.3., $S$ is commutative i.e. $a\alpha x = x \alpha a$.

Hence $S$ is a completely regular $\Gamma$-semi group.

Conversely, let $S$ is completely regular $\Gamma$-semi group. Then for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x \beta a)$ and $a\alpha x = x \alpha a$.

Consider 

$$a\alpha x \beta a = a$$

$$\Rightarrow x\alpha a \alpha x \beta a = x \alpha a$$

$$\Rightarrow x\alpha (a\alpha x) \beta a = x \alpha a$$

(\because S is completely regular)

$$\Rightarrow x\alpha (x \alpha a \beta a) = x \alpha a$$

$$\Rightarrow x\alpha a \beta a = a$$

(\because S is cancellative)

\therefore S is a left regular $\Gamma$-semi group.
Similarly, we can prove that a cancellative $\Gamma$-semi group $S$ is right regular $\Gamma$-semi group if and only if $S$ is completely regular $\Gamma$-semi group.

2.2.6. Theorem: A cancellative $\Gamma$-semi group $S$ is left regular $\Gamma$-semi group if and only if $S$ is $E$-inversive $\Gamma$-semi group.

Proof: Let $S$ be a cancellative $\Gamma$-semi group. Assume that $S$ is left regular $\Gamma$-semi group. Then, by the Theorem 2.2.3., $S$ is commutative

Consider $x\alpha x\beta a = a \Rightarrow x\beta (x\alpha a)\beta a = x\beta a \Rightarrow x\beta (a\alpha x)\beta a = x\beta a$

$\Rightarrow (x\beta a)\alpha (x\beta a) = x\beta a \Rightarrow x\beta a \in E(S)$

Similarly, $x\alpha a\beta a = a \Rightarrow (x\alpha a\beta a)\alpha x = a\alpha x \Rightarrow (x\alpha a)\beta (x\alpha a) = a\alpha x$

$\Rightarrow (a\alpha x)\beta (a\alpha x) = a\alpha x \Rightarrow a\alpha x \in E(S)$

$\Rightarrow a$ is an $E$-inversive element in $S$

$\therefore S$ is a $E$-inversive $\Gamma$-semi group.

Conversely, assume that $S$ is $E$-inversive $\Gamma$-semi group. Then for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $(a\alpha x)\beta (a\alpha x) = a\alpha x$

Consider $(a\alpha x)\beta (a\alpha x) = a\alpha x$

$\Rightarrow (a\alpha x\beta a)\alpha x = a\alpha x$

$\Rightarrow a\alpha x\beta a = a$ \hspace{1cm} (\because S$ is cancellative$)$

$\Rightarrow (a\alpha x)\beta a = a$

$\Rightarrow (x\alpha a)\beta a = a$ \hspace{1cm} (\because S$ is commutative$)$

$\Rightarrow x\alpha a\beta a = a$

$\therefore S$ is a left regular $\Gamma$-semi group.

Similarly, we can prove that a cancellative $\Gamma$-semi group $S$ is right regular $\Gamma$-semi group if and only if $S$ is $E$-inversive $\Gamma$-semi group.
2.2.7. **Theorem:** A cancellative left regular $\Gamma$-semi group $S$ is commuting regular $\Gamma$-semi group.

**Proof:** Assume that $S$ is cancellative left regular $\Gamma$-semi group. By the Theorem 2.2.5., $S$ is regular $\Gamma$-semi group.

Let $a, b \in S$. Then there exists $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha\beta a = a$ and $b\alpha\beta b = b$.

Consider $a\alpha b = (a\alpha\beta a)\alpha(b\alpha\beta b)$

$= a\alpha\beta(a\alpha b)\alpha\beta b$

$= a\alpha\beta(b\alpha a)\alpha\beta b$

$= a\alpha(x\beta b)\alpha(a\alpha)\beta b$

$= a\alpha(b\beta x)\alpha(x\alpha a)\beta b$

$= (a\alpha b)\beta(x\alpha x)\alpha a\beta b$

$= (a\alpha b)\beta x\alpha(a\beta b)$

$= (a\alpha b)\beta x\alpha(a\beta b)$

$\therefore S$ is a commuting regular $\Gamma$-semi group.

2.3. **Regular $\Gamma$-semi groups satisfying the identity** $a\alpha b\beta c = c\alpha b$:

This section deals with the structures of regular $\Gamma$-semi groups which satisfy the identity $a\alpha b\beta c = c\alpha b$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. The results obtained in this section based on the results of Yamada.M. and Kimura.N. [47].

2.3.1. **Theorem:** A regular $\Gamma$-semi group $S$ satisfying the identity $a\alpha b\beta c = c\alpha b$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, is one of the following.

(i) Commutative $\Gamma$-semi group.

(ii) Left (right) regular $\Gamma$-semi group.

(iii) Completely regular $\Gamma$-semi group.
(iv) Commuting regular $\Gamma$-semi group.

**Proof:** Let $S$ be a regular $\Gamma$-semi group satisfying the identity $aab\beta c = cab$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Since $S$ is regular $\Gamma$-semi group, for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$

(i) To prove $S$ is commutative

Consider $aab = cab\beta a \Rightarrow aab = (cab)\beta a \Rightarrow aab = (aab\beta c)\beta a \Rightarrow a\alpha b = aab\beta(c\beta a)$

$\Rightarrow a\alpha b = aab\beta d\beta a\alpha c \Rightarrow a\alpha b = (aab\beta d)\beta a\alpha c \Rightarrow a\alpha b = (dab)\beta a\alpha c \Rightarrow a\alpha b = (d\alpha (b\beta a\alpha c)\beta a\alpha c) \Rightarrow a\alpha b = (d\alpha (b\beta a\alpha c)\beta a\alpha c) \Rightarrow a\alpha b = (d\alpha a\beta b)\beta a \Rightarrow a\alpha b = b\alpha a$

$\therefore S$ is a commutative $\Gamma$-semi group.

(ii) Since $S$ is a regular $\Gamma$-semi group, for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$

Consider $a = a\alpha x\beta a \Rightarrow a = (x\alpha a)\beta a \Rightarrow a = x\alpha a\beta a \quad (\therefore S$ is commutative)

$\therefore S$ is a left regular $\Gamma$-semi group.

Similarly, $a = a\alpha x\beta a \Rightarrow a = a\alpha (x\beta a) \Rightarrow a = a\alpha a\alpha x \quad (\therefore S$ is commutative)

$\therefore S$ is a right regular $\Gamma$-semi group.

(iii) Since $S$ is a regular $\Gamma$-semi group, for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$ and by the condition (i) we have $a\alpha x = a\alpha x$

$\therefore S$ is a completely regular $\Gamma$-semi group.

(iv) To prove $S$ is commuting regular $\Gamma$-semi group, we need to prove that $a\alpha b = b\alpha a(\beta\gamma) b\alpha a$

Consider $a\alpha b = x\alpha b\beta a$

$= (x\alpha b)\beta a$
\[(a\alpha b\beta x)\beta a\]
\[(a\alpha b)\beta (x\beta a)\]
\[(a\alpha b)\beta(b\beta a\alpha x)\]
\[(a\alpha b)\beta b\beta(a\alpha x)\]
\[(a\alpha b)\beta b\beta(x\alpha a)\]
\[(a\alpha b)\beta(b\beta x)\alpha a\]
\[(a\alpha b)\beta(x\beta b)\alpha a\]
\[(a\alpha b)\beta x\beta(b\alpha a)\]
\[(b\alpha a)\beta x\beta(b\alpha a)\]

\[a\alpha b = (b\alpha a)\beta x\beta(b\alpha a)\]

\(\therefore S\) is a commuting regular \(\Gamma\) - semi group.

2.3.2. Theorem: A regular \(\Gamma\) - semi group \(S\) satisfying the identity \(a\alpha b\beta c = c\alpha b\) for all \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\) is quasi- separative \(\Gamma\) - semi group.

Proof: Let \(S\) be a regular \(\Gamma\) - semi group with the identity \(a\alpha b\beta c = c\alpha b\). Then, by the Theorem 2.3.1., \(S\) is a left regular \(\Gamma\) - semi group.

To prove that \(S\) is quasi separative \(\Gamma\) -semi group, we shall prove that for any \(a, b \in S\) and \(\alpha \in \Gamma\) such that \(a\alpha a = a\alpha b = b\alpha b \Rightarrow a = b\)

Consider \(a\alpha a = a\alpha b \Rightarrow a\alpha(a\alpha x\beta a) = a\alpha(b\alpha y\beta b)\)

\[\Rightarrow a\alpha(a\alpha x)\beta a = (a\alpha b\alpha y)\beta b\]
\[\Rightarrow a\alpha(c\alpha x\beta a)\beta a = yab\beta b\]
\[\Rightarrow a\alpha\alpha(x\beta a\beta a) = yab\beta b\]
\[\Rightarrow a\alpha(c)\alpha a = b \quad \therefore S\text{ is left regular}\]
\[\Rightarrow a\alpha(c\alpha z\beta c)\alpha a = b\]
\[(a\alpha c\alpha z)\beta c\alpha a = b\]
\[(z\alpha c)\beta c\alpha a = b\]
\[(z\alpha c\beta c)\alpha a = b\]
\[\Rightarrow c\alpha a = b\]  \(\because S\) is left regular

\[\Rightarrow c\alpha (a\alpha x\beta a) = b\]
\[\Rightarrow (c\alpha a\alpha x)\beta a = b\]
\[\Rightarrow x\alpha a\beta a = b\]
\[\Rightarrow a = b\]

Consider  \(b\alpha b = b\alpha a \Rightarrow (b\alpha y\beta b)c\alpha b = b\alpha (a\alpha x\beta a)\)
\[\Rightarrow (b\alpha y)\beta b\alpha b = (b\alpha a\alpha x)\beta a\]
\[\Rightarrow (c\alpha y\beta b)\beta b\alpha b = (x\alpha a)\beta a\]
\[\Rightarrow c\alpha (y\beta b\beta b)c\alpha b = (x\alpha a\beta a)\]
\[\Rightarrow c\alpha b\beta b = a\]
\[\Rightarrow b = a\]

\[\because a\alpha a = a\alpha b = b\alpha b \Rightarrow a = b\]

Hence  \(S\) is a quasi-separative \(\Gamma\) - semi group.

2.3.3. Theorem: A regular \(\Gamma\) - semi group  \(S\) with an identity  \(a\alpha b\beta c = c\alpha b\) is weakly separative \(\Gamma\) - semi group

Proof: Let  \(S\) be a regular \(\Gamma\) - semi group satisfying the identity  \(a\alpha b\beta c = c\alpha b\)

To prove that  \(S\) is weekly separative, we shall prove that  \(a\alpha a = b\alpha a = b\alpha b \Rightarrow a = b\)

By the Theorem 2.3.2., we have  \(a\alpha a = a\alpha b = b\alpha b \Rightarrow a = b\) and by the Theorem 2.3.1.,  \(S\) is commutative  \(\because a\alpha a = b\alpha a = b\alpha b \Rightarrow a = b\)

Hence  \(S\) is a weekly separative \(\Gamma\) - semi group.
2.3.4. **Theorem:** A regular $\Gamma$-semi group $S$ with an identity $a\alpha b\beta c = c\alpha b$ is separative $\Gamma$-semi group.

**Proof:** Let $S$ be a regular $\Gamma$-semi group satisfying the identity $a\alpha b\beta c = c\alpha b$.

To prove that $S$ is separative, we shall prove that $a\alpha a = a\alpha b$ and $b\alpha b = b\alpha a$.

\[ \Rightarrow a = b \text{ and } a\alpha a = b\alpha a \text{ and } b\alpha b = a\alpha b \Rightarrow a = b. \]

Let $a\alpha a = a\alpha b$ and $b\alpha a = b\alpha b$. By the Theorem 2.3.1., $S$ is commutative.

\[ \therefore a\alpha a = a\alpha b = b\alpha a = b\alpha b \text{ and by the Theorem 2.3.3. } S \text{ is weakly separative}. \]

ie. $a\alpha a = a\alpha b = b\alpha a = b\alpha b \Rightarrow a = b$

Similarly, we can prove that $a\alpha a = b\alpha a$ and $b\alpha b = b\alpha a \Rightarrow a = b$

\[ \therefore S \text{ is a separative } \Gamma \text{- semi group}. \]

2.3.5. **Theorem:** Let $S$ be a regular $\Gamma$-semi group satisfying the identity $a\alpha b\beta c = c\alpha b$. Then $S$ is permutable $\Gamma$-semi group.

**Proof:** Let $S$ be a regular $\Gamma$-semi group satisfying the identity $a\alpha b\beta c = c\alpha b$.

Since $S$ is regular $\Gamma$-semi group, for any $a, b \in S$ there exists $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$ and $b = b\alpha y\beta b$.

Consider

\[ a\alpha b\beta c = (a\alpha x\beta a)\alpha b\beta c \]

\[ = a\alpha (x\beta a\alpha b)\beta c \]

\[ = a\alpha b\beta a\beta c \]

\[ = (a\alpha b\alpha y)\beta (b\beta a\beta c) \]

\[ = (y\alpha b)\beta b\alpha a\beta c \]

\[ = (y\alpha b\beta b)\alpha a\beta c \]

\[ a\alpha b\beta c = b\alpha a\beta c \]

\[ \therefore S \text{ is left permutable } \Gamma \text{- semi group}. \]
Similarly, consider \( a\alpha b\beta c = (a\alpha x\beta a)\alpha b\beta c \)
\[
= (a\alpha x\beta a)\alpha b\beta c \\
= a\alpha (b)\beta c \\
= a\alpha (b\alpha y\beta b)\beta c \\
= (a\alpha b\alpha y)\beta b\beta c \\
= (\gamma \alpha b\beta b)\beta c \\
= b\beta c \\
\]
\( a\alpha b\beta c = a\alpha c\beta b \)
\[
\therefore S \text{ is right permutable } \Gamma - \text{semi group.}
\]

Hence \( S \) is permutable \( \Gamma - \text{semi group.} \)

2.3.6. Theorem: Let \( S \) be regular \( \Gamma - \text{semi group satisfying the identity } a\alpha b\beta c = cab \). Then \( S \) is a \( \Gamma - \text{medial.} \)

Proof: From the Theorem 2.3.5., \( S \) is permutable. i.e. \( a\alpha b\beta c = a\alpha c\beta b \) for all \( a, b, c \in S \) and 
\[
\alpha, \beta \in \Gamma \quad \text{For} \quad a\alpha b\beta c\gamma d = (a\alpha b\beta c)\gamma d = (a\alpha c\beta b)\gamma d \\
\]
\( a\alpha b\beta c\gamma d = a\alpha c\beta b\gamma d \)
\[
\text{Hence } S \text{ is } \Gamma - \text{medial.}
\]

2.3.7. Theorem: A regular \( \Gamma - \text{semi group } S \) satisfying the identity \( a\alpha b\beta c = cab \) is conditionally commutative \( \Gamma - \text{semi group.} \)

Proof: Let \( S \) be a regular \( \Gamma - \text{semi group satisfying the identity } a\alpha b\beta c = cab \) for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \). Since \( S \) is regular \( \Gamma - \text{semi group, for any } a, b \in S \) there exists \( x, y \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = a\alpha x\beta a \) and \( b = b\alpha y\beta b \). Let \( a\alpha b = b\alpha a \). Then we shall prove that \( a\alpha z\alpha b = b\alpha z\alpha a \) for some \( z \in S \)
Consider \( a\alpha b = (a\alpha x\beta a)\alpha b \)
\[= a\alpha(x\beta a)ab\]
\[= a\alpha z\beta(a\alpha x)ab\]
\[= a\alpha z\beta(x\alpha a)ab\]
\[= a\alpha(z\beta x)a\alpha a\beta\]
\[= a\alpha(x\beta z)a\alpha a\beta\]
\[= a\alpha x\beta(z\alpha a)ab\]
\[= a\alpha(x\beta z)a\alpha a\beta\]
\[= a\alpha x\beta(z\alpha a)ab\]
\[= a\alpha x\beta(a\alpha z)ab\]
\[= (a\alpha x\beta a)\alpha z\alpha ab\]
\[a\alpha a = a\alpha z\alpha ab\]

And
\[b\alpha a = b\alpha y\beta b\alpha a\]
\[= b\alpha(y\beta b)\alpha a\]
\[= b\alpha(z\beta b\alpha y)\alpha a\]
\[= b\alpha(z\beta b)\alpha y\alpha a\]
\[= b\alpha(b\beta z)\alpha y\alpha a\]
\[= bab\beta(z\alpha y)\alpha a\]
\[= (bab\beta y)\alpha z\alpha a\]
\[b\alpha a = b\alpha z\alpha a\]

Hence \[a\alpha b = b\alpha a \Rightarrow a\alpha z\alpha ab = b\alpha z\alpha a\]

\[\therefore S\] is conditionally commutative \(\Gamma\) - semi group.

2.3.8. **Theorem:** A regular \(\Gamma\) - semi group \(S\) satisfying the identity \(a\alpha b\beta c = c\alpha b\) is L-commutative and R-commutative \(\Gamma\) - semi group.
**Proof:** Let $S$ be a regular $\Gamma$-semi group satisfying the identity $aab\beta c = cab$ for all $a,b,c \in S$ and $\alpha, \beta \in \Gamma$. Let $a,b \in S$. Then there exists $x,y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$ and $b = b\alpha y\beta b$

Consider $b\alpha a\beta x = b\alpha (a\alpha x\beta a)\beta x$

$= b\alpha (a\alpha x)\beta a\beta x$

$= b\alpha (b\alpha x\beta a)\beta a\beta x$

$= b\alpha b\alpha (x\beta a\beta a)\beta x$

$= (b\alpha b\alpha a)\beta x$

$b\alpha a\beta x = a\alpha b\beta x$

Hence $S$ is L-commutative $\Gamma$-semi group.

Similarly, we can prove that a regular $\Gamma$-semi group $S$ satisfying the identity $aab\beta c = cab$ is R-commutative $\Gamma$-semi group.