CHAPTER 2

VALUED INNER PRODUCT SPACE

A detailed investigation is presented in this chapter for inequalities in valued inner product space. In addition, the numerical solutions involved in the computation of various inequalities are also discussed.

2.1 INTRODUCTION

This chapter consists of two sections. The first section describes about the inequalities in valued inner product space. Algebraic Number Theory involves using techniques from Algebra and finite group theory to gain deeper understanding of number fields. Theory of equations comprises a major part of traditional Algebra. Motivated by the concept of valuation in Algebraic Number Theory, an interesting notion of valued inner product space is introduced in this section. There are many inequalities between means. Some interesting inequalities on valued inner product space combined with the theory of equations discussed in this section.

Number theory may briefly be defined as the study of the properties of integers. It will be found later on that some properties of integers hold good only when the integers are positive and prime. The second section deals with the solutions of inner product of vectors. Pythagorean triplets provide the existence of solutions for those inequalities.

2.2 SOME INEQUALITIES IN VALUED INNER PRODUCT SPACE

We introduce the notion of valued norm and valued inner product on linear space as follows:
**Definition 2.1** Let $X$ be a real linear space over a field $K$. For every $u, v \in X$ and any $\alpha \in K$ a real valued function $\| \cdot \|$ on $X$ satisfying the following:

(i) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$,

(ii) $\|\alpha u\| = \|\alpha\|\|u\|$, 

(iii) $\|u + v\| \leq \|u\| + \|v\|$

is called a valued norm and the pair $(X, \| \cdot \|)$ is called a valued normed linear space.

**Definition 2.2** Let $X$ be a real linear space over a field $K$. For every $u, v, w \in X$ and $\alpha, \beta \in K$ a real valued function $(\cdot, \cdot)$ on $V \times V$ satisfying the following:

(i) $(u, u) \geq 0$ and $(u,u) = 0$ iff $u = 0$,

(ii) $(u, v) = (v, u)$,

(iii) $(\alpha u + \beta v, w) = \|\alpha\|(u, w) + \|\beta\|(v, w)$

is called a valued inner product and the pair $(X, (\cdot, \cdot))$ is called a valued inner product space.

**Theorem 2.1 Arithmatic mean of inner product of vectors of first kind**

Let $(X, (\cdot, \cdot))$ be a valued inner product space. Then for any nonzero vectors in $X$ whose elements are greater than or equal to 1, it follows that

\[
\frac{(u_1, u_2) + (v_1, v_2)}{2} \geq \frac{(u_1, v_1) + (u_2, v_2)}{2} \text{ if } \|u_1\| < \|v_2\| \text{ and } \|u_2\| < \|v_1\| \text{ (or)} \\
\|u_1\| > \|v_2\| \text{ and } \|u_2\| > \|v_1\|.
\]

**Proof**

Consider

\[
\frac{(u_1, u_2) + (v_1, v_2)}{2} - \frac{(u_1, v_1) + (u_2, v_2)}{2} \leq \frac{1}{2} (\|u_1\|\|u_2\| + \|v_1\|\|v_2\|) - \frac{1}{2} (\|u_1\|\|v_1\| + \|u_2\|\|v_2\|).
\]

(By Cauchy-Schwarz inequality)
\[
\frac{1}{2}(\|u_1\|\|u_2\| - \|v_1\|) + \|v_2\|(\|v_1\| - \|u_2\|).
\]
\[
= \frac{1}{2}(\|u_1\|\|u_2\| - \|v_1\| - \|v_2\|)(\|u_2\| - \|v_1\|)).
\]
We conclude that \(\frac{(u_1,u_2)+(v_1,v_2)}{2} \geq \frac{(u_1,v_1)+(u_2,v_2)}{2}\).

**Theorem 2.2** Arithmetic mean of inner product of vectors of second kind

Let \((X, (\cdot, \cdot))\) be a valued inner product space. Then for any nonzero vectors in \(X\) whose elements are greater than or equal to 1, we have

\[
\frac{(u_1,u_2)+(v_1,v_2)}{2} \leq \frac{(u_1,v_1)+(u_2,v_2)}{2}
\]

if \(\|u_1\| < \|v_2\|\) and \(\|v_1\| < \|u_2\|\) (or)

\(\|u_1\| > \|v_2\|\) and \(\|v_1\| > \|u_2\|\).

**Proof**

Consider \(\frac{(u_1,v_1)+(u_2,v_2)}{2} - \frac{(u_1,u_2)+(v_1,v_2)}{2}\)

\[
\leq \frac{1}{2}(\|u_1\|\|v_1\| + \|u_2\|\|v_2\|) - \frac{1}{2}(\|u_1\||\|u_2\| + \|v_1\||\|v_2\|).
\]

(By Cauchy-Schwarz inequality)

\[
= \frac{1}{2}\{\|u_1\|(|v_1| - |u_2|) + |v_2|(|u_2| - |v_1|)\}.
\]

\[
= \frac{1}{2}\{\|u_1\|(|v_1| - |u_2|) - |v_2|(|v_1| - |u_2|)\}.
\]

\[
= \frac{1}{2}\{(|u_1| - |v_2|)(|v_1| - |u_2|)\} \geq 0.
\]

We conclude that \(\frac{(u_1,u_2)+(v_1,v_2)}{2} \leq \frac{(u_1,v_1)+(u_2,v_2)}{2}\).

**Example 2.1** Arithmetic mean of inner product of vectors of first kind

Let \((X, (\cdot, \cdot))\) be a valued inner product space and \(u_1, u_2, v_1\) and \(v_2\) are elements in \(X\).

**Case (i)** Let \(\|u_1\| < \|v_2\|\) and \(\|u_2\| < \|v_1\|\).

\(u_1 = (1,2).\ |u_1| = \sqrt{1^2 + 2^2} = \sqrt{5}.\ u_2 = (3,4).\ |u_2| = \sqrt{3^2 + 4^2} = \sqrt{25}.
\)

\((u_1, u_2) = 1(3) + 2(4) = 11.\)

\(v_1 = (5,6).\ |v_1| = \sqrt{5^2 + 6^2} = \sqrt{61}.\ v_2 = (7,8).\ |v_2| = \sqrt{7^2 + 8^2} = \sqrt{113}.
\)

\((v_1, v_2) = 5(7) + 6(8) = 83.\)
\[
\frac{(u_1,u_2)+(v_1,v_2)}{2} = \frac{11+83}{2} = 47.
\]
\[
(u_1, v_1) = 1(5) + 2(6) = 17. (u_2, v_2) = 3(7) + 4(8) = 53.
\]
\[
\frac{(u_1,v_1)+(u_2,v_2)}{2} = \frac{17+53}{2} = 35.
\]

**Case (ii)** Let \(||u_1|| > ||v_2||\) and \(||u_2|| > ||v_1||\).
\[
u_1 = (3,4). ||u_1|| = \sqrt{3^2 + 4^2} = \sqrt{25}. u_2 = (7,8). ||u_2|| = \sqrt{7^2 + 8^2} = \sqrt{113}.
\]
\[
(v_1, v_2) = 5(1) + 6(2) = 17.
\]
\[
\frac{(u_1,u_2)+(v_1,v_2)}{2} = \frac{53+17}{2} = 35.
\]
\[
(u_1, v_1) = 3(5) + 4(6) = 39. (u_2, v_2) = 7(1) + 8(2) = 23.
\]
\[
\frac{(u_1,v_1)+(u_2,v_2)}{2} = \frac{39+23}{2} = 31.
\]

We conclude that \(\frac{(u_1,u_2)+(v_1,v_2)}{2} \geq \frac{(u_1,v_1)+(u_2,v_2)}{2}\).

**Example 2.2 Arithmetic mean of inner product of vectors of second kind**

Let \((X, (\cdot, \cdot))\) be a valued inner product space and \(u_1, u_2, v_1\) and \(v_2\) are elements in \(X\).

**Case (i)** Let \(||u_1|| < ||v_2||\) and \(||v_1|| < ||u_2||\).
\[
u_1 = (1,2). ||u_1|| = \sqrt{5}. u_2 = (7,8). ||u_2|| = \sqrt{113}.
\]
\[
(v_1, v_2) = 5(3) + 6(4) = 39. (v_1, v_2) = 3(4). ||v_2|| = \sqrt{25}.
\]
\[
\frac{(u_1,u_2)+(v_1,v_2)}{2} = \frac{23+39}{2} = 31.
\]
\[
(u_1, v_1) = 1(5) + 2(6) = 17. (u_2, v_2) = 7(3) + 8(4) = 53.
\]
\[
\frac{(u_1,v_1)+(u_2,v_2)}{2} = \frac{17+53}{2} = 35.
\]

**Case (ii)** Let \(||u_1|| > ||v_2||\) and \(||v_1|| > ||u_2||\).
\[
u_1 = (5,6). ||u_1|| = \sqrt{61}. u_2 = (1,2). ||u_2|| = \sqrt{5}.
\]
\[(u_1, u_2) = 5(1) + 6(2) = 17.\]
\[v_1 = (7, 8), \|v_1\| = \sqrt{113}. v_2 = (3, 4), \|v_2\| = \sqrt{25}.\]
\[(u_1, v_2) = 7(3) + 8(4) = 53.
\[
\frac{(u_1, u_2) + (v_1, v_2)}{2} = \frac{17 + 53}{2} = 35.
\]
\[(u_1, v_1) = 5(7) + 6(8) = 83. (u_2, v_2) = 1(3) + 2(4) = 11.
\[
\frac{(u_1, v_1) + (u_2, v_2)}{2} = \frac{83 + 11}{2} = 47.
\]
We conclude that \[\frac{(u_1, u_2) + (v_1, v_2)}{2} \leq \frac{(u_1, v_1) + (u_2, v_2)}{2}\].

**Theorem 2.3** Cauchy-Schwarz inequalities can be derived from the arithmetic mean of inner product of vectors of first and second kind.

**Proof**

Case (i) Arithmetic mean of inner product of vectors of first kind
\[
\frac{(u_1, u_2) + (v_1, v_2)}{2} \geq \frac{(u_1, v_1) + (u_2, v_2)}{2}
\]
if \[\|u_1\| < \|v_2\|\] and \[\|u_2\| < \|v_1\|\] (or)
\[\|u_1\| > \|v_2\|\] and \[\|u_2\| > \|v_1\|\].

Consider \[\frac{(u_1, v_1) + (u_2, v_2)}{2} \leq \frac{(u_1, u_2) + (v_1, v_2)}{2}\].

We know that, Geometric mean \(\leq\) Arithmetic mean.

Then \[((u_1, v_1)(u_2, v_2))^{1/2} \leq [(u_1, u_2)(v_1, v_2)]^{1/2}\].

Squaring and put \[u_1 = u_2\] & \[v_1 = v_2\].
\[(u_1, v_1)(u_1, v_1) = (u_1, u_1)(v_1, v_1).
\[(u_1, v_1)^2 \leq \|u_1\|^2\|v_1\|^2.

Taking positive square root, we get
\[|(u_1, v_1)| \leq \|u_1\|\|v_1\|\].

Case (ii) Arithmetic mean of inner product of vectors of second kind
\[
\frac{(u_1, u_2) + (v_1, v_2)}{2} \leq \frac{(u_1, v_1) + (u_2, v_2)}{2}
\]
if \[\|u_1\| < \|v_2\|\] and \[\|v_1\| < \|u_2\|\] (or)
\[\|u_1\| > \|v_2\|\] and \[\|v_1\| > \|u_2\|\].

We know that, Geometric mean \(\leq\) Arithmetic mean.

Then \[((u_1, u_2)(v_1, v_2))^{1/2} \leq [(u_1, v_1)(u_2, v_2)]^{1/2}\].
Squaring and put \( u_1 = v_1 \) & \( u_2 = v_2 \).
\((u_1, u_2)(u_1, u_2) \leq (u_1, u_1)(u_2, u_2)\).
\((u_1, u_2)^2 \leq \|u_1\|^2\|u_2\|^2\).
Taking positive square root, we get
\(|(u_1, u_2)| \leq \|u_1\|\|u_2\|\).

**Theorem 2.4** Arithmetic mean of inner product of vectors of first kind never depends upon the relation between \( \|u_2\| \) and \( \|v_2\| \).

**Proof**

Arithmetic mean of inner product of vectors of first kind
\[
\frac{(u_1, u_1 + v_1, v_2)(u_2, v_2)}{2} \geq \frac{(u_1, v_1) + (u_2, v_2)}{2}
\]
if \( \|u_1\| < \|v_2\| \) and \( \|u_2\| < \|v_1\| \) (or)
\( \|u_1\| > \|v_2\| \) and \( \|u_2\| > \|v_1\| \).
Consider
\[
\frac{(u_1, u_1 + u_2, v_2)}{2} \leq \frac{(u_1, u_2) + (v_1, v_2)}{2}.
\]
We know that, Geometric mean \( \leq \) Arithmetic mean.

Then \([ (u_1, v_1)(u_2, v_2) ]^{1/2} \leq \frac{(u_1, u_2) + (v_1, v_2)}{2}\).

Squaring and put \( u_1 = u_2 \) & \( v_1 = v_2 \).

\( 4(u_2, v_2)(u_2, v_2) \leq \{(u_2, u_2) + (v_2, v_2)\}^2 \).

\( 4(u_2, v_2)^2 \leq \{\|u_2\|^2 + \|v_2\|^2\}^2 \).

Taking positive square root, we get
\( 2|(u_2, v_2)| \leq \|u_2\|^2 + \|v_2\|^2 \).
\( 0 \leq \|u_2\|^2 + \|v_2\|^2 - 2|(u_2, v_2)| \).
\( \leq \|u_2\|^2 + \|v_2\|^2 - 2\|u_2\|\|v_2\| \).
\( = (\|u_2\| - \|v_2\|)^2 \) or \( (\|v_2\| - \|u_2\|)^2 \).

That is \( (\|u_2\| - \|v_2\|)^2 \geq 0 \) or \( (\|v_2\| - \|u_2\|)^2 \geq 0 \).

Taking positive square root, we get \( \|u_2\| - \|v_2\| \geq 0 \) or \( \|v_2\| - \|u_2\| \geq 0 \).

This implies that \( \|u_2\| \geq \|v_2\| \) or \( \|u_2\| \leq \|v_2\| \).

We conclude that, Arithmetic mean of inner product of vectors of first kind never depends upon the relation between \( \|u_2\| \) and \( \|v_2\| \).
Theorem 2.5 Arithmetic mean of inner product of vectors of second kind never depends upon the relation between $\|u_1\|$ and $\|u_2\|$.

Proof

Arithmetic mean of inner product of vectors of second kind

$$\frac{(u_1,u_2)+(v_1,v_2)}{2} \leq \frac{(u_1,v_1)+(u_2,v_2)}{2}$$

if $\|u_1\| < \|v_2\|$ and $\|v_1\| < \|u_2\|$ (or)

$\|u_1\| > \|v_2\|$ and $\|v_1\| > \|u_2\|$.

We know that, Geometric mean $\leq$ Arithmetic mean.

Then $[(u_1,u_2)(v_1,v_2)]^{1/2} \leq \frac{(u_1,v_1)+(u_2,v_2)}{2}$.

Squaring and put $u_1 = v_1$ & $u_2 = v_2$.

$4(u_1,u_2)(u_1,u_2) \leq ((u_1,u_1) + (u_2,u_2))^2$.

$4(u_1,u_2)^2 \leq \{\|u_1\|^2 + \|u_2\|^2\}^2$.

Taking positive square root, we get

$$2|(u_1,u_2)| \leq \|u_1\|^2 + \|u_2\|^2.$$

$$0 \leq \|u_1\|^2 + \|u_2\|^2 - 2|(u_1,u_2)|.$$

$$\leq \|u_1\|^2 + \|u_2\|^2 - 2\|u_1\|\|u_2\|.$$

$$= (\|u_1\| - \|u_2\|)^2 \text{ or } (\|u_2\| - \|u_1\|)^2.$$

That is $(\|u_1\| - \|u_2\|)^2 \geq 0$ or $(\|u_2\| - \|u_1\|)^2 \geq 0$.

Taking positive square root, we get $\|u_1\| - \|u_2\| \geq 0$ or $\|u_2\| - \|u_1\| \geq 0$.

This implies that $\|u_1\| \geq \|u_2\|$ or $\|u_1\| \leq \|u_2\|$.

We conclude that, Arithmetic mean of inner product of vectors of second kind never depends upon the relation between $\|u_1\|$ and $\|u_2\|$.

Theorem 2.6 From the Arithmetic mean of inner product of vectors, we have

$$4(u_1,v_1)(u_2,v_2)(v_1,v_2) < (\|u_1\|^4 + \|u_2\|^4)(\|u_1\|^2 + \|u_2\|^2).$$

Proof

Arithmetic mean of inner product of vectors of first and second kind is

$$\frac{(u_1,v_1)+(u_2,v_2)}{2} \leq \frac{(u_1,u_2)+(v_1,v_2)}{2} \text{ & } \frac{(u_1,u_2)+(v_1,v_2)}{2} \leq \frac{(u_1,v_1)+(u_2,v_2)}{2}.$$
We know that, Geometric mean ≤ Arithmetic mean.

Then \([u_1, v_1](u_2, v_2)\) \(\leq \frac{1}{2}(u_1, u_2) + (v_1, v_2)\) and
\([u_1, u_2](v_1, v_2)\) \(\leq \frac{1}{2}(u_1, v_1) + (u_2, v_2)\).

Multiplying together, we get
\([u_1, v_1](u_2, v_2)(u_1, u_2)(v_1, v_2)\) \(\leq \frac{1}{4}[(u_1, u_2) + (v_1, v_2)][(u_1, v_1) + (u_2, v_2)]\).

Apply the tools of second kind \(u_1 = v_1\) and \(u_2 = v_2\) at right hand side and squaring,
\(16(u_1, v_1)(u_2, v_2)(u_1, u_2)(v_1, v_2)\)
\(\leq [(u_1, u_2) + (u_1, u_2)]^2[(u_1, v_1) + (u_2, v_2)]^2\).
\(\leq [2(u_1, u_2)]^2[\|u_1\|^2 + \|u_2\|^2]^2\).
\(4(u_1, v_1)(u_2, v_2)(v_1, v_2)\leq (u_1, u_2)[\|u_1\|^2 + \|u_2\|^2]^2\).
\(\leq (\|u_1\| \|u_2\|)[\|u_1\|^2 + \|u_2\|^2][\|u_1\|^2 + \|u_2\|^2]\).
\(= (\|u_1\|^3 \|u_2\| + \|u_1\| \|u_2\|^3)[\|u_1\|^2 + \|u_2\|^2]\).
\(< (\|u_1\|^4 + \|u_2\|^4)(\|u_1\|^2 + \|u_2\|^2)\).
(By the result, \(a^3b + ab^3 < a^4 + b^4\))

**Theorem 2.7 Inner product of Arithmetic mean of vectors of first kind**

Let \((X, (\cdot, \cdot))\) be a valued inner product space. Then for any nonzero vectors in \(X\) whose elements are greater than or equal to 1, we have
\(\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right) \leq \frac{1}{2}(u_1, u_2) + (v_1, v_2)\) if \(\|u_1\| < \|v_1\|\) and \(\|u_2\| < \|v_2\|\) (or)
\(\|u_1\| > \|v_1\|\) and \(\|u_2\| > \|v_2\|\).

**Proof**

Consider \(\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right) - \frac{1}{2}(u_1, u_2) + (v_1, v_2)\)
\(\leq \frac{1}{4}(\|u_1 + v_1\| \|u_2 + v_2\|) - \frac{1}{2}(\|u_1\| \|u_2\| + \|v_1\| \|v_2\|)\).
(By Cauchy-Schwarz inequality)
\[
\|\mathbf{u} + \mathbf{v}\| \leq \frac{1}{4} (\|\mathbf{u}\| + \|\mathbf{v}\|)(\|\mathbf{u}\| + \|\mathbf{v}\|) - \frac{1}{2} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|).
\]

(By triangle inequality)

\[
= \frac{1}{4} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|)
- \frac{1}{2} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|).
\]

\[
= \frac{1}{4} (-\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|\|\mathbf{u}\| - \|\mathbf{v}\|\|\mathbf{v}\|)
= \frac{1}{4} (\|\mathbf{u}\| (\|\mathbf{v}\| - \|\mathbf{u}\|) - \|\mathbf{v}\| (\|\mathbf{v}\| + \|\mathbf{u}\|))
= \frac{1}{4} (\|\mathbf{u}\| - \|\mathbf{v}\|) (\|\mathbf{v}\| - \|\mathbf{u}\|)
\]

Hence \(\left(\frac{\mathbf{u} + \mathbf{v}}{2}, \frac{\mathbf{u} + \mathbf{v}}{2}\right) \leq \frac{(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v})}{2}\).

**Theorem 2.8** Inner product of Arithmetic mean of vectors of second kind

Let \((X, (\cdot, \cdot))\) be a valued inner product space. Then for any nonzero vectors in \(X\) whose elements are greater than or equal to 1, we have

\[\left(\frac{\mathbf{u} + \mathbf{v}}{2}, \frac{\mathbf{u} + \mathbf{v}}{2}\right) \geq \frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2}\] if \(\|\mathbf{u}\| < \|\mathbf{v}\|\) and \(\|\mathbf{v}\| < \|\mathbf{u}\|\) (or)

\(\|\mathbf{u}\| > \|\mathbf{v}\|\) and \(\|\mathbf{v}\| > \|\mathbf{u}\|\).

**Proof**

Consider \(\frac{(\mathbf{u}, \mathbf{u}) + (\mathbf{v}, \mathbf{v})}{2} - \left(\frac{\mathbf{u} + \mathbf{v}}{2}, \frac{\mathbf{u} + \mathbf{v}}{2}\right)\)

\[
\leq \frac{1}{2} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|) - \frac{1}{4} (\|\mathbf{u}\| + \|\mathbf{v}\|)(\|\mathbf{u}\| + \|\mathbf{v}\|).
\]

(By Cauchy-Schwarz inequality)

\[
\leq \frac{1}{2} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|) - \frac{1}{4} (\|\mathbf{u}\| + \|\mathbf{v}\|)(\|\mathbf{u}\| + \|\mathbf{v}\|).
\]

(By triangle inequality)

\[
= \frac{1}{2} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|)
- \frac{1}{4} (\|\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|)
\]

\[
= \frac{1}{4} (\|\mathbf{u}\|\|\mathbf{u}\| - \|\mathbf{u}\|\|\mathbf{v}\| - \|\mathbf{v}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|)
\]

\[
\leq \frac{1}{4} (\|\mathbf{u}\|\|\mathbf{u}\| - \|\mathbf{u}\|\|\mathbf{v}\| - \|\mathbf{v}\|\|\mathbf{u}\| + \|\mathbf{v}\|\|\mathbf{v}\|).
\]
\[
\begin{align*}
&= \frac{1}{4} \left( ||u_1|| (||u_2|| - ||v_2||) - ||v_1|| (||u_2|| - ||v_2||) \right) \\
&= \frac{1}{4} \left( (||u_1|| - ||v_1||) (||u_2|| - ||v_2||) \right) \\
&= -\frac{1}{4} \left( (||u_1|| - ||v_1||) (||v_2|| - ||u_2||) \right) \leq 0.
\end{align*}
\]

Hence \( \frac{(u_1, u_2) + (v_1, v_2)}{2} \leq \left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) \).

**Example 2.3** Arithmetic mean of inner product of vectors of first kind

Let \( (X, \langle \cdot, \cdot \rangle) \) be a valued inner product space and \( u_1, u_2, v_1 \) and \( v_2 \) are elements in \( X \).

**Case (i)** Let \( ||u_1|| < ||v_1|| \) and \( ||u_2|| < ||v_2|| \).

\[
\begin{align*}
u_1 &= (1,2). \quad ||u_1|| = \sqrt{5}. \quad u_2 = (3,4). \quad ||u_2|| = \sqrt{25}.
\end{align*}
\]

\( (u_1, u_2) = 1(3) + 2(4) = 11. \)

\[
\begin{align*}
v_1 &= (5,6). \quad ||v_1|| = \sqrt{61}. \quad v_2 = (7,8). \quad ||v_2|| = \sqrt{113}.
\end{align*}
\]

\( (v_1, v_2) = 5(7) + 6(8) = 83. \quad \frac{(u_1, u_2) + (v_1, v_2)}{2} = \frac{11 + 83}{2} = 47. \)

\[
\left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) = 3(5) + 4(6) = 39.
\]

We conclude that \( \left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) \leq \left( \frac{(u_1, u_2) + (v_1, v_2)}{2} \right) \).

**Case (ii)** Let \( ||u_1|| > ||v_1|| \) and \( ||u_2|| > ||v_2|| \).

\[
\begin{align*}
u_1 &= (5,6). \quad ||u_1|| = \sqrt{61}. \quad u_2 = (7,8). \quad ||u_2|| = \sqrt{113}.
\end{align*}
\]

\( v_1 = (3,4). \quad ||v_1|| = \sqrt{25}. \quad v_2 = (1,2). \quad ||v_2|| = \sqrt{5}. \)

\( (u_1, u_2) = 5(7) + 6(8) = 83. \quad (v_1, v_2) = 3(1) + 4(2) = 11. \)

\[
\begin{align*}
\frac{(u_1, u_2) + (v_1, v_2)}{2} &= \frac{83 + 11}{2} = 47.
\end{align*}
\]

\[
\begin{align*}
\frac{u_1 + v_1}{2} &= \frac{(5,6) + (3,4)}{2} = (4,5). \quad \frac{u_2 + v_2}{2} = \frac{(7,8) + (1,2)}{2} = (4,5).
\end{align*}
\]

\[
\left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) = 4(4) + 5(5) = 41.
\]

We conclude that \( \left( \frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2} \right) \leq \left( \frac{(u_1, u_2) + (v_1, v_2)}{2} \right) \).
Example 2.4 Arithmetic mean of inner product of vectors of second kind

Let \((X, \langle \cdot, \cdot \rangle)\) be a valued inner product space and \(u_1, u_2, v_1\) and \(v_2\) are elements in \(X\).

**Case (i)** Let \(\|u_1\| < \|v_1\|\) and \(\|v_2\| < \|u_2\|\).

\[
\begin{align*}
u_1 &= (1,2). \quad \|u_1\| = \sqrt{5}. \quad u_2 = (7,8). \quad \|u_2\| = \sqrt{113}.
(u_1, u_2) &= 1(7) + 2(8) = 23.

v_1 &= (5,6). \quad \|v_1\| = \sqrt{61}. \quad v_2 = (3,4). \quad \|v_2\| = \sqrt{25}.
(v_1, v_2) &= 5(3) + 6(4) = 39. \quad \frac{(u_1, u_2) + (v_1, v_2)}{2} = \frac{23 + 39}{2} = 31. \quad \frac{u_1 + v_1}{2} = \frac{(1,2) + (5,6)}{2} = (3,4). \quad \frac{u_2 + v_2}{2} = \frac{(7,8) + (3,4)}{2} = (5,6).

\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right) &= \left(\frac{1 + 5}{2}, \frac{2 + 2}{2}\right) = (3,5) + 4(6) = 39.
\end{align*}
\]

**Case (ii)** Let \(\|u_1\| > \|v_1\|\) and \(\|v_2\| > \|u_2\|\).

\[
\begin{align*}
u_1 &= (5,6). \quad \|u_1\| = \sqrt{61}. \quad u_2 = (1,2). \quad \|u_2\| = \sqrt{5}.
(v_1, v_2) &= 5(3) + 6(4) = 39. \quad \frac{(u_1, u_2) + (v_1, v_2)}{2} = \frac{23 + 39}{2} = 31. \quad \frac{u_1 + v_1}{2} = \frac{(5,6) + (3,4)}{2} = (4,5). \quad \frac{u_2 + v_2}{2} = \frac{(7,8) + (1,2)}{2} = (4,5).

\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right) &= \left(\frac{5 + 3}{2}, \frac{6 + 4}{2}\right) = (4,4) + 5(5) = 41.
\end{align*}
\]

We conclude that \(\left(\frac{u_1 + v_1}{2}, \frac{u_2 + v_2}{2}\right) \geq \frac{(u_1, u_2) + (v_1, v_2)}{2}\).

### 2.3 A NUMBER APPROACH TO INEQUALITIES IN VALUED INNER PRODUCT SPACE

In this section, we make available solutions of the inequalities discussed above in a valued inner product space as follows:

**Theorem 2.9** Inner product of Arithmetic mean of vectors of first kind

Let \((X, \langle \cdot, \cdot \rangle)\) be a valued inner product space. Then for any nonzero vectors...
in $X$ whose elements are greater than or equal to 1, we have

$$\left(\frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2}\right) \leq \frac{(u_{11}, u_{12}) + (u_{21}, u_{22})}{2}$$

if $u_{11}, u_{12}, u_{21}$ and $u_{22}$ are symmetric and either $\|u_{11}\|$ (or) $\|u_{22}\| < \|u_{12}\|$ and $\|u_{11}\|$ (or) $\|u_{22}\| > \|u_{12}\|$.

**Proof**

Consider $\left(\frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2}\right) - \frac{(u_{11}, u_{12}) + (u_{21}, u_{22})}{2}$

$$\leq \frac{1}{4}(\|u_{11} + u_{12}\|) ||u_{21} + u_{22}|| - \frac{1}{2}(\|u_{11}\| ||u_{12}|| + \|u_{21}\| ||u_{22}||).$$

(By Cauchy-Schwarz inequality)

$$\leq \frac{1}{4}(\|u_{11}\| + ||u_{12}\|) (||u_{21}|| + ||u_{22}||) - \frac{1}{2}(\|u_{11}\| ||u_{12}|| + \|u_{21}\| ||u_{22}||).$$

(By triangle inequality)

$$= \frac{1}{4}(\|u_{11}\| ||u_{21}|| + \|u_{11}\| ||u_{22}|| + \|u_{12}\| ||u_{21}|| + \|u_{12}\| ||u_{22}|| - \frac{1}{2}(||u_{11}|| ||u_{12}|| + \|u_{21}\| ||u_{22}||)).$$

Since $u_{11}, u_{12}, u_{21}$ and $u_{22}$ are symmetric. Then put $u_{12} = u_{21}$.

$$= \frac{1}{4}(\|u_{11}\| ||u_{22}|| + \|u_{12}\| ||u_{22}|| - \frac{1}{2}(||u_{11}|| ||u_{12}|| + \|u_{21}\| ||u_{22}||)).$$

Either $\|u_{11}\| ||u_{22}|| < \|u_{12}\|$ and $\|u_{11}\| ||u_{22}|| > \|u_{12}\|$,

We conclude that $\left(\frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2}\right) \leq \frac{(u_{11}, u_{12}) + (u_{21}, u_{22})}{2}$. 


Theorem 2.10 Inner product of Arithmetic mean of vectors of second kind

Let \((X, \langle \cdot, \cdot \rangle)\) be a valued inner product space. Then for any nonzero vectors in \(X\) whose elements are greater than or equal to 1, we have

\[
\frac{u_{11} + u_{12}}{2} + \frac{u_{21} + u_{22}}{2} \geq \frac{(u_{11}, u_{12}) + (u_{21}, u_{22})}{2}
\]

if \(u_{11}, u_{12}, u_{21}\) and \(u_{22}\) are symmetric and either \(\|u_{11}\| \leq \|u_{12}\|\) (or) \(\|u_{11}\| \geq \|u_{22}\|\).

Proof

Consider \(\left(\frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2}\right)\)

\[
\leq \frac{1}{2} (\|u_{11}\| \|u_{12}\| + \|u_{21}\| \|u_{22}\|) - \frac{1}{4} (\|u_{11} + u_{12}\| \|u_{21} + u_{22}\|).
\]

(By Cauchy-Schwarz inequality)

\[
\leq \frac{1}{2} (\|u_{11}\| \|u_{12}\| + \|u_{21}\| \|u_{22}\|) - \frac{1}{4} (\|u_{11}\| + \|u_{12}\|) (\|u_{21}\| + \|u_{22}\|).
\]

(By triangle inequality)

\[
= \frac{1}{4} (2 \|u_{11}\| \|u_{12}\| + 2 \|u_{21}\| \|u_{22}\|)
\]

\[
- \frac{1}{4} (\|u_{11}\| \|u_{21}\| + \|u_{11}\| \|u_{22}\| + \|u_{12}\| \|u_{21}\| + \|u_{12}\| \|u_{22}\|).
\]

\[
= \frac{1}{4} (\|u_{11}\| (\|u_{12}\| - \|u_{21}\| - \|u_{22}\|) + \|u_{12}\| (\|u_{11}\| - \|u_{21}\| - \|u_{22}\|)
\]

\[
+ 2 \|u_{21}\| \|u_{22}\|).
\]

Since \(u_{11}, u_{12}, u_{21}\) and \(u_{22}\) are symmetric. Then put \(u_{12} = u_{21}\).

\[
= \frac{1}{4} (\|u_{11}\| \|u_{22}\| + \|u_{12}\| (\|u_{11}\| - \|u_{22}\|) + 2 \|u_{12}\| \|u_{22}\|).
\]

\[
= \frac{1}{4} (\|u_{11}\| \|u_{22}\| + \|u_{12}\| \|u_{11}\| - \|u_{12}\|^2 - \|u_{12}\| \|u_{22}\| + 2 \|u_{12}\| \|u_{22}\|).
\]

\[
= \frac{1}{4} (\|u_{11}\| (\|u_{22}\| - \|u_{22}\|) + \|u_{12}\| (-\|u_{12}\| + \|u_{22}\|)).
\]

\[
= \frac{1}{4} (\|u_{11}\| (-\|u_{22}\| + \|u_{12}\|) + \|u_{12}\| (\|u_{22}\| - \|u_{12}\|)).
\]

\[
= \frac{1}{4} (\|u_{11}\| - \|u_{12}\|) (\|u_{12}\| - \|u_{22}\|).
\]

\[
= (-) \frac{1}{4} (\|u_{11}\| - \|u_{12}\|) (\|u_{22}\| - \|u_{12}\|).
\]
Either $\|u_{11}\|$ and $\|u_{22}\| < \|u_{12}\|$ (or) $\|u_{11}\|$ and $\|u_{22}\| > \|u_{12}\|$, 
We conclude that
\[
\frac{(u_{11},u_{12})+(u_{21},u_{22})}{2} \leq \left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right).
\]

**Theorem 2.11 Inner product of Arithmetic mean of vectors**

Let $(X,(\bullet,\bullet))$ be a valued inner product space. Then for any nonzero vectors in $X$ whose elements are greater than or equal to 1, we have
\[
\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) \geq \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2}
\]
if $u_{11}$, $u_{12}$, $u_{21}$ and $u_{22}$ are scalar symmetric.

**Proof**

Case (i) Inner product of Arithmetic mean of vectors of first kind
\[
\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) - \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2} \leq \frac{1}{4}(\|u_{11}\| - \|u_{12}\|)(\|u_{22}\| - \|u_{12}\|).
\]
Since $u_{11}$, $u_{12}$, $u_{21}$ and $u_{22}$ are scalar symmetric.

Then put $u_{12} = u_{21}$ and $u_{11} = u_{22}$.
\[
\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) - \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2} \leq \frac{1}{4}(\|u_{11}\| - \|u_{12}\|)^2.
\]
We observe that
\[
\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) \geq \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2}.
\]

Case (i) Inner product of Arithmetic mean of vectors of second kind
\[
\frac{(u_{11},u_{12})+(u_{21},u_{22})}{2} - \left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) \leq -\frac{1}{4}(\|u_{11}\| - \|u_{12}\|)(\|u_{22}\| - \|u_{12}\|).
\]
Since $u_{11}$, $u_{12}$, $u_{21}$ and $u_{22}$ are scalar symmetric.

Then put $u_{12} = u_{21}$ and $u_{11} = u_{22}$.
\[
\frac{(u_{11},u_{12})+(u_{21},u_{22})}{2} - \left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) \leq -\frac{1}{4}(\|u_{11}\| - \|u_{12}\|)^2 \leq 0.
\]
We observe that
\[
\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{22}}{2}\right) \geq \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2}.
\]

**Example 2.5 Symmetric**

Let $u_{11} = (3,4)$; $\|u_{11}\| = 5$. $u_{12} = (5,12)$. $\|u_{12}\| = 13$.
\[(u_{11},u_{12}) = 3(5) + 4(12) = 63.
\]
Let $u_{21} = (5,12)$; $\|u_{11}\| = 13$. $u_{12} = (9,12)$. $\|u_{12}\| = 15$. 

\( (u_{21}, u_{22}) = 5(9) + 12(12) = 189. \frac{(u_{11} u_{12}) + (u_{21} u_{22})}{2} = 126. \)
\[
\frac{u_{11} + u_{12}}{2} = \frac{(3,4)+(5,12)}{2} = (4,8). \frac{u_{21} + u_{22}}{2} = \frac{(5,12)+(9,12)}{2} = (7,12).
\]
\[
\left( \frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2} \right) = 124. \text{ Therefore } \left( \frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2} \right) \leq \left( \frac{u_{11} u_{12}) + (u_{21} u_{22})}{2} \right).
\]

**Example 2.6 Scalar Symmetric**

Let \( u_{11} = (3,4); \| u_{11} \| = 5. u_{12} = (5,12). \| u_{12} \| = 13. \)
\[
( u_{11}, u_{12} ) = 3(5) + 4(12) = 63.
\]

Let \( u_{21} = (5,12); \| u_{11} \| = 13. u_{12} = (3,4). \| u_{12} \| = 5. \)
\[
( u_{21}, u_{22} ) = 63. \frac{(u_{11} u_{12}) + (u_{21} u_{22})}{2} = 63.
\]
\[
\frac{u_{11} + u_{12}}{2} = \frac{(3,4)+(5,12)}{2} = (4,8). \frac{u_{21} + u_{22}}{2} = \frac{(5,12)+(3,4)}{2} = (4,8).
\]
\[
\left( \frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2} \right) = 80. \text{ Therefore } \left( \frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2} \right) \geq \left( \frac{u_{11} u_{12}) + (u_{21} u_{22})}{2} \right).
\]

**Theorem 2.12** The minimal positive prime solution in \( R^2 \) for the inequality
\[
\left( \frac{u_{11} + u_{12}}{2}, \frac{u_{21} + u_{22}}{2} \right) \geq \left( \frac{u_{11} u_{12}) + (u_{21} u_{22})}{2} \right)
\]
of positive vectors which is greater than 1 is \( u_{11} = (3,4) \) and \( u_{12} = (5,12) \) if \( u_{11}, u_{12}, u_{21} \) and \( u_{22} \) are scalar symmetric.

**Proof**

Let \( u_{11} = (x_1, y_1) \) and \( u_{12} = (x_2, y_2) \). Since \( u_{12} = u_{21} \) and \( u_{11} = u_{22} \).

Then \( \left( \frac{u_{11} + u_{12}}{2}, \frac{u_{12} + u_{11}}{2} \right) \geq \left( \frac{u_{11} u_{12}) + (u_{12} + u_{11})}{2} \right). \)
\[
\left( \frac{u_{11} + u_{12}}{2}, \frac{u_{11} + u_{12}}{2} \right) \geq \left( \frac{u_{11} u_{12}) + (u_{11} u_{12})}{2} \right).
\]
\[
\left( \frac{u_{11} + u_{12}}{2}, \frac{u_{11} + u_{12}}{2} \right) \geq \frac{2(u_{11} u_{12})}{2}.
\]
\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right), \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \right) \geq (x_1 x_2 + y_1 y_2).
\]
\[
\left( \frac{x_1 + x_2}{2} \right)^2 + \left( \frac{y_1 + y_2}{2} \right)^2 \geq (x_1 x_2 + y_1 y_2).
\]
\[
(x_1 + x_2)^2 + (y_1 + y_2)^2 \geq 4(x_1 x_2 + y_1 y_2).
\]
\[
(x_1^2 + x_2^2 + 2x_1 x_2 + y_1^2 + y_2^2 + 2y_1 y_2 \geq 4(x_1 x_2 + y_1 y_2).
\]
\[
(x_1^2 + y_1^2) + (x_2^2 + y_2^2) \geq 2(x_1 x_2 + y_1 y_2).
\]
We have the minimal primitive solutions for the Pythagorean equation $x^2 + y^2 = z^2$ in $R^2$ is (3,4) and (5,12).

Since $(x_1^2 + y_1^2) + (x_2^2 + y_2^2) = (3^2 + 4^2) + (5^2 + 12^2) = 184$ and $2(x_1x_2 + y_1y_2) = 2[(3 \times 5) + (4 \times 12)] = 126$.

**Theorem 2.13** The minimal positive prime solution in $R^3$ for the inequality

$\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{21}+u_{11}}{2}\right) \geq \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2}$ of positive vectors which is greater than 1 has 3 and 7 as vector length.

**Proof**

Let $u_{11} = (x_1, y_1, z_1)$ and $u_{12} = (x_2, y_2, z_2)$. Since $u_{12} = u_{21}$ & $u_{11} = u_{22}$.

Then $\left(\frac{u_{11}+u_{12}}{2}, \frac{u_{12}+u_{11}}{2}\right) \geq \frac{(u_{11},u_{12})+(u_{21},u_{22})}{2}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$, \$ \frac{\frac{x_1+y_1}{2}+\frac{z_1+z_2}{2}}{2} \geq \frac{x_1}{x_2} + y_1y_2 + z_1z_2$. \$

We have the minimal primitive solutions for the Pythagorean equation $x^2 + y^2 + z^2 = w^2$ in $R^3$ is (1,2,2) and (2,3,6), since $w_1 = 3$ and $w_2 = 7$.

**Theorem 2.14** The prime solution in $R^4$ for the Inner product of Arithmetic mean of vectors has vector length is a required prime.

**Proof**

The proof is directly obtained from every positive integer can be represented as the sum of four squares.

We conclude from this theorem is the vectors in $R^4$ is the minimum possible space for any required prime length.