Chapter 5

$k$-th best element rationalization with full domain

5.1 Introduction

Here we would like to find characterization results for a choice function when it is in general $k$-th best element rationalizable, where $k$ is any positive integer. Again throughout this chapter it will be assumed that the domain of a choice function contains all non-empty subsets of $X$, i.e., $C : D \mapsto \Sigma$ where $D = \Sigma$. With the assumption of full domain, for $k = 1$, the rationalizability conditions are already there in the literature for a long time. We therefore focus on cases with $k \geq 2$. Characterization results will be discussed for cases when the binary relation is an ordering as well as when it is reflexive, connected and acyclic.

5.2 Axioms

We now introduce some axioms. We wish to use them in characterization of a $k$-rationalizable choice function. Axioms A.5.1 and A.5.2 are used for characterization of a choice function that is $k$-rationalizable by an ordering. Axioms A.5.3 and A.5.4 characterize a choice function that is $k$-rationalizable by a reflexive, connected and acyclic binary relation.
A.5.1: \((\forall S \in \Sigma)[O_S \geq k \rightarrow C(S) = \{x \in S \mid O_{P_x \cap S} = k - 1\}]\)

A.5.2: \((\forall S, T \in \Sigma)[O_S < k \rightarrow [T \subseteq S \land C(S) \cap T \neq \emptyset \rightarrow C(T) = C(S) \cap T]]\)

A.5.3: \((\forall S \in \Sigma)[O_S \geq k \rightarrow C(S) = \{x \in S \mid O_{S^P} = k - 1\}]\)

A.5.4: \((\forall S \in \Sigma)[O_S < k \rightarrow C(S) = \{x \in S \mid O_{S^P} = O_S - 1\}]\)

It will be proved later that if the binary relation \(R_2\) over a set is reflexive, connected and acyclic then the order of that set ensures the existence of a best element of that order. For example, if \(R_2\) over the set \(S\) is reflexive, connected and acyclic and \(O_S = n\), where \(n \in N\), then there exists an \(n\)-th best element in \(S\). A.5.1 requires that whenever there is a \(k\)-th best element in a set, an element will be chosen if and only if the set of all elements preferred to it is of order \(k - 1\). Intuitively the idea is, if the chosen element is a \(k\)-th best element then among the unchosen preferred elements there should not be any \(k\)-th best element. Moreover, because in the process of choosing this \(k\)-th best element we have eliminated \(k - 1\) consecutive and distinct sets of best elements, it should be the case that this set of unchosen preferred elements has a \((k - 1)\)-th best element. In case of a \(k\)-rationalizable choice function if a \(k\)-th best element does not exist in a set then the chosen elements are essentially the worst elements. Axiom A.5.2 requires that for all such sets Arrow’s axiom holds. Considering the fact that Arrow’s axiom is necessary and sufficient for worst element rationalizability of a choice function by an ordering, axiom A.5.2 seems plausible. The idea behind the axiom A.5.3 is similar to that of A.5.1. Difference between these two axioms is rather technical. If the binary relation \(R_2\) is acyclic then \(P_x \cap S\) is a subset of \(S^P_x\). With a transitive \(R_2\) they become equal. Axiom A.5.4 requires that in the absence of \(k\)-th best element in a set, the chosen elements are those for which the order of the set of preferred elements (the preference relation being \(T(P_2 | S)\)) is one less than that of the mother set. Intuitively, according to the axiom, the set of unchosen preferred elements lacks one preference level than the original set. This missing preference level is constituted by the chosen elements; which makes the chosen elements the set of worst elements in the
set. So the spirit behind axiom A.5.2 and A.5.4 are essentially same even though they look quite different.

5.3 \textit{k-Rationalization}

5.3.1 Necessary and sufficient condition for a choice function to be \(k\)-rationalizable by an ordering

Lemma 5.1 If \(k \geq 3\) and A.5.1 and A.5.2 are satisfied then \(R_2\) is transitive.

Proof: Let there be \(x, y\) and \(z\) in \(X\) such that \(xR_2y\) and \(yR_2z\). Suppose \(zP_2x\).

We know by definition, \(xR_2y \iff y \in C(\{x, y\})\), \(yR_2z \iff z \in C(\{y, z\})\) and \(zP_2x \iff C(\{x, z\}) = \{x\}\). Also \(\{x, y, z\} \in \Sigma\). Let \(\{x, y, z\} = S\).

There can be four cases:

1. \(C(\{x, y\}) = \{y\}, C(\{y, z\}) = \{y, z\}, C(\{x, z\}) = \{x\}\) and therefore \(O_S = 3\).

2. \(C(\{x, y\}) = \{x, y\}, C(\{y, z\}) = \{z\}, C(\{x, z\}) = \{x\}\) and therefore \(O_S = 3\).

3. \(C(\{x, y\}) = \{x, y\}, C(\{y, z\}) = \{y, z\}, C(\{x, z\}) = \{x\}\) and therefore \(O_S = 2\).

4. \(C(\{x, y\}) = \{y\}, C(\{y, z\}) = \{z\}, C(\{x, z\}) = \{x\}\) and therefore \(O_S = 3\).

First let us consider \(k = 3\).

Suppose case 1 holds. Then \(O_S = 3 = k\). Therefore by A.5.1, \(C(S) = \{x \in S \mid O_{P_x \cap S} = 2\}\).

We have \(P_x \cap S = \{z\}\), \(P_y \cap S = \{x\}\) and \(P_z \cap S = \emptyset\).

It turns out then that \(C(S) = \emptyset\), which contradicts the assumption of non-emptiness of choice sets. Therefore case 1 does not hold.
Case 2 is analogous to case 1.

Suppose case 3 holds. Then $O_S = 2 < k$.
Suppose $z \in C(S)$.
By A.5.2, $\{x, z\} \subseteq S \land z \in C(S) \rightarrow z \in C(\{x, z\})$. But this contradicts $C(\{x, z\}) = \{x\}$. Therefore, $z \notin C(S)$.
Suppose $y \in C(S)$.
By A.5.2, $\{y, z\} \subseteq S \land y \in C(S) \land z \notin C(S) \rightarrow C(\{y, z\}) = \{y\}$. But this is contradictory to $C(\{y, z\}) = \{y, z\}$. Therefore, $y \notin C(S)$.
Suppose $x \in C(S)$.
By A.5.2, $\{x, y\} \subseteq S \land x \in C(S) \land y \notin C(S) \rightarrow C(\{x, y\}) = \{x\}$. This again contradicts $C(\{x, y\}) = \{x, y\}$. Therefore, $x \notin C(S)$.
It then turns out that $C(S) = \emptyset$, which is a contradiction. So case 3 does not hold.

Suppose case 4 holds. Then $O_S = 3 = k$. Therefore by A.5.1, $C(S) = \{x \in S | O_{P_x \cap S} = 2\}$.
We have $P_x \cap S = \{z\}$, $P_y \cap S = \{x\}$ and $P_z \cap S = \{y\}$.
It turns out again that $C(S) = \emptyset$ and once more we get a contradiction. Therefore case 4 does not hold.
All four possible cases are leading to contradictions. Therefore we conclude that $\sim z P_2 x$. Also from our assumption that $D = \Sigma$ it follows that $R_2$ is connected. So $x R_2 z$. Therefore with $k = 3$, $R_2$ is transitive if A.5.1 and A.5.2 are satisfied by the choice function.

Now let $k > 3$.

Suppose case 1 holds. Then $O_S = 3 < k$.
Suppose $x \in C(S)$.
By A.5.2, $\{x, y\} \subseteq S \land x \in C(S) \rightarrow x \in C(\{x, y\})$. But we know that $C(\{x, y\}) = \{y\}$. Therefore, $x \notin C(S)$.
Suppose $z \in C(S)$.
By A.5.2, \( \{x, z\} \subseteq S \land z \in C(S) \rightarrow z \in C(\{x, z\}) \). This contradicts \( C(\{x, z\}) = \{x\} \). Therefore, \( z \notin C(S) \).

Suppose \( y \in C(S) \).

By A.5.2, \( \{y, z\} \subseteq S \land y \in C(S) \land z \notin C(S) \rightarrow C(\{y, z\}) = C(S) \cap \{y, z\} = \{y\} \). This is contradictory to \( C(\{y, z\}) = \{y, z\} \). Therefore, \( y \notin C(S) \).

It turns out that \( C(S) = \emptyset \), which is a contradiction. Therefore, case 1 does not hold.

Case 2 is analogous to case 1.

Case 3 also does not hold and the proof is same as with \( k = 3 \).

Suppose case 4 holds. Then \( O_S = 3 < k \).

Suppose \( x \in C(S) \).

By A.5.2, \( \{x, y\} \subseteq S \land x \in C(S) \rightarrow x \in C(\{x, y\}) \).

But we know \( C(\{x, y\}) = \{y\} \). Therefore, \( x \notin C(S) \).

Suppose \( y \in C(S) \).

By A.5.2, \( \{y, z\} \subseteq S \land y \in C(S) \rightarrow y \in C(\{y, z\}) \).

This contradicts with \( C(\{y, z\}) = \{z\} \). Therefore, \( y \notin C(S) \).

Suppose \( z \in C(S) \).

By A.5.2, \( \{x, z\} \subseteq S \land z \in C(S) \rightarrow z \in C(\{x, z\}) \).

This again is a contradiction as we know \( C(\{x, z\}) = \{x\} \). Therefore, \( z \notin C(S) \).

It turns out again that \( C(S) = \emptyset \), which is a contradiction. Therefore case 4 does not hold. Again all four possible cases lead to contradictions. Therefore, \( \sim z P_2 x \). Since \( R_2 \) is connected we have \( x R_2 z \). Therefore with \( k > 3 \), \( R_2 \) is transitive if A.5.1 and A.5.2 are satisfied by the choice function.

Hence the lemma is proved.

**Lemma 5.2** For any \( S \) in \( \Sigma \), \( O_S = j \) and \( R_2 \) over \( S \) is acyclic iff \( G_j(S, R_2) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R_2) = S \) where \( j \in N \).
Proof: Let us consider any arbitrary \( S \) in \( \Sigma \) and let \( O_S = j \). Also let \( R_2 \) over \( S \) be acyclic. We shall first prove that, \( G_j(S, R_2) \neq \emptyset \) \( \land \bigcup_{i=1}^{j} G_i(S, R_2) = S \).

Let \( j = 1 \). Then \( O_S = 1 \).
\[
O_S = 1 \rightarrow (\forall x, y \in S)(C(\{x, y\}) = \{x, y\}) \rightarrow G(S, R_2) = S \tag{5.1}
\]

Let \( j > 1 \).
\[
O_S = j \rightarrow \text{there exist distinct } z_1, z_2, ..., z_j \text{ in } S \text{ such that } (\forall i \in \{1, 2, ..., j-1\})[C(\{z_i, z_{i+1}\}) = \{z_{i+1}\}]
\]
\( R_2 \) is acyclic. Therefore without any loss of generality let \( z_i \in G_{l_i}(S, R_2) \) for all \( i \in \{1, 2, ..., j\} \).
\[
C(\{z_1, z_2\}) = \{z_2\} \rightarrow z_1 P_2 z_2
\]
\( \rightarrow l_2 > l_1 \)
\[
C(\{z_2, z_3\}) = \{z_3\} \rightarrow z_2 P_2 z_3
\]
\( \rightarrow l_3 > l_2 \)

Proceeding this way we get \( C(\{z_{j-1}, z_j\}) = \{z_j\} \).
\[
C(\{z_{j-1}, z_j\}) = \{z_j\} \rightarrow z_{j-1} P_2 z_j
\]
\( \rightarrow l_j > l_{j-1} \)

The minimum value that any \( l_i \) can take is 1. Therefore \( l_1 \geq 1 \).
\[
l_1 \geq 1 \rightarrow l_j \geq j
\]
Therefore, \( G_j(S, R_2) \neq \emptyset \). \( \tag{5.2} \)

Now suppose \( G_{j+1}(S, R_2) \neq \emptyset \). Then there exists some \( x \in S \) such that \( x \in G_{j+1}(S, R_2) \). Also \( x \in G_{j+1}(S, R_2) \) implies that \( x \notin G_j(S, R_2) \).
\[
x \notin G_j(S, R_2) \land x \in G_{j+1}(S, R_2)
\]
\( \rightarrow (\exists y_1 \in S)[y_1 \in G_j(S, R_2) \land y_1 P_2 x] \)
\[
y_1 \neq x \text{ as } P_2 \text{ is asymmetric. Also, } y_1 P_2 x \text{ implies that, } C(\{y_1, x\}) = \{x\}.
\]
Again \( y_1 \in G_j(S, R_2) \) implies that \( y_1 \notin G_{j-1}(S, R_2) \).
\[
y_1 \notin G_{j-1}(S, R_2) \land y_1 \in G_j(S, R_2)
\]
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\[ (\exists y_2 \in S) [y_2 \in G_{j-1}(S, R_2) \wedge y_2 P_2 y_1] \]
\[ y_2 \neq y_1 \text{ as } P_2 \text{ is asymmetric. } y_2 \neq x \text{ as } P_2 \text{ is asymmetric. Also, } y_2 P_2 y_1 \text{ implies that, } C(\{y_2, y_1\}) = \{y_1\}. \]
Again, \( y_2 \in G_{j-1}(S, R_2) \) implies that, \( y_2 \notin G_{j-2}(S, R_2) \).

\[ y_2 \notin G_{j-2}(S, R_2) \wedge y_2 \in G_{j-1}(S, R_2) \]
\[ \rightarrow (\exists y_3 \in S) [y_3 \in G_{j-2}(S, R_2) \wedge y_3 P_2 y_2] \]
\[ y_3 \neq y_2 \text{ as } P_2 \text{ is asymmetric. } y_3 \neq y_1 \text{ as } P_2 \text{ is asymmetric. } y_3 \neq x \text{ as } R_2 \text{ is acyclic. Also, } y_3 P_2 y_2 \text{ implies that, } C(\{y_3, y_2\}) = \{y_2\}. \]

Proceeding thus we get, \( (\exists y_j \in S) [y_j \in G_{1}(S, R_2) \wedge y_j P_2 y_{j-1}] \)
where \( y_j \neq y_{j-1}, y_j \neq y_{j-2}, \ldots, y_j \neq x \). Also, \( C(\{y_j, y_{j-1}\}) = \{y_{j-1}\} \).

\[ C(\{y_1, x\}) = \{x\} \wedge C(\{y_2, y_1\}) = \{y_1\} \wedge C(\{y_3, y_2\}) = \{y_2\} \wedge \cdots \wedge C(\{y_j, y_{j-1}\}) = \{y_{j-1}\} \]
\[ \rightarrow O_S \geq j + 1 \]
But this contradicts with, \( O_S = j \). Therefore, \( G_{j+1}(S, R_2) = \emptyset \).

\[ R_2 \text{ is acyclic } \wedge G_{j+1}(S, R_2) = \emptyset \rightarrow \bigcup_{i=1}^{j} G_i(S, R_2) = S \]
(5.3)

Following 5.1, 5.2 and 5.3 it is proved that for any \( S \) in \( \Sigma \), if \( O_S = j \) and \( R_2 \)
over \( S \) is acyclic then \( G_j(S, R_2) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R_2) = S \) where \( j \in N \).

Next we prove the converse. Let there be a set \( S \) in \( \Sigma \) such that \( G_j(S, R_2) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R_2) = S \) for some \( j \in N \). We shall prove that \( O_S = j \) and \( R_2 \)

is acyclic.

Suppose \( R_2 \) is not acyclic. Then there exist \( x_1, x_2, \ldots, x_n \) in \( S \) such that,
\[ x_1 P_2 x_2 \wedge x_2 P_2 x_3 \wedge \cdots \wedge x_{n-1} P_2 x_n \wedge x_n P_2 x_1 \]
Therefore, \( (\forall i \in \{1, 2, \ldots, j\})(\forall l \in \{1, 2, \ldots, n\})(x_l \notin G_i(S, R_2)) \).

Then, \( \bigcup_{i=1}^{j} G_i(S, R_2) \neq S \). But this is a contradiction.

Therefore \( R_2 \) is acyclic.

\[ G_j(S, R_2) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R_2) = S \]
\[ G_k(S, R_2) = \emptyset \text{ for all } k > j \]
\[ O_S = k \text{ for all } k > j \]
\[ O_S \leq j \]  
\[ O_S = k < j \rightarrow G_j(S, R_2) = \emptyset \]

But this is a contradiction. Therefore,

\[ O_S \geq j. \]  
\[ (5.4) \land (5.5) \rightarrow O_S = j \]

Hence the lemma is proved.

**Corollary 5.1** For any \( S \in \Sigma \), if \( G_k(S, R_2) \neq \emptyset \) and \( R_2 \) over \( S \) is reflexive, connected and acyclic then \( O_S \geq k \).

**Proof:** \( R_2 \) over \( S \) is reflexive, connected and acyclic and \( G_k(S, R_2) \neq \emptyset \) imply that there exists \( j \geq k \) such that, \( G_j(S, R_2) \neq \emptyset \) \( \land \sum_{i=1}^{j} G_i(S, R_2) = S \).

Hence, by lemma 5.2, \( O_S = j \). Therefore, \( O_S \geq k \).

**Proposition 5.1** For all \( k \geq 3 \), a choice function \( C \) is \( k \)-rationalizable by an ordering if \( C \) satisfies A.5.1 and A.5.2.

**Proof:** Let \( k \geq 3 \) and also let A.5.1 and A.5.2 be satisfied by the choice function \( C \). It will be shown that \( R_2 \) is an ordering \( k \)-rationalization of the choice function \( C \).

From lemma 5.1 we know that with \( k \geq 3 \) and A.5.1 and A.5.2 satisfied \( R_2 \) is transitive. Also \( R_2 \) is reflexive and connected as \( D = \Sigma \). Therefore \( R_2 \) is an ordering. It remains to show that \( R_2 \) is a \( k \)-rationalization.

**Case 1:** Let \( G_j(S, R_2) \neq \emptyset \) \( \land \sum_{i=1}^{j} G_i(S, R_2) = S \); where \( 1 \leq j < k \).

We shall prove that \( C(S) = G_j(S, R_2) \).
By lemma 5.2, $O_S = j < k$.

Let $x \in C(S)$.

$x \in C(S)$

$\rightarrow (\forall T \subseteq S)(x \in T \rightarrow x \in C(T))$ by A.5.2

$\rightarrow (\forall z \in S)(x \in C(\{x, z\}))$

$\rightarrow (\forall z \in S)(zR_2x)$

$G_j(S, R_2) \neq \emptyset \land \bigcup_{i=1}^{j} G_i(S, R_2) = S \land (\forall z \in S)(zR_2x)$

$\rightarrow x \in G_j(S, R_2)$ \hspace{1cm} [\because R_2 \text{ is transitive}]

Therefore,

$C(S) \subseteq G_j(S, R_2)$. \hspace{1cm} (5.6)

Let $x$ be the $j$th best element of $S$ with respect to $R_2$.

$\bigcup_{i=1}^{j} G_i(S, R_2) = S \land x \in G_j(S, R_2)$

$\rightarrow (\forall y \in S)(yR_2x)$

$\rightarrow (\forall y \in S)(x \in C(\{x, y\}))$ \hspace{1cm} [by definition of $R_2$] \hspace{1cm} (5.7)

Suppose $x \notin C(S)$. Then $z \in C(S)$.

$\{x, z\} \subseteq S \land x \notin C(S) \land z \in C(S) \rightarrow C(\{x, z\}) = \{z\}$ \hspace{1cm} by A.5.2

But this is contradictory to (5.7). So, $x \in C(S)$. Therefore,

$G_j(S, R_2) \subseteq C(S)$. \hspace{1cm} (5.8)

(5.6) $\land$ (5.8) $\rightarrow C(S) = G_j(S, R_2)$

**Case 2:** Let $G_k(S, R_2) \neq \emptyset$. We shall prove that $C(S) = G_k(S, R_2)$.

By corollary 5.1, $O_S \geq k$.

Let $x \in C(S)$. Then by A.5.1 we have, $O_{P_zS} = k - 1$. Also $(\exists j \in N)(x \in G_j(S, R_2))$ as $R_2$ is an ordering. 48
Suppose, \( x \in G_j(S, R_2) \) where \( j < k \).

\[
x \in G_j(S, R_2) \\
\rightarrow [\forall y \in S - \bigcup_{i=1}^{j-1} G_i(S, R_2)] [x R_2 y] \\
\rightarrow [\forall y \in S - \bigcup_{i=1}^{j-1} G_i(S, R_2)] [y \in C(\{x, y\})] \\
\rightarrow P_x \cap S \subseteq \bigcup_{i=1}^{j-1} G_i(S, R_2)
\]

Let, \( \bigcup_{i=1}^{j} G_i(S, R_2) = S' \). Clearly, \( G_{j-1}(S', R_2) \neq \emptyset \) and \( \bigcup_{i=1}^{j-1} G_i(S', R_2) = S' \).

By lemma 5.2, the order of the set \( S' = j - 1 \).

\( j < k \rightarrow O_{P_x \cap S} < k - 1 \)

But this is a contradiction.

Hence, \( j \geq k \). (5.9)

Suppose, \( x \in G_j(S, R_2) \) where \( j > k \).

\[
x \in G_j(S, R_2) \\
\rightarrow [\forall y \in \bigcup_{i=1}^{j-1} G_i(S, R_2)] [y P_2 x] \quad \text{[\( \because \) } R_2 \text{ is transitive}] \\
\rightarrow [\forall y \in \bigcup_{i=1}^{j-1} G_i(S, R_2)] [C(\{x, y\}) = \{x\}] \\
\rightarrow \bigcup_{i=1}^{j-1} G_i(S, R_2) \subseteq P_x \cap S
\]

By lemma 5.2, the order of the set \( \bigcup_{i=1}^{j} G_i(S, R_2) = j - 1 \).

Therefore, \( O_{P_x \cap S} \geq j - 1 > k - 1 \). This again is a contradiction.

Hence, \( j \leq k \). (5.10)

From (5.9) and (5.10) we conclude, \( j = k \) and accordingly, \( x \in G_k(S, R_2) \).

Therefore,

\( C(S) \subseteq G_k(S, R_2) \). (5.11)
Now we prove that $G_k(S, R_2) \subseteq C(S)$. Let $x \in G_k(S, R_2)$.

$x \in G_k(S, R_2)$

$\rightarrow [\forall y \in \bigcup_{i=1}^{k-1} G_i(S, R_2)][y P_2 x] \land [\forall y \in S - \bigcup_{i=1}^{k-1} G_i(S, R_2)][x R_2 y]$

$\rightarrow [\forall y \in \bigcup_{i=1}^{k-1} G_i(S, R_2)][C(\{x, y\}) = \{x\}] \land [\forall y \in S - \bigcup_{i=1}^{k-1} G_i(S, R_2)][y \in C(\{x, y\})]$

$\rightarrow P_x \cap S = \bigcup_{i=1}^{k-1} G_i(S, R_2)$

By lemma 5.2 the order of the set $\bigcup_{i=1}^{k-1} G_i(S, R_2) = k - 1$.

It follows then by A.5.1, $x \in C(S)$.

Therefore,

$G_k(S, R_2) \subseteq C(S)$.  \hspace{1cm} (5.12)

(5.11) $\land$ (5.12) $\rightarrow C(S) = G_k(S, R_2)$

Following case 1 and case 2 we conclude that $R_2$ is a $k$-rationalization of the choice function $C$ if $k \geq 3$ and A.5.1 and A.5.2 are satisfied. Hence the proposition is proved.

**Proposition 5.2** For all $k \geq 3$, a choice function $C$ satisfies A.5.1 and A.5.2 if $C$ is $k$-rationalizable by an ordering.

**Proof:** Let $k \geq 3$ and let $R$ be a $k$-th best element ordering rationalization of the choice function $C$. We show that A.5.1 and A.5.2 hold.

$R$ is a $k$-rationalization with $k \geq 3$. Therefore the following statement holds:

$(x, y) \in R \leftrightarrow y \in C(\{x, y\}) \leftrightarrow (x, y) \in R_2$

Therefore $R = R_2$.

Let $O_S = j \geq k$. We know that $R_2$ is an ordering. By lemma 5.2, $G_j(S, R_2) \neq \emptyset$, which implies $G_k(S, R_2) \neq \emptyset$. $R_2$ is $k$-rationalization. Therefore,
\( C(S) = G_k(S, R_2) \) \hspace{1cm} (5.13)

\( x \in C(S) \)
\[\rightarrow [\forall y \in \bigcup_{i=1}^{k-1} G_i(S, R_2)] \left[ \forall y \in \bigcup_{i=1}^{k-1} G_i(S, R_2) \right] \left[ y R_2 y \right] \] (from 5.13)
\[\rightarrow [\forall y \in \bigcup_{i=1}^{k-1} G_i(S, R_2)] \left[ C(\{x, y\}) \right] = \{x\} \land [\forall y \in S - \bigcup_{i=1}^{k-1} G_i(S, R_2)] [y \in C(\{x, y\})] \]
\[\rightarrow P_x \cap S = \bigcup_{i=1}^{k-1} G_i(S, R_2) \]

Let \( S' = \bigcup_{i=1}^{k-1} G_i(S, R_2) \). Clearly, \( G_{k-1}(S', R_2) \neq \emptyset \) and \( \bigcup_{i=1}^{k-1} G_i(S', R_2) = S' \).
Therefore, \( O_{S'} = O_{P_x \cap S} = k - 1 \). \hspace{1cm} (5.14)

Next we prove the converse. Let, \( x \in S \) and \( O_{P_x \cap S} = k - 1 \). Let, \( P_x \cap S = S' \). Then \( O_{S'} = k - 1 \). Also, \( R_2 \) is an ordering. By lemma 5.2 we have,
\( G_{k-1}(S', R_2) \neq \emptyset \) and \( \bigcup_{i=1}^{k-1} G_i(S', R_2) = S' \).

\( S' \subseteq P_x \rightarrow (\forall y \in S') \left[ C(\{x, y\}) = \{x\} \right] \)
\[\rightarrow (\forall y \in S') (y R_2 x) \]
\( P_x \cap S = S' \)
\[\rightarrow (\forall y \in S) [C(\{x, y\}) = \{x\} \rightarrow y \in S'] \]
\[\rightarrow (\forall y \in S - S') (y \in C(\{x, y\})) \]
\[\rightarrow (\forall y \in S - S') (x R_2 y) \]

\( S' \subseteq S \land G_{k-1}(S', R_2) \neq \emptyset \land \bigcup_{i=1}^{k-1} G_i(S', R_2) = S' \land (\forall y \in S') (y R_2 x) \land (\forall y \in S - S') (x R_2 y) \rightarrow x \in G_k(S, R_2) \)
\[\rightarrow x \in C(S) \] [by 5.13] \hspace{1cm} (5.15)

From 5.14 and 5.15 we say that A.5.1 holds.
Now we show that A.5.2 is also satisfied. Let $O_S = j < k$. We know that $R_2$ is an ordering. By lemma 5.2, $G_j(S, R_2) \neq \emptyset$ and $\bigcup_{i=1}^{j} G_i(S, R_2) = S$ where $j < k$. Therefore, by definition of $k$-rationalization, $C(S) = G_j(S, R_2)$. (5.16)

Let $T \subseteq S$ and $C(S) \cap T \neq \emptyset$.

$T \subseteq S \rightarrow O_T = l \leq j$

By lemma 5.2, $G_i(T, R_2) \neq \emptyset$ and $\bigcup_{i=1}^{l} G_i(T, R_2) = T$ where $l \leq j < k$. Therefore, by definition of $k$-rationalization, $C(T) = G_i(T, R_2)$. (5.17)

Let $x \in C(T)$. Also let $y \in C(S) \cap T$.

$y \in C(S) \rightarrow y \in G_j(S, R_2)$ \hspace{1cm} [by 5.16]

$y \in G_j(S, R_2) \cap \bigcup_{i=1}^{j} G_i(S, R_2) = S \rightarrow (\forall z \in S)(zR_2y)$

$\rightarrow xR_2y$

$x \in C(T) \rightarrow x \in G_i(T, R_2)$ \hspace{1cm} [by 5.17]

$x \in G_i(T, R_2) \cap \bigcup_{i=1}^{l} G_i(T, R_2) = T \rightarrow (\forall z \in T)(zR_2x)$

$\rightarrow yR_2x$

$yR_2x \cap xR_2y \rightarrow xI_2y$

$R_2$ is an ordering. Therefore, \((\forall z \in S)[(zR_2y \leftrightarrow zR_2x) \land (yR_2z \leftrightarrow xR_2z)]\).

$x, y \in S \land y \in G_j(S, R_2) \rightarrow x \in G_j(S, R_2)$

$\rightarrow x \in C(S)$

Therefore, $C(T) \subseteq C(S) \cap T$. (5.18)

Let $x \in C(S) \cap T$.

$x \in C(S) \rightarrow x \in G_j(S, R_2)$

$\rightarrow (\forall y \in S)(yR_2x)$

$\rightarrow (\forall y \in T)(yR_2x)$

$\rightarrow x \in G_i(T, R_2)$

$\rightarrow x \in C(T)$

Therefore $C(S) \cap T \subseteq C(T)$ (5.19)
5.18 ∧ 5.19 → \( C(S) ∩ T = C(T) \)
Therefore A.5.2 holds.

Hence the proposition is proved.

**Theorem 5.1** If \( k \geq 3 \) then there exist a \( k \)-th best element ordering rationalization of a choice function \( C \) iff A.5.1 and A.5.2 are satisfied.

**Proof:** Proposition 5.1 and proposition 5.2 together establish the theorem.

5.3.2 Necessary and sufficient condition for a choice function to be \( k \)-rationalizable by a reflexive, connected and acyclic binary relation

**Lemma 5.3** If \( R_2 \) is acyclic then \( x \in G_i(S, R_2) \) iff \( O_{S^P} = i - 1 \) where \( i \in N \).

**Proof:** Let \( R_2 \) be acyclic. Let \( x \in G_i(S, R_2) \).
If \( i = 1 \) then \( S^P_x = \emptyset \) and \( O_{S^P} = 0 \). Now let \( i > 1 \).
\( x \in G_i(S, R_2) \)
\( \rightarrow [\exists y_{i-1} \in G_{i-1}(S, R_2)](y_{i-1}P_2x) \)
\( \rightarrow [\exists y_{i-2} \in G_{i-2}(S, R_2)](y_{i-2}P_2y_{i-1}) \)
\( \vdots \)
\( \rightarrow [\exists y_1 \in G_1(S, R_2)](y_1P_2y_{i-1}) \)

[Notice that all \( y_1, y_2, ..., y_{i-1} \) and \( x \) are distinct as \( G_j(S, R_2) \) for different values of \( j \in N \) are mutually exclusive sets.]
\( \rightarrow O_{S^P_x} \geq i - 1 \) \hfill (5.20)

Suppose \( y \in S^P_x \) and \( y \in G_i(S, R_2) \).
\( y \in S^P_x \rightarrow \exists n \in N, \exists z_1, z_2, ..., z_n \in S \) such that \( z_1 = y \) and \( z_n = x \) and \( z_{t-1}P_2z_t \) for all \( t \in \{2, 3, ..., n\} \)
\( z_{n-1}P_2x \rightarrow z_{n-1} \in G_{l_{n-1}}(S, R_2) \) and \( l_{n-1} < i \)
\[ z_{n-2} P_2 z_{n-1} \to z_{n-2} \in G_{l_{n-2}}(S, R_2) \text{ and } l_{n-2} < l_{n-1} \]

\[ y P_2 z_2 \to y \in G_l(S, R_2) \text{ and } l < l_2 \]

Therefore, \( l < i \) and \( y \in S_x^P \to y \in \bigcup_{t=1}^{i-1} G_t(S, R_2) \)

\[ \to O_{S_x^P} \leq i - 1 \]  \hspace{1cm} (5.21)

From (5.20) and (5.21) we get \( O_{S_x^P} = i - 1 \).

The converse of the lemma is trivial. Hence the lemma is proved.

**Theorem 5.2** If \( k \geq 2 \) then there exists a \( k \)-rationalization of the choice function \( C \) if and only if \( C \) satisfies A.5.3 and A.5.4.

**Proof:** Let \( k \geq 2 \) and also let A.5.3 and A.5.4 be satisfied.

We first show that \( R_2 \) is acyclic.

Suppose \( R_2 \) is not acyclic. Then there exists \( y_1, y_2, \ldots, y_n \) in \( X \) such that

\[ y_1 P_2 y_2 \wedge y_2 P_2 y_3 \wedge \cdots \wedge y_{n-1} P_2 y_n \wedge y_n P_2 y_1. \]

Consider the set \( S = \{y_1, \ldots, y_n\} \).

\[ (\forall x \in S)(S_x^P = S) \to O_{S_x^P} = O_S \]

If \( O_S \geq k \) then A.5.3 is violated. If \( O_S < k \) then A.5.4 is violated. So \( R_2 \) is acyclic.

Now we prove sufficiency of A.5.3 and A.5.4. We may have 2 cases.

**Case 1:** Let \( G_j(S, R_2) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R_2) = S \) where \( 1 \leq j < k \).

By lemma 5.2, \( O_S = j < k \).

Therefore by A.5.4, \( C(S) = \{z \in S \mid O_{S_x^P} = j - 1\} \)

\[ x \in C(S) \to O_{S_x^P} = j - 1. \]

By lemma 5.3, \( x \in G_j(S, R_2) \) and hence \( C(S) \subseteq G_j(S, R_2) \). \hspace{1cm} (5.22)

Let \( x \in G_j(S, R_2) \). By lemma 5.3, \( O_{S_x^P} = j - 1 \) and hence, \( x \in C(S) \).

Therefore, \( G_j(S, R_2) \subseteq C(S) \). \hspace{1cm} (5.23)
From (5.22) and (5.23), $C(S) = G_j(S, R_2)$.

**Case 2:** Let $G_k(S, R_2) \neq \emptyset$.

By corollary 5.1, $G_k(S, R_2) \neq \emptyset \rightarrow O_S \geq k$.

Therefore by A.5.3, $C(S) = \{ z \in S \mid O_{S^*} = k - 1 \}$.

$x \in C(S) \rightarrow O_{S^*} = k - 1$

$R_2$ is acyclic. So without any loss of generality let $x \in G_i(S, R_2)$.

By lemma 5.3, $O_{S^*} = i - 1$. Therefore $i = k$ and $x \in G_k(S, R_2)$. Hence, $C(S) \subseteq G_k(S, R_2)$.

Let $x \in G_k(S, R_2)$. By the lemma 5.3, $O_{S^*} = k - 1$. Therefore, $x \in C(S)$.

Hence, $G_k(S, R_2) \subseteq C(S)$. (5.24)

From (5.24) and (5.25), $C(S) = G_k(S, R_2)$.

Now we prove the necessity part.

$R$ is an acyclic $k$-rationalization. With $D = \Sigma$ and $k \geq 2$ it is immediate that $R = R_2$.

Let $O_S \geq k$. By lemma 5.2, $G_k(S, R_2) \neq \emptyset$. Therefore, $C(S) = G_k(S, R_2)$.

By lemma 5.3, $x \in G_k(S, R_2) \leftrightarrow O_{S^*} = k - 1$. Hence A.5.3 holds.

Let $O_S < k$. Then we have 2 cases.

1. $G_j(S, R_2) \neq \emptyset \land G_{j+1}(S, R_2) = \emptyset$ where $1 < j < k$.

Acyclicity of $R_2$ implies $\bigcup_i G_i(S, R_2) = S$ and hence $O_S = j$. By definition of $k$-rationalization $C(S) = G_j(S, R_2)$. Therefore by lemma 5.3, $x \in C(S) \leftrightarrow O_{S^*} = O_S - 1$.

2. $G_2(S, R_2) = \emptyset$

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By definition of $k$-rationalization $C(S) = G_1(S, R_2)$. Acyclicity of $R_2$ implies $G_1(S, R_2) = S$. Therefore for all $x \in S$, we have $S^P_x = \emptyset$ and $O_{s^P} = 0$. Again, $G_1(S, R_2) = S$ implies that $O_S = 1$ and therefore $x \in C(S) \iff O_{s^P} = O_S - 1$.

Therefore A.5.4 is satisfied.

Hence the theorem is proved.

5.4 Independence of the axioms

We now briefly discuss the mutual independence of the axioms used in the characterization result. Examples 1, 2 and 3 show the mutual independence of axioms A.5.1 and A.5.2. Examples 4, 5 and 6 show the mutual independence of axioms A.5.3 and A.5.4.

Example 5.4.1

For any $k \geq 3$ consider $k$ distinct elements $x_1, x_2, ..., x_k$. Let $X = \{x_1, x_2, ..., x_k\}$. Let, $C(X) = X$. Consider any nonempty $S \subset X$. Let $S = \{x_i, x_j, ..., x_n\}$. Let $C(S) = \{x_a\}$, such that $a = \max\{i, j, ..., n\}$. We can check that such a choice function violates A.5.1 but satisfies A.5.2.

Example 5.4.2

Let $k = 3$. $X = \{x, y, z, w\}$

$\forall a \in X)[(C(\{a\}) = \{a\}]$

$C(\{x, y, z, w\}) = \{w\}$, $C(\{x, y, z\}) = \{x, y, z\}$, $C(\{x, y, w\}) = \{w\}$, $C(\{x, z, w\}) = \{w\}$, $C(\{y, z, w\}) = \{y, z, w\}$, $C(\{x, y\}) = \{y\}$, $C(\{x, z\}) = \{z\}$, $C(\{x, w\}) = \{w\}$, $C(\{y, z\}) = \{y, z\}$, $C(\{y, w\}) = \{w\}$, $C(\{z, w\}) = \{w\}$

A.5.2 is violated but A.5.1 is satisfied.
Example 5.4.3

Let $k \geq 4$. $X = \{x, y, z, w\}$

$(\forall a \in X)[(C(\{a\}) = \{a\}]$

$C(\{x, y, z, w\}) = \{w\}, C(\{x, y, z\}) = \{x, y, z\}, C(\{x, y, w\}) = \{w\}, C(\{x, z, w\}) = \{w\}, C(\{y, z, w\}) = \{y, z, w\}, C(\{x, y\}) = \{y\}, C(\{x, z\}) = \{z\}, C(\{x, w\}) = \{w\}, C(\{y, z\}) = \{z\}, C(\{y, w\}) = \{w\}, C(\{z, w\}) = \{w\}$

A.5.2 is violated but A.5.1 is satisfied.

Example 5.4.4

$X = \{a, b, c\}$

$C(\{a\}) = \{a\}, C(\{b\}) = \{b\}, C(\{c\}) = \{c\}, C(\{a, b\}) = \{a, b\}, C(\{b, c\}) = \{b, c\}, C(\{a, c\}) = \{a, c\}, C(\{a, b, c\}) = \{a\}$

For all values of $k \geq 2$ A.5.4 is violated. A.5.3 is satisfied trivially.

Example 5.4.5

$X = \{a, b, c\}$

$C(\{a\}) = \{a\}, C(\{b\}) = \{b\}, C(\{c\}) = \{c\}, C(\{a, b\}) = \{b\}, C(\{b, c\}) = \{c\}, C(\{a, c\}) = \{c\}, C(\{a, b, c\}) = \{c\}$

For $k = 2$ A.5.3 is violated and A.5.4 is satisfied.

Example 5.4.6

For any $k \geq 3$ consider $k$ distinct elements $x_1, x_2, ..., x_k$. Let $X = \{x_1, x_2, ..., x_k\}$.

Let, $C(X) = X$. Consider any nonempty $S \subset X$. Let $S = \{x_i, x_j, ..., x_n\}$. Let $C(S) = \{x_a\}$, such that $a = \max\{i, j, ..., n\}$. We can check that such a choice function violates A.5.3 but satisfies A.5.4.