Chapter 2

Definitions and notation

We wish to discuss here some basic concepts and ideas that are very commonly used in social choice theory. We would like to introduce some basic definitions and notation which we are going to use in subsequent chapters of this thesis.

2.1 Choice sets and choice functions

We denote the set of all positive integers by $N$. Let the non-empty finite universal set of alternatives be denoted by $X$. We denote the power set of $X$ (i.e. the set of all subsets of $X$) by $2^X$. Let $\Sigma$ be the set of all non-empty subsets of $X$ i.e., $\Sigma = 2^X - \{\emptyset\}$.

A choice function is defined as a function, $C : D \rightarrow \Sigma$ such that $C(A) \subseteq A$ for all $A \in D$ where $\emptyset \neq D \subseteq \Sigma$. In other words, to every set $A$ belonging to the domain $D$, which is a distinguished nonempty collection of nonempty subsets of $X$, we assign a set $C(A)$ which is a subset of $A$ itself. $C(A)$ is called the choice set of the set $A$. We illustrate the idea with an example.

Example 2.1.1

Let $X = \{a, b, c\}$.

$2^X = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \emptyset\}$
\[ \Sigma = 2^X - \{\emptyset\} ; \quad D = \{\{a\}, \{a, b\}, \{a, b, c\}\} \]

\[ C(\{a\}) = \{a\}, \quad \] \[ C(\{a, b\}) = \{a\}, \quad \] \[ C(\{a, b, c\}) = \{a\}. \quad C \text{ is a choice function defined over the domain } D. \]

### 2.2 Binary relation

A binary relation \( R \) over a set \( S \) is a subset of the Cartesian product \( S \times S \). For any binary relation \( R \) defined over a set \( S \), the asymmetric and symmetric parts of \( R \), designated by \( P \) and \( I \) respectively, are defined as follows,

\[ (\forall x, y \in S)[xPy \iff xRy \land \neg yRx] \]
\[ (\forall x, y \in S)[xIy \iff xRy \land yRx]. \]

A binary relation \( R \) defined over a set \( S \) is said to be:
- **reflexive** iff \((\forall x \in S)(xRx)\)
- **connected** iff \((\forall x, y \in S)(x \neq y \implies xRy \lor yRx)\)
- **transitive** iff \((\forall x, y, z \in S)(xRy \land yRz \implies xRz)\)
- **acyclic** iff \((\forall n \in \mathbb{N} - \{1, 2\})(\forall x_1, x_2, ..., x_n \in S)(x_1Px_2 \land x_2Px_3 \land ... \land x_{n-1}Px_n \implies \neg x_nPx_1)\).

A binary relation \( R \) is said to be an **ordering** if and only if it is reflexive, connected and transitive.

A binary relation \( R \) is said to be a **quasi-ordering** if and only if it is reflexive and transitive.

Consider two binary relations \( R_1 \) and \( R_2 \) on \( X \). A **composition** thereof is defined by,

\[ R_1R_2 = \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in R_1 \land (z, y) \in R_2\}. \]

For any binary relation \( R \) we define the following infinite sequence of binary relations:
\[ R^1 = R, \quad R^2 = RR, \quad R^3 = RR^2, \ldots, \quad R^t = RR^{t-1} \; ; \; t \in N - \{1\} \]

Let \( T(R) = \bigcup_{n \in N} R^n \). \( T(R) \) is said to be the transitive closure of \( R \).

A binary relation \( R \) on \( X \) is \( T \)-Consistent iff,

\[ (\forall x, y \in X)[(x, y) \in T(R) \rightarrow (x, y) \in R \lor (y, x) \notin R]. \]

### 2.3 Derivation of \( k \)-th best element

Let \( R \) be a binary relation defined over a set \( S \). An element \( x \in S \) is said to be a best element (or 1st best element) in \( S \) with respect to \( R \) if and only if,

\[ (\forall y \in S)(xRy). \]

Let the set of all such best elements in \( S \) be \( G(S, R) \).

#### Example 2.3.1

Let \( X = \{a, b, c\} \) and \( R \) be a binary relation on \( X \).

\[ R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\} \]

Here the only best element in \( X \) is \( a \) and accordingly the set of best elements is \( G(X, R) = \{a\} \).

We define that an element \( x \in S - G(S, R) \) is a second best element in \( S \) with respect to \( R \) if and only if, \( (\forall y \in S - G(S, R))(xRy) \). The set of all 2nd best elements in \( S \) is defined as \( G(S - G(S, R), R) \). In the example above we have \( X - G(X, R) = \{b, c\} \). We see that \( b \) is the only second best element here and \( G(X - G(X, R), R) = \{b\} \).

We say an element \( x \in S - (G(S, R) \cup G(S - G(S, R), R)) \) is a third best element in \( S \) with respect to \( R \) if and only if, \( (\forall y \in S - (G(S, R) \cup G(S - G(S, R), R)))(xRy) \). Also \( G(S - (G(S, R) \cup G(S - G(S, R), R)), R) \) is the set of all third best elements in \( S \). Coming back to the previous example we see that \( X - (G(X, R) \cup G(X - G(X, R), R)) = \{c\} \). Also \( c \) is the only third best
element and \( G(S - (G(S, R) \cup G(S - G(S, R), R)), R) = \{c\} \).

To make things look less complicated, we introduce the following notation system. We define,

\[
G(S, R) = G_1(S, R)
G(S - G(S, R), R) = G_2(S, R)
G(S - (G(S, R) \cup G(S - G(S, R), R)), R) = G_3(S, R)
\]
and so on. Following our notation we define,

\[
G_i(S, R) = G(S - \bigcup_{j=1}^{i-1} G_j(S, R), R) ; \ i \geq 2 , i \in N.
\]

Also we define \( x \in S - \bigcup_{i=1}^{k-1} G_i(S, R) \) to be a \( k \)-th best element in \( S \) with respect to \( R \) if and only if, \((\forall y \in S - \bigcup_{i=1}^{k-1} G_i(S, R))(xRy) \) where \( k \geq 2, k \in N \).

**Example 2.3.2**

Let \( S = \{a, b, c, d, e\} \) and let \( R \) be a binary relation on \( S \).

\[
R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (a,c), (a,d), (a,e), (b,a), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)\}
\]

Clearly,

\[
G_1(S, R) = \{a,b\} \quad S - G_1(S, R) = \{c,d,e\}
G_2(S, R) = \{c\} \quad S - \bigcup_{i=1}^{2} G_i(S, R) = \{d,e\}
G_3(S, R) = \{d\} \quad S - \bigcup_{i=1}^{3} G_i(S, R) = \{e\}
G_4(S, R) = \{e\}
\]

### 2.4 \( k \)-Rationalizability of a choice function

A binary relation \( R \) rationalizes a choice function \( C \) if and only if \( C(S) = G(S, R) \), for all \( S \in D \).
Example 2.4.1

Let \( X = \{a, b, c\} \). \( C : \Sigma \rightarrow \Sigma \) is a choice function.
\[
\begin{align*}
C(\{a\}) &= \{a\}, \\
C(\{b\}) &= \{b\}, \\
C(\{c\}) &= \{c\}, \\
C(\{a, b\}) &= \{a\}, \\
C(\{b, c\}) &= \{b\}, \\
C(\{a, b, c\}) &= \{a\}
\end{align*}
\]

Consider \( R \) on \( X \):
\[
R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}
\]

\( R \) rationalizes the choice function \( C \).

A binary relation \( R \) is said to be a 2-rationalization (second best element rationalization) of a choice function \( C \) if and only if
\[
\begin{align*}
C(S) &= G_2(S, R) \quad \text{if } G_2(S, R) \neq \emptyset \\
&= G_1(S, R) \quad \text{if } G_2(S, R) = \emptyset
\end{align*}
\]

for all \( S \in D \).

Example 2.4.2

Let \( X = \{a, b, c, d\} \) and \( C : D \subseteq \Sigma \rightarrow \Sigma \) be a choice function.
\[
D = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, b, c, d\}\}
\]
\[
\begin{align*}
C(\{a\}) &= \{a\}, \\
C(\{b\}) &= \{b\}, \\
C(\{c\}) &= \{c\}, \\
C(\{d\}) &= \{d\}, \\
C(\{a, b\}) &= \{b\}, \\
C(\{b, c\}) &= \{c\}, \\
C(\{a, b, c, d\}) &= \{b\}
\end{align*}
\]

Consider \( R \) on \( X \):
\[
R = \{(a, a), (b, b), (c, c), (a, d), (a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}
\]

\( R \) is a 2-rationalization of the choice function \( C \).

A binary relation \( R \) is a k-rationalization (k-th best element rationalization), \( k \in N \), of a choice function \( C \) iff,
\[
\begin{align*}
C(S) &= G_k(S, R) \quad \text{if } G_k(S, R) \neq \emptyset \\
&= G_j(S, R) \quad \text{if } G_j(S, R) \neq \emptyset \land G_{j+1}(S, R) = \emptyset \text{ where } 1 < j < k; j, k \in N \\
&= G_1(S, R) \quad \text{if } G_2(S, R) = \emptyset
\end{align*}
\]
for all $S \in D$.

**Example 2.4.3**

Let $X = \{a, b, c, d\}$ and $C : D \subseteq \Sigma \mapsto \Sigma$ be a choice function.

$D = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{c, b\}, \{a, b, c, d\}\}$

$C(\{a, b\}) = \{b\}, \ C(\{b, c\}) = \{c\}, \ C(\{a, b, c\}) = \{c\}, \ C(\{a, b, d\}) = \{d\}$,

$C(\{c, b, d\}) = \{d\}, \ C(\{a, b, c, d\}) = \{c\}$

Consider $R$ on $X$:

$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$

$R$ is a 3-rationalization (third best element rationalization) of the choice function $C$.

**Example 2.4.4**

Let $X = \{a, b, c, d, e\}$ and

$D = \{\{a\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{b, c, d, e\}, \{a, b, c, d, e\}\}$

$C : D \mapsto \Sigma$ is a choice function.

$C(\{a\}) = \{a\}, \ C(\{a, b\}) = \{b\}, \ C(\{c, d\}) = \{c, d\}, \ C(\{a, b, c\}) = \{c\}$,

$C(\{a, c, d\}) = \{c, d\}, \ C(\{c, d, e\}) = \{e\}, \ C(\{a, b, c, d\}) = \{c, d\}, \ C(\{b, c, d, e\}) = \{e\}$,

$C(\{a, b, c, d, e\}) = \{c, d\}$

Consider $R$ on $X$:

$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e)$,

$(d, c), (d, e)\}$

$R$ is a 3-rationalization of the choice function $C$.

### 2.5 A derived binary relation $R_2$

We derive a binary relation $R_2$ over $X$ as follows:

$$(\forall x, y \in X)((xR_2y) \iff y \in C(\{x, y\})).$$
P₂ and I₂ respectively are the asymmetric and symmetric parts of R₂.

For any set S ∈ Σ we define,

\[ B_S = \{ x \in S \mid (\forall y \in S)(y \in C(\{x, y\})) \} \]

For any x ∈ X we define,

\[ P_x = \{ y \in X \mid C(\{x, y\}) = \{x\} \}. \]

Clearly \( B_S \) is the set of best elements in \( S \) with respect to \( R₂ \) and \( P_x \) is the set of all elements in \( X \) that are preferred to \( x \) with respect to \( R₂ \).

For all \( S \in \Sigma \) and for all \( x, y \in S \), \((x, y) \in P₂|S \leftrightarrow x, y \in S \land C(\{x, y\}) = \{y\} \).

\( P₂|S \) is the restriction of \( P₂ \) (asymmetric part of \( R₂ \)) over the set \( S \).

For all \( S \in \Sigma \) and for all \( x \in S \), \( S_x^P = \{ y \in S \mid (y, x) \in T(P₂|S) \} \).

\( T(P₂|S) \) is the transitive closure of \( P₂|S \).

Example 2.5.1

Let \( X = \{a, b, c, d, e\} \). Let the choice function \( C \) be the following: \( C(\{a\}) = \{a\}, C(\{a, b\}) = \{b\}, C(\{a, c\}) = \{c\}, C(\{a, d\}) = \{d\}, C(\{b, c\}) = \{c\}, C(\{c, d\}) = \{d\}, C(\{c, e\}) = \{e\}, C(\{d, e\}) = \{e\}, C(\{a, e\}) = \{a, e\}, C(\{b, e\}) = \{b, e\} \).

\( R₂ = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, e), (c, d), (c, e), (d, e), (e, a), (e, b)\} \)

\( P₂ = \{(a, b), (a, c), (a, d), (b, c), (b, e), (c, d), (c, e), (d, e), (e, a), (e, b)\} \)

Consider the set \( X \). Clearly, \( B_X = \{a\} \). Consider the set \( X' = X - \{a\} \).

One may check that, for all non-empty subsets \( Y \) of \( X' \), \( B_Y = \emptyset \).

\( P_a = \emptyset, P_b = \{a\}, P_c = \{a, b\}, P_d = \{a, c\}, P_e = \{c, d\} \).

Consider the sets \( S = \{a, b, e\} \) and \( T = \{a, c, e\} \).

\( P₂|S = \{(a, b)\} \) and \( P₂|T = \{(a, c), (c, e)\} \).

\( S_x^P = \emptyset \) and \( T_x^P = \{a, c\} \). Notice that \( a \in S, T \) yet \( a \notin S_x^P \) but \( a \in T_x^P \).
2.6 Order of a set

For any set $S \subseteq X$, the order of the set $S$, denoted as $O_S$, is defined as follows:

$$
O_S = \begin{cases} 
0 & \text{iff } S = \emptyset \\
1 & \text{iff } (\forall x, y \in S)\{\{x, y\} \in D \rightarrow C(\{x, y\}) = \{x, y\}\} \\
n & \text{iff otherwise}
\end{cases}
$$

where $n$ is the largest value of $m$ such that there exist distinct $z_1, z_2, \ldots, z_m$ in $S$ and $(\forall i \in \{1, 2, \ldots, m-1\})[C(\{z_i, z_{i+1}\}) = \{z_{i+1}\}]$.

The following exposition will make the idea of $O_S$ more intuitive. For any set $S$ and a choice function $C$ we first derive the binary relation $P_2$ (i.e., asymmetric part of $R_2$). If $S$ is an empty set then it is a trivial case and we consider this set with order zero. If, however, $S$ is non-empty but $P_2$ restricted over $S$ is empty then $S$ is a set with order one. If $P_2$ restricted over $S$ is non-null then we look for the longest chain of elements that are consecutively joined by $P_2$ such that no element occurs more than once. The number of elements present in this longest chain is the value of $O_S$. In other words, $O_S$ gives us the largest number of consecutive preference levels present in the set $S$. We illustrate the idea by considering the following examples.

Example 2.6.1

Let $S = \{a, b, c\}$ and the choice function be $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{a, c\}) = \{a\}$.

The longest chain of elements joined by $P_2$ that we may get here such that no element occurs more than once, can be, $(aP_2b \land bP_2c)$ or $(bP_2c \land cP_2a)$ or $(cP_2a \land aP_2b)$. In all cases we have three distinct consecutive preference levels. So the order of the set is 3.

Example 2.6.2

Let $S = \{a, b, c, d, e\}$ and the choice function be $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{d, e\}) = \{e\}$.
The longest chain can be constituted by the elements $a$, $b$ and $c$. We have three distinct and consecutive preference levels here with $a$ at the top, $b$ in the middle and $c$ at the bottom. The order of $S$ is 3.

Example 2.6.3

Let $S = \{a, b, c, d\}$. Let the choice function $C$ be the following: $C(\{a, b\}) = \{b\}$, $C(\{b, c\}) = \{c\}$, $C(\{c, d\}) = \{d\}$, $C(\{a, c\}) = \{a, c\}$, $C(\{a, d\}) = \{a, d\}$, $C(\{b, d\}) = \{b, d\}$, $C(\{a, b, c, d\}) = \{a, b, c, d\}$.

The longest chain can be constituted by the elements $a$, $b$, $c$, $d$. The order of $S$ is 4.