CHAPTER 7

FLUID QUEUE DRIVEN BY AN $M/M/1$ QUEUE
SUBJECT TO BERNOULLI SCHEDULE
CONTROLLED VACATION AND VACATION
INTERRUPTION

This chapter deals with the stationary analysis of a fluid queue driven by an $M/M/1$ queuing model subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. The governing system of differential difference equations are solved using matrix geometric method in the Laplacian domain. The resulting solutions are then inverted to obtain an explicit expression for the joint steady state probabilities of the content of the buffer and the state of the background queueing model. Numerical illustrations are added to depict the convergence of the stationary buffer content distribution subject to suitable stability conditions.

7.1 MODEL DESCRIPTION

Consider an $M/M/1$ queueing model with Poisson arrival and exponential distribution service times. When the system becomes empty, the server begins a vacation of random length, and takes an ordinary vacation with probability $p$ or a working vacation with probability $1 - p$, where $0 \leq p \leq 1$. In an ordinary vacation, the server will stop working even if there are new arrivals during the vacation period. In a working vacation, customers are served at a lower rate $\mu_v < \mu$. Further, in a working vacation, it is assumed that at the
instants of service completion, the vacation is either interrupted and the server is resumed to a regular busy period with probability \(1 - q\) or it is continued with probability \(q\). When the vacation periods ends and the system is nonempty, a new busy period starts. The ordinary vacation times and the working vacation times are also assumed to be exponentially distributed with parameter \(\theta\) and \(\theta_v\), respectively. Define

\[ J(t) = \begin{cases} 
0, & \text{if the server is in a working vacation at time } t \\
1, & \text{if the server is in an ordinary vacation at time } t \\
2, & \text{if the server is in a regular busy period at time } t 
\end{cases} \]

It is well known that the process \(\{(X(t), J(t)), t \geq 0\}\) is a Quasi Birth and Death process with state space given by

\[ S = \{(0,0)\} \cup \{(0,1)\} \cup \{(n,j), n = 1, 2, \ldots \text{ and } j = 0, 1, 2\}. \]

The state transition diagram of the background queueing model is given in Figure 7.1. Let

\[ \pi_{n,j} = \lim_{t \to \infty} P\{X(t) = n, J(t) = j\}, \quad (n,j) \in S, \]

represent the steady state probabilities of the background queueing model. Further, let \(\pi_0 = (\pi_{0,0}, \pi_{0,1})\) and \(\pi_n = (\pi_{n,0}, \pi_{n,1}, \pi_{n,2})\) for \(n \geq 1\). Then, the stationary probability vector is denoted by

\[ \pi = (\pi_0, \pi_1, \pi_2, \ldots). \]

It is readily seen that the system of equations governing the background queueing model under steady state can be written in the form of matrix as

\[ \pi Q = 0 \quad \text{and} \quad \pi_0 e_1 + \sum_{n=1}^{\infty} \pi_n e_2 = 1, \quad (7.1) \]
Figure 7.1: State Transition Diagram of an M/M/1 Queue subject to Bernoulli Schedule Controlled Vacation and Vacation Interruption

where \( e_1 = (1, 1)^T \), \( e_2 = (1, 1, 1)^T \) and \( Q = \begin{pmatrix} B_0 & A_0 \\ C_0 & B & A \\ C & B & A \\ \vdots & \vdots & \vdots \end{pmatrix} \). Note that,

\[
B_0 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix},\ A_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},\ C_0 = \begin{pmatrix} \mu_v & 0 \\ 0 & 0 \\ (1-p)\mu & p\mu \end{pmatrix},
\]

\[
A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},\ B = \begin{pmatrix} -(\lambda + \mu_v + \theta_v) & 0 & \theta_v \\ 0 & -(\lambda + \theta) & \theta \\ 0 & 0 & -(\lambda + \mu) \end{pmatrix}
\]

and

\[
C = \begin{pmatrix} q\mu_v & 0 & (1-q)\mu_v \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}.
\]

Then, the steady state probabilities of the
background queueing model are given by Zhang & Shi (2009) as

\[
\pi_{n,0} = \frac{(1-p)\left(1-\frac{\lambda}{\mu}\right)}{\lambda - \mu, \lambda} K^n \chi^n, \quad n = 0, 1, 2, \ldots
\]

\[
\pi_{n,1} = \frac{p\left(1-\frac{\lambda}{\mu}\right)}{\lambda} K \left(\frac{\lambda}{\lambda + \theta}\right)^n, \quad n = 0, 1, \ldots
\]

and \( \pi_{n,2} = \frac{(1-\frac{\lambda}{\mu})}{\mu} K \sum_{k=0}^{n-1} \left(\frac{\lambda}{\mu}\right)^{n-k-1} \left[p \left(\frac{\lambda}{\lambda + \theta}\right)^k + (1-p)\chi^k\right], n = 1, 2, \ldots \)

subject to the stability conditions \( \frac{\lambda}{\mu} < 1 \) and \( \chi = \frac{(\lambda+\mu_0+\lambda_2)-\sqrt{(\lambda+\mu_0+\lambda_2)^2-4\lambda\mu_0}}{2\lambda\mu_0} < 1 \),
where \( K = \left[p \frac{\lambda+\theta}{\mu} + (1-p)\frac{(\mu-\lambda_2)}{\mu(1-\chi)(\lambda-\mu_0)}\right] \).

\[7.2\] ANALYSIS OF FLUID QUEUE

This section deals with the stationary analysis of a fluid queue modulated by an \( M/M/1 \) queueing model subject to Bernoulli Schedule controlled Vacation and Vacation Interruption. Let \( C(t) \) be the content of the buffer at time \( t \). Furthermore, it is assumed that the content of the buffer increases at the rate \( r \), when there are customers in the background queueing model, while the buffer content decreases at the rate \( r_0 \), when the system is empty. The rate at which the content of the buffer varies with time is given by

\[
\frac{dC(t)}{dt} = \begin{cases} 
0, & (X(t), J(t)) = (0,0), C(t) = 0 \\
r_0, & (X(t), J(t)) = (0,0), C(t) > 0 \\
r_0, & (X(t), J(t)) = (0,1), C(t) > 0 \\
r, & (X(t), J(t)) = (n,j), n \geq 1 \text{ and } j = 0, 1, 2
\end{cases}
\]

where \( r_0 < 0 \) and \( r > 0 \). Clearly the 3-dimensional process \( \{(X(t), J(t), C(t)), t \geq 0\} \) represents a fluid queue driven by an \( M/M/1 \)
queue with Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. As the content of the buffer varies dynamically, it is necessary that the net effective rate of the fluid remains negative to ensure the stability of the process in a long run. Hence, the stability condition is given by

\[ d = r_0(\pi_{0,0} + \pi_{0,1}) + r \sum_{n=1}^{\infty} \sum_{j=0}^{2} \pi_{n,j} < 0. \]

Define the joint probability distribution functions of the Markov process \( \{(X(t), J(t), C(t)), t \geq 0\} \) under steady state as

\[ F_{n,j}(x) = \lim_{t \to +\infty} Pr\{X(t) = n, J(t) = j, C(t) \leq x\}, \quad x > 0, \ (n, j) \in S. \]

Using standard methods, the system of differential difference equations that governs the process \( \{(X(t), J(t), C(t)), t \geq 0\} \) are given by

\[
\begin{align*}
    r_0 \frac{dF_{0,0}(x)}{dx} & = -\lambda F_{0,0}(x) + \mu F_{1,0}(x) + (1 - p)\mu F_{1,2}(x), \\
    r_0 \frac{dF_{0,1}(x)}{dx} & = -\lambda F_{0,1}(x) + p\mu F_{1,2}(x), \\
    r \frac{dF_{n,0}(x)}{dx} & = -(\lambda + \mu_v + \theta_v)F_{n,0}(x) + \lambda F_{n-1,0}(x) + q\mu_v F_{n+1,0}(x) \quad n \geq 1, \\
    r \frac{dF_{n,1}(x)}{dx} & = -(\lambda + \theta)F_{n,1}(x) + \lambda F_{n-1,1}(x) \quad n \geq 1, \quad (7.2) \\
    r \frac{dF_{1,2}(x)}{dx} & = -(\lambda + \mu)F_{1,2}(x) + \theta F_{1,1}(x) + \theta_v F_{1,0}(x) + (1 - q)\mu_v F_{2,2}(x) + \mu F_{2,1}(x), \\
    r \frac{dF_{n,2}(x)}{dx} & = -(\lambda + \mu)F_{n,2}(x) + \theta F_{n,1}(x) + \theta_v F_{n,0}(x) + (1 - q)\mu_v F_{n+1,0}(x) \\
    & \quad + \lambda F_{n-1,2}(x) + \mu F_{n+1,2}(x) \quad n \geq 2.
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
    F_{0,0}(0) & = a_1, \\
    F_{0,1}(0) & = a_2, \quad \text{and} \\
    F_{n,j}(0) & = 0, \quad (n, j) \in S \setminus \{(0,0) \cup (0,1)\}. \quad (7.3)
\end{align*}
\]
The constants $a_1$ and $a_2$ are such that $0 < a_1 < 1$ and $0 < a_2 < 1$.

### 7.3 SOLUTION METHODOLOGY

This section presents explicit expressions for the joint steady state probabilities of the background queueing model and the content of the buffer in terms of modified Bessel function of the first kind. The governing system of equations in the Laplace domain are expressed as a system of matrix equations. The minimal non-negative solution of the matrix quadratic equation is determined. The stationary joint probability distributions are expressed in terms of this minimal non-negative solution and are shown to satisfy the governing system of matrix equations. In this sequel, define

$$F_0(x) = (F_{0,0}(x), F_{0,1}(x)) \quad \text{and} \quad F_n(x) = (F_{n,0}(x), F_{n,1}(x), F_{n,2}(x)), \quad n = 1, 2, \ldots.$$  

Let $F(x) = (F_0(x), F_1(x), F_2(x), \cdots)$. Then, the governing system of equations represented by equation (7.2) can be written in matrix form as

$$F'(x) \Lambda = F(x)Q,$$  

(7.4)

where $\Lambda = \begin{pmatrix} \Sigma' \\ \Sigma' \\ \Sigma' \\ \ddots \end{pmatrix}$, $\Sigma' = \begin{pmatrix} r & 0 \\ 0 & r_0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}$.

Let $\hat{F}(s) = (\hat{F}_0(s), \hat{F}_1(s), \hat{F}_2(s), \cdots)$ represents the Laplace transform of $F(x)$. Then, the Laplace transform of the equation (7.4) yields

$$\hat{F}(s)(Q - s \Lambda) = (\bar{\alpha} \ 0 \ 0 \ \cdots ),$$

where $\bar{\alpha} = (r_0a_1, \ r_0a_2)$. The governing system of differential difference
equations in the Laplace domain are then given by

\[
\hat{F}_0(s)(B_0 - s\Sigma') + \hat{F}_1(s)C_0 = -\bar{a}, \tag{7.5}
\]

\[
\hat{F}_0(s)A_0 + \hat{F}_1(s)(B - s\Sigma) + \hat{F}_2(s)C = 0, \tag{7.6}
\]

and \[
\hat{F}_{n-1}(s)A + \hat{F}_n(s)(B - s\Sigma) + \hat{F}_{n+1}(s)C = 0 \quad \text{for} \quad n = 2, 3, \cdots. \tag{7.7}
\]

Our objective is to solve the above system of matrix difference equations to obtain explicit expressions for the stationary probability distribution and hence determine the stationary buffer content distribution. Towards this end, we present a Lemma below followed by a Theorem.

**Lemma 1** The matrix quadratic equation

\[
A + R(s)(B - s\Sigma) + R^2(s)C = 0, \tag{7.8}
\]

has the minimal nonnegative solution given by

\[
R(s) = \begin{pmatrix}
\chi(s) & 0 & \psi_1(s) \\
0 & \psi(s) & \psi_2(s) \\
0 & 0 & z(s)
\end{pmatrix},
\]

where

\[
\chi(s) = \frac{(\lambda + \mu_v + \theta_v + sr) - \sqrt{(\lambda + \mu_v + \theta_v + sr)^2 - 4\lambda \mu_v}}{2\mu_v},
\]

\[
\psi(s) = \frac{\lambda}{\lambda + \theta + sr},
\]
\[ \psi_1(s) = \frac{\chi(s)z(s)((1-q)\mu, \chi(s) + \theta_v)}{\lambda - \mu \chi(s)z(s)}, \]
\[ \psi_2(s) = \frac{\theta z(s)}{\lambda + \theta + sr - \mu z(s)}, \]
and \[ z(s) = \frac{(\lambda + \mu + sr) - \sqrt{(\lambda + \mu + sr)^2 - 4\lambda \mu}}{2\mu}. \]

**Proof**

Since \( A, (B - s\Sigma) \) and \( C \) are all upper triangular matrices, we can assume that the solution \( R(s) \) has the same structure as

\[
R(s) = \begin{pmatrix}
    r_{11}(s) & r_{12}(s) & r_{13}(s) \\
    0 & r_{22}(s) & r_{23}(s) \\
    0 & 0 & r_{33}(s)
\end{pmatrix}.
\]

Note that, the element in the first row and second column of all the matrices of \( A, (B - s\Sigma) \) and \( C \) are zero, therefore \( r_{12}(s) \) in \( R(s) \) is also zero. Substituting for \( R(s) \) into equation (7.8) leads to

\[
\lambda - (\lambda + \mu_v + \theta_v + sr)r_{11}(s) + q\mu_v(r_{11}(s))^2 = 0, \quad (7.9)
\]
\[
\lambda - (\lambda + \mu + sr)r_{33}(s) + \mu(r_{33}(s))^2 = 0, \quad (7.10)
\]
\[
\lambda - (\lambda + \theta + sr)r_{22}(s) = 0, \quad (7.11)
\]

\[
(1-q)\mu_v(r_{11}(s))^2 + \mu r_{13}(s)r_{11}(s) + \mu r_{13}(s)r_{33}(s) + \theta_v r_{11}(s) \\
- (\lambda + \mu + sr)r_{13}(s) = 0, \quad (7.12)
\]

and

\[
\theta r_{22}(s) - (\lambda + \mu + sr)r_{23}(s) + \mu[r_{22}(s)r_{23}(s) + r_{23}(s)r_{33}(s)] = 0. \quad (7.13)
\]
The solution of equation (7.9) and equation (7.10) are given by

\[ r_{11}(s) = \frac{\lambda + \mu + \Theta v + sr}{2q\mu_v} \pm \frac{\sqrt{(\lambda + \mu + \Theta v + sr)^2 - 4\lambda q\mu_v}}{2q\mu_v}, \]

and

\[ r_{33}(s) = \frac{\lambda + \mu + sr}{2\mu} \pm \frac{\sqrt{(\lambda + \mu + sr)^2 - 4\lambda\mu}}{2\mu}. \]

Let \( \chi(s) (\chi_1(s)) \) and \( z(s) (z_1(s)) \) denote the negative (positive) roots of \( r_{11}(s) \) and \( r_{33}(s) \) respectively. Considering the root that lies inside the unit circle, \( \chi(s) \) and \( z(s) \) are taken for further analysis. Further, \( z(s) \) satisfies the following relations.

\[ sr + \lambda + \mu(1 - z(s)) = \frac{\lambda}{z(s)} = \mu + \frac{sr}{1 - z(s)}. \]  \( (7.14) \)

From equation (7.11), we get

\[ r_{22}(s) = \frac{\lambda}{\lambda + \Theta + sr} = \psi(s). \]

Substituting for \( r_{11}(s) \) and \( r_{33}(s) \) into equation (7.12) yields

\[ r_{13}(s) = \frac{\chi(s)(1 - q)\mu, \chi(s) + \Theta_v}{\lambda + \mu + sr - \mu(\chi(s) + z(s))}. \]

Using the relation given by equation (7.14) in the above leads to

\[ r_{13}(s) = \frac{\chi(s)z(s)(1 - q)\mu, \chi(s) + \Theta_v}{\lambda - \mu\chi(s)z(s)} = \psi_1(s). \]

Again, substituting for \( r_{22}(s) \) and \( r_{33}(s) \) into equation (7.13) leads to

\[ r_{23}(s) = \frac{\lambda\Theta}{(sr + \lambda + \mu)(sr + \lambda + \Theta) - \lambda\mu - \mu z(s)(sr + \Theta + \lambda)}. \]

Using the relation given by equation (7.14) in the above equation yields

\[ r_{23}(s) = \frac{\Theta z(s)}{\lambda + \Theta + sr - \mu z(s)} = \psi_2(s). \]
This completes the proof.

Note that $R^n(s)$ for $n = 1, 2, 3, \cdots$ can be simplified as

$$R^n(s) = \begin{pmatrix}
\chi^n(s) & 0 & \psi_1(s) \sum_{i=1}^n \chi^{n-i}(s) z^{i-1}(s) \\
0 & \psi^n(s) & \psi_2(s) \sum_{j=1}^n \psi^{n-j}(s) z^{j-1}(s) \\
0 & 0 & z^n(s)
\end{pmatrix}.$$  \hspace{1cm} (7.15)

Also $R(0) = R$, where $R$ is given by $R = \begin{pmatrix}
\chi & 0 & \frac{\lambda - \mu \chi}{\mu} \\
0 & \frac{\lambda}{\lambda + \theta} & \frac{\lambda}{\mu} \\
0 & 0 & \frac{\lambda}{\mu}
\end{pmatrix}$, with

$$\chi = \frac{(\lambda + \mu + \theta) - \sqrt{(\lambda + \mu + \theta)^2 - 4 \lambda \mu \theta}}{2 \mu \theta}.$$

**Theorem 1** If $d < 0$, then the stationary joint probability distributions of the content of the buffer and the state of the background queueing model in Laplace domain are given by

$$\hat{F}_n(s) = \hat{F}_0(s)e^{R^n(s)} \quad \text{for} \quad n = 1, 2, 3, \cdots,$$

$$\text{and} \quad \hat{F}_0(s) = \frac{a}{s \Sigma - B_0 - eR(s)C_0},$$  \hspace{1cm} (7.16)\hspace{1cm} (7.17)

where $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

**Proof**

Assume

$$\hat{F}_n(s) = \hat{F}_{n-1}(s)R(s) \quad \text{for} \quad n = 1, 2, 3, \cdots.$$  \hspace{1cm} (7.18)

Then, it can be recursively written as

$$\hat{F}_n(s) = \hat{F}_0(s)e^{R^n(s)} \quad \text{for} \quad n = 1, 2, 3, \cdots.$$
Substituting equation (7.16) into the equation (7.6) leads to

\[ \hat{F}_0(s)A_0 + \hat{F}_1(s)(B - s\Sigma) + \hat{F}_2(s)C = \hat{F}_0(s)eA + \hat{F}_0(s)eR(s)(B - s\Sigma) + \hat{F}_0(s)eR^2(s)C, \]

\[ = \hat{F}_0(s)e[A + R(s)(B - s\Sigma) + R^2(s)C], \]

\[ = 0 \quad \text{(by Lemma 1)}. \]

Similarly, substituting equation (7.18) into the equation (7.7) leads to

\[ \hat{F}_{n-1}(s)A + \hat{F}_n(s)(B - s\Sigma) + \hat{F}_{n+1}(s)C = \hat{F}_{n-1}(s)A + \hat{F}_{n-1}(s)R(s)(B - s\Sigma) + \hat{F}_{n-1}(s)R^2(s)C, \]

\[ = \hat{F}_{n-1}(s)[A + R(s)(B - s\Sigma) + R^2(s)C], \]

\[ = 0 \quad \text{(by Lemma 1)}. \]

From equation (7.5), we get

\[ \hat{F}_0(s)(B_0 - s\Sigma') + \hat{F}_0(s)eR(s)C_0 = -\bar{a}. \]

Therefore

\[ \hat{F}_0(s) = \frac{\bar{a}}{s\Sigma' - B_0 - eR(s)C_0}. \quad \text{(7.19)} \]

Hence all the joint stationary probabilities, \( \hat{F}_n(s) \) for \( n = 1, 2, 3, \ldots \) are in terms of \( \hat{F}_0(s) \), where \( \hat{F}_0(s) \) is given by equation (7.19). This completes the proof.

We now present an explicit expressions for the components of \( \hat{F}_0(s) \)
and \( \hat{F}_n(s) \). Consider

\[
\begin{align*}
eR(s)C_0 & = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi(s) & 0 & \psi_1(s) \\ 0 & \psi(s) & \psi_2(s) \\ 0 & 0 & z(s) \end{pmatrix} \begin{pmatrix} \mu_v & 0 \\ 0 & 0 \\ (1-p)\mu & \mu \end{pmatrix}, \\
& = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_v \chi(s) + (1-p)\mu \psi_1(s) & p\mu \psi_1(s) \\ (1-p)\mu \psi_2(s) & p\mu \psi_2(s) \\ (1-p)\mu z(s) & p\mu z(s) \end{pmatrix}, \\
& = \begin{pmatrix} \mu_v \chi(s) + (1-p)\mu \psi_1(s) & p\mu \psi_1(s) \\ (1-p)\mu \psi_2(s) & p\mu \psi_2(s) \end{pmatrix}.
\end{align*}
\]

and hence

\[
\begin{align*}
s \Sigma' - B_0 - eR(s)C_0 & = \begin{pmatrix} sr_0 & 0 \\ 0 & sr_0 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} - \begin{pmatrix} \mu_v \chi(s) + (1-p)\mu \psi_1(s) & p\mu \psi_1(s) \\ (1-p)\mu \psi_2(s) & p\mu \psi_2(s) \end{pmatrix}, \\
& = \begin{pmatrix} sr_0 + \lambda - \mu \chi(s) - (1-p)\mu \psi_1(s) & -p\mu \psi_1(s) \\ -(1-p)\mu \psi_2(s) & sr_0 + \lambda - p\mu \psi_2(s) \end{pmatrix}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{1}{s \Sigma' - B_0 - eR(s)C_0} & = W(s) \begin{pmatrix} sr_0 + \lambda - p\mu \psi_2(s) & p\mu \psi_1(s) \\ (1-p)\mu \psi_2(s) & sr_0 + \lambda - \mu \chi(s) - (1-p)\mu \psi_1(s) \end{pmatrix}, \\
& = (7.20)
\end{align*}
\]
where

\[ W(s) = \frac{1}{[sr_0 + \lambda - \mu \chi(s) - (1 - p)\mu \psi_1(s)][sr_0 + \lambda - p\mu \psi_2(s)] - (1 - p)\mu \psi_2(s)\mu \psi_1(s)}. \]

which can be rewritten as

\[ W(s) = \frac{1}{(sr_0 + \lambda - p\mu \psi_2(s))(sr_0 + \lambda - \mu \chi(s))} \sum_{k=0}^{\infty} \left( \frac{\psi_1(s)(1 - p)\mu(sr_0 + \lambda)}{(sr_0 + \lambda - p\mu \psi_2(s))(sr_0 + \lambda - \mu \chi(s))} \right)^k, \]

and hence

\[ W(s) = \frac{1}{(sr_0 + \lambda - p\mu \psi_2(s))(sr_0 + \lambda - \mu \chi(s))} \sum_{k=0}^{\infty} \psi_3^k(s), \tag{7.21} \]

where

\[ \psi_3(s) = \frac{\psi_1(s)(1 - p)\mu(sr_0 + \lambda)}{(sr_0 + \lambda - p\mu \psi_2(s))(sr_0 + \lambda - \mu \chi(s))}. \]

Substituting equation (7.20) in equation (7.19) gives

\[ \hat{F}_{0}(s) = W(s) \begin{pmatrix} a_1r_0 & a_2r_0 \\ sr_0 + \lambda - p\mu \psi_2(s) & p\mu \psi_1(s) \\ (1 - p)\mu \psi_2(s) & sr_0 + \lambda - \mu \chi(s) - (1 - p)\mu \psi_1(s) \end{pmatrix}, \]

which further yields

\[ \left( \hat{F}_{0,0}(s) \hat{F}_{0,1}(s) \right) = W(s) \begin{pmatrix} a_1r_0(sr_0 + \lambda - p\mu \psi_2(s) + a_2r_0(1 - p)\mu \psi_2(s) \\ a_1r_0p\mu \psi_1(s) + a_2r_0(sr_0 + \lambda - \mu \chi(s) - (1 - p)\mu \psi_1(s)) \end{pmatrix}. \]
Upon simplification, we obtain

\[
\hat{F}_{0,0}(s) = \frac{1}{(sr_0 + \lambda - p\mu\psi_2(s))(sr_0 + \lambda - \mu_r\chi(s))} \sum_{k=0}^{\infty} \psi_3^k(s)
\]

\[
[a_1 r_0 (sr_0 + \lambda - p\mu\psi_2(s) + a_2 r_0 (1 - p)\mu\psi_2(s))],
\]

and hence

\[
\hat{F}_{0,0}(s) = \sum_{k=0}^{\infty} \psi_3^k(s) \left[ \frac{a_1 r_0}{(sr_0 + \lambda - \mu_r\chi(s))} + \frac{a_2 r_0 (1 - p)\mu\psi_2(s)}{(sr_0 + \lambda - p\mu\psi_2(s))(sr_0 + \lambda - \mu_r\chi(s))} \right],
\]

\[
= \sum_{k=0}^{\infty} (\psi_3(s))^k \left[ \frac{a_1 r_0}{sr_0 + \lambda - \mu_r\chi(s)} + a_2 r_0 (1 - p)\mu\psi_2(s) D(s) \right].
\]

Therefore

\[
\hat{F}_{0,0}(s) = \sum_{k=0}^{\infty} (\psi_3(s))^k \left[ a_1 \sum_{l=0}^{\infty} \left( \frac{\mu_r}{r_0} \right)^l \frac{\chi^l(s)(s + \lambda)}{sr_0 + \lambda - \mu_r\chi(s)} + a_2 r_0 (1 - p)\mu\psi_2(s) D(s) \right],
\]

(7.22)

where

\[
D(s) = \frac{1}{(sr_0 + \lambda - p\mu\psi_2(s))(sr_0 + \lambda - \mu_r\chi(s))}.
\]

Similarly,

\[
\hat{F}_{0,1}(s) = E(s)[a_1 r_0 p\mu\psi_1(s) + a_2 r_0 sr_0 + \lambda - \mu_r\chi(s) - (1 - p)\mu\psi_1(s)],
\]

\[
= \frac{1}{(sr_0 + \lambda - p\mu\psi_2(s))(sr_0 + \lambda - \mu_r\chi(s))} \sum_{k=0}^{\infty} \psi_3^k(s)
\]

\[
[\psi_1(s)(a_1 r_0 p\mu - a_2 r_0 (1 - p)\mu) + a_2 r_0 (sr_0 + \lambda - \mu_r\chi(s))],
\]

\[
= \sum_{k=0}^{\infty} \psi_3^k(s) \left[ \frac{\psi_1(s)(a_1 r_0 p\mu - a_2 r_0 (1 - p)\mu)}{(sr_0 + \lambda - p\mu\psi_2(s))(sr_0 + \lambda - \mu_r\chi(s))} \right.
\]

\[
+ \frac{a_2 r_0}{(sr_0 + \lambda - p\mu\psi_2(s))}.\]

Hence

\[ \hat{F}_{0,1}(s) = \sum_{k=0}^{\infty} \psi_3^k(s) \left[ (a_1 r_0 p \mu - a_2 r_0 (1 - p) \mu) \psi_1(s) D(s) \right. \]

\[ \left. + a_2 \sum_{k=0}^{\infty} \left( \frac{p \mu}{r_0} \right)^k \frac{\psi_2^k(s)}{(s + \frac{\lambda}{r_0})^{k+1}} \right]. \tag{7.23} \]

From equation (7.16), we get

\[ [\hat{F}_{n,0}(s) \quad \hat{F}_{n,1}(s) \quad \hat{F}_{n,2}(s)] = (\hat{F}_{0,0}(s) \quad \hat{F}_{0,1}(s)) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \]

\[ \begin{pmatrix} \chi^n(s) & 0 & \psi_1(s) \sum_{i=1}^{n} \chi^{n-i}(s) z^{i-1}(s) \\ 0 & \psi^n(s) & \psi_2(s) \sum_{j=1}^{n} \psi^{n-j}(s) z^{j-1}(s) \\ 0 & 0 & z^n(s) \end{pmatrix}, \]

which can be written as

\[ \hat{F}_{n,0}(s) = \chi^n(s) \hat{F}_{0,0}(s), \tag{7.24} \]

\[ \hat{F}_{n,1}(s) = \psi^n(s) \hat{F}_{0,1}(s), \tag{7.25} \]

and

\[ \hat{F}_{n,2}(s) = \psi_1(s) \sum_{i=1}^{n} \chi^{n-i}(s) z^{i-1}(s) \hat{F}_{0,0}(s) + \psi_2(s) \sum_{j=1}^{n} \psi^{n-j}(s) z^{j-1}(s) \hat{F}_{0,1}(s). \tag{7.26} \]

Having determined all the joint steady state probabilities in the Laplace domain, we now present the explicit analytical solution by inverting using transform techniques. With \( \alpha = \frac{2 \sqrt{\lambda \mu}}{r} \) and \( \beta_v = \frac{2 \sqrt{\lambda \mu v}}{r} \), inversion of
equations (7.24), (7.25), (7.22), (7.23) and (7.26) yields

\[
F_{n,0}(x) = \frac{nI_n(\beta_0)\beta_0^n}{x} \left(\frac{r}{2q\mu v}\right)^n e^{-\left(\frac{r}{\mu v}\right)x} \ast F_{0,0}(x) \quad n = 1, 2, 3, \ldots, \tag{7.27}
\]

\[
F_{n,1}(x) = \left(\frac{\lambda}{r}\right)^n e^{-\left(\frac{r}{\mu v}\right)x} \frac{x^{n-1}}{(n-1)!} \ast F_{0,1}(x) \quad n = 1, 2, 3, \ldots, \tag{7.28}
\]

\[
F_{0,0}(x) = \sum_{k=0}^{\infty} \delta(x)^k \ast \left[ a_1 \sum_{l=0}^{\infty} \left(\frac{\mu v}{r_0}\right)^{l} e^{-\left(\frac{r}{r_0}\right)x} \frac{x^l}{l!} \ast I_l(\beta \alpha)\beta_0^l \left(\frac{r}{2q\mu v}\right)^l e^{-\left(\frac{r}{\mu v}\right)x} \ast F_{0,0}(x) \right] + a_2 r_0 (1-p) \mu \psi_2(x) \ast D(x), \tag{7.29}
\]

\[
F_{0,1}(x) = \sum_{k=0}^{\infty} \delta(x)^k \ast \left[ (a_1 r_0 p \mu - a_2 r (1-p) \mu) \psi_1(x) \ast D(x) \right. \\
+ a_2 \sum_{k=0}^{\infty} \left(\frac{\mu v}{r_0}\right)^k e^{-\left(\frac{r}{r_0}\right)x} \frac{x^k}{k!} \ast \psi_2(x)^k \right], \text{ and} \tag{7.30}
\]

\[
F_{n,2}(x) = \left(\psi_1(x) \ast \sum_{i=1}^{n} \left(\frac{r}{2q\mu v}\right)^{n-i} \frac{(n-i)I_{n-i}(\beta_0)\beta_0^{n-i}}{x} e^{-\left(\frac{r}{\mu v}\right)x} \ast \left(\frac{r}{2\mu}\right)^{i-1} \right. \\
\left. \frac{(i-1)I_{i-1}(\alpha x)\alpha^{i-1}}{x} e^{-\left(\frac{r}{\mu v}\right)x} \right) + \left(\psi_2(x) \ast \sum_{j=1}^{n} \left(\frac{\lambda}{r}\right)^{n-j} e^{-\left(\frac{r}{\mu v}\right)x} \right. \\
\left. \frac{x^{n-j-1}}{(n-j-1)!} \ast \left(\frac{r}{2\mu}\right)^{j-1} \frac{(j-1)I_{j-1}(\alpha x)\alpha^{j-1}}{x} e^{-\left(\frac{r}{\mu v}\right)x} \ast F_{0,1}(x) \right) \quad n \geq 1. \tag{7.31}
\]
where

$$\psi_3(x) = (1 - p) \mu \psi_1(x) \ast \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(p\mu_j)^i \mu_j^j}{r_0^{i+j+1}} e^{-\left(\frac{x}{\lambda_0}\right)^x} \frac{x^{i+j}}{(i+j)!} \ast \psi_2(x)^j \ast \frac{j I_j(\beta_i x) \beta_i^j}{x} \left(\frac{r}{2q\mu_v}\right)^j e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x},$$

$$D(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(p\mu_j)^i \mu_j^j}{r_0^{i+j+2}} e^{-\left(\frac{x}{\lambda_0}\right)^x} \frac{x^{i+j+1}}{(i+j+1)!} \ast \psi_2(x)^i \ast \frac{j I_j(\beta_i x) \beta_i^j}{x} \left(\frac{r}{2q\mu_v}\right)^j e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x},$$

$$\psi_1(x) = (1 - q) \left(\mu \sum_{k=0}^{\infty} \frac{\mu^k (k + 2) I_{k+2}(\beta_i x) \beta_i^{k+2}}{\lambda_{k+1}} \left(\frac{r}{2q\mu_v}\right)^k e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x} + \theta_v \left(\sum_{k=0}^{\infty} \frac{\mu^k (k + 1) I_{k+1}(\beta_i x) \beta_i^{k+1}}{\lambda_{k+1}} \right) \left(\frac{r}{2q\mu_v}\right)^k e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x} \ast \left(\frac{r}{2\mu}\right)^{k+1} \frac{(k + 1) I_{k+1}(\alpha x) \alpha^{k+1}}{x} e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x}\right),$$

and

$$\psi_2(x) = \theta \sum_{k=0}^{\infty} \frac{\mu^k}{\lambda_{k+1} k!} e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x} \frac{x^k}{k!} \ast \left(\frac{r}{2\mu}\right)^{k+1} \frac{(k + 1) I_{k+1}(\alpha x) \alpha^{k+1}}{x} e^{-\left(\frac{\lambda_i + \alpha_0}{r}\right)x}.$$ 

Thus all the joint steady state probabilities of the state of the system and the content of the buffer are explicitly obtained in terms of modified Bessel function of the first kind.

**Remark: 1** When \( p = 0 \) and \( q = 0 \), the model under consideration reduces to a fluid queue driven by an \( M/M/1 \) queue with working vacation and vacation interruption discussed by Xu et al (2012). With \( a_2 = 0 \), the expression for \( F_{0,0}(s) \)
from equation (7.17) yields

$$
\hat{F}_{0,0}(s) = \frac{a_1 r_0}{\lambda + sr_0 - \mu \chi(s) - \mu \psi_1(s)},
$$

(7.32)

where

$$
\chi(s) = \frac{\lambda}{\lambda + \mu + \theta v + sr},
$$

$$
\psi_1(s) = \frac{\chi(s)[\mu, \chi(s) + \theta v]}{\mu[z_1(s) - \chi(s)]},
$$

and

$$
z_1(s) = \frac{(\lambda + \mu + sr) + \sqrt{(\lambda + \mu + sr)^2 - 4\lambda \mu}}{2\mu}.
$$

Observe that equation (7.32) is seen to coincide with first expression in equation (6) of Xu et al (2012).

**Remark: 2** When \( p = 1 \), the model under consideration reduces to a fluid queue modulated by an \( M/M/1 \) queue with multiple exponential vacation discussed by Wang et al (2010). Let \( a_1 = 0 \) and \( z(s) = \frac{\lambda}{\mu} z_0(s) \), where \( z_0(s) = \frac{(\lambda + \mu + sr) - \sqrt{(\lambda + \mu + sr)^2 - 4\lambda \mu}}{2\lambda} \). Then the expression for \( \hat{F}_{0,1}(s) \) from equation (7.17) becomes

$$
\hat{F}_{0,1}(s) = \frac{a_2 r_0}{\lambda + sr_0 - \mu \psi_2(s)},
$$

where

$$
\psi_2(s) = \frac{\lambda z_0(s)}{\mu(\lambda + \theta + sr - \lambda z_0(s))}.
$$

Substituting \( \psi_2(s) \) in the above equation and after certain simplification, we obtain

$$
\hat{F}_{0,1}(s) = \frac{a_2 r_0}{\frac{\lambda z_0(s)}{\mu(\lambda + \theta + sr - \lambda z_0(s))}},
$$
and hence
\[ \hat{F}_{0,1}(s) = \frac{a_2r_0[\lambda + \theta + sr - \lambda z_0(s)]}{(sr_0 + \lambda)(sr + \lambda + \theta - \lambda z_0(s)) - \lambda \theta z_0(s)}, \]
\[ = \frac{a_2r_0[sr + \lambda(1 - z_0(s)) + \theta]}{(sr_0 + \lambda)(sr + \lambda(1 - z_0(s)) + \theta) - \lambda \theta z_0(s)}. \quad (7.33) \]

Observe that equation (7.33) is seen to coincide with equation (16) of Wang et al (2010).

### 7.4 BUFFER CONTENT DISTRIBUTION

The stationary buffer content distribution of the fluid model under consideration is given by
\[ F(x) = \sum_{n=0}^{\infty} F_{n,0}(x) + \sum_{n=1}^{\infty} F_{n,1}(x) + \sum_{n=1}^{\infty} F_{n,2}(x). \]

Taking Laplace transform of the above equation yields
\[ \hat{F}(s) = \sum_{n=0}^{\infty} \hat{F}_{n,0}(s) + \sum_{n=0}^{\infty} \hat{F}_{n,1}(s) + \sum_{n=1}^{\infty} \hat{F}_{n,2}(s). \]

In matrix notation, the above equation can be rewritten as
\[ \hat{F}(s) = \hat{F}_0(s)e_1 + \sum_{n=1}^{\infty} \hat{F}_n(s)e_2. \]

Then,
\[ \hat{F}(s) \quad = \quad \hat{F}_0(s)e_1 + \hat{F}_0(s)e \sum_{n=1}^{\infty} (R(s))^n e_2, \quad (\text{from equation (7.16)}) \]
\[ \quad = \quad \hat{F}_0(s)e \sum_{n=0}^{\infty} (R(s))^n e_2, \]
\[ \quad = \quad \hat{F}_0(s)e[I - R(s)]^{-1}e_2, \quad (7.34) \]
where

\[
[I - R(s)]^{-1} = \begin{pmatrix}
\frac{1}{1-\chi(s)} & 0 & \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} \\
0 & \frac{1}{1-z(s)} & \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} \\
0 & \frac{sr+\lambda+\theta}{sr+\theta} & \frac{1}{1-z(s)}
\end{pmatrix}.
\]

Consider

\[
e[I - R(s)]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{1-\chi(s)} & 0 & \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} \\
0 & \frac{1}{1-z(s)} & \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} \\
0 & \frac{sr+\lambda+\theta}{sr+\theta} & \frac{1}{1-z(s)}
\end{pmatrix} \begin{pmatrix} 1 \\
1 \\
1
\end{pmatrix},
\]

and hence

\[
e[I - R(s)]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\frac{1}{1-\chi(s)} + \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} \\
\frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} + \frac{1}{1-z(s)} \\
\frac{sr+\lambda+\theta}{sr+\theta} + \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)}
\end{pmatrix},
\]

\[
e[I - R(s)]^{-1} = \frac{1}{1-\chi(s)} + \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} + \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)}.
\]

(7.35)

Substituting equation (7.35) in equation (7.34) yields

\[
\hat{F}(s) = (\hat{F}_0(s, \hat{F}_0(s)) \begin{pmatrix}
\frac{1}{1-\chi(s)} + \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} \\
\frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} + \frac{1}{1-z(s)} \\
\frac{sr+\lambda+\theta}{sr+\theta} + \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)}
\end{pmatrix},
\]

Upon simplification, we get

\[
\hat{F}(s) = \frac{1}{1-\chi(s)} \hat{F}_0(s) + \frac{\psi_1(s)}{(1-\chi(s))(1-z(s))} \hat{F}_0(s) + \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} \hat{F}_0(s) + \frac{1}{(sr+\theta)} \hat{F}_0(s) + \frac{\psi_2(s)(sr+\lambda+\theta)}{(1-z(s))(sr+\theta)} \hat{F}_0(s),
\]
Therefore

\[ \hat{F}(s) = \sum_{k=0}^{\infty} (\chi(s))^k \hat{F}_{0,0}(s) + \psi_1(s) \sum_{k=0}^{\infty} (\chi(s))^k \sum_{j=0}^{\infty} (z(s))^j \hat{F}_{0,0}(s) \]

\[ + \hat{F}_{0,1}(s) + \psi_2(s) \sum_{k=0}^{\infty} (z(s))^k \hat{F}_{0,1}(s) \]

\[ + \frac{\lambda}{sr + \theta} \hat{F}_{0,1}(s) + \psi_2(s) \sum_{k=0}^{\infty} (z(s))^k \frac{\lambda}{sr + \theta} \hat{F}_{0,1}(s). \]

which on inversion leads to

\[ F(x) = \sum_{k=0}^{\infty} \left( \frac{r}{2q\mu_v} \right)^k \frac{kI_k(\beta_v x)\beta_v^k}{x} e^{-\left(\frac{\lambda \mu_v + \theta_v}{r}\right)x} * F_{0,0}(x) + \psi_1(x) \]

\[ \ast \sum_{k=0}^{\infty} \left( \frac{r}{2q\mu_v} \right)^k \frac{kI_k(\beta_v x)\beta_v^k}{x} e^{-\left(\frac{\lambda \mu_v + \theta_v}{r}\right)x} \ast \sum_{j=0}^{\infty} \left( \frac{r}{2\mu} \right)^j \frac{jI_j(\alpha x)\alpha^j}{x} e^{-\left(\frac{\lambda \mu}{r}\right)x} \]

\[ \ast F_{0,0}(x) + F_{0,1}(x) + \frac{\lambda}{r} e^{-\left(\frac{\mu}{r}\right)x} * F_{0,1}(x) + \psi_2(x) \]

\[ \ast \sum_{k=0}^{\infty} \left( \frac{r}{2\mu} \right)^k \frac{kI_k(\alpha x)\alpha^k}{x} e^{-\left(\frac{\lambda \mu}{r}\right)x} * F_{0,1}(x) \]

\[ \ast \psi_2(x) \ast \sum_{k=0}^{\infty} \left( \frac{r}{2\mu} \right)^k \frac{kI_k(\alpha x)\alpha^k}{x} e^{-\left(\frac{\lambda \mu}{r}\right)x} \ast \frac{\lambda}{r} e^{-\left(\frac{\mu}{r}\right)x} * F_{0,1}(x). \]

(7.36)

where \( F_{0,0}(x) \) and \( F_{0,1}(x) \) are given by equation (7.29) and equation (7.30) respectively.

### 7.5 NUMERICAL ILLUSTRATIONS

This section illustrates the variation of the buffer content distribution against the content of the buffer for \( \lambda = 1, \mu = 2, \mu_v = 1.1, \theta = 0.9, \theta_v = 0.6, \)

\( p = 0.5, \ q = 0.5, \ r = 1 \) and varying values of \( r_0 \). The choice of \( r_0 \) is relatively high as compared to \( r \) because of our assumptions that \( r \) happens when the background queueing model is nonempty and \( r_0 \) happens otherwise. To
compensate for the rarity in the occurrence of \( r_0 \), it is assumed to be larger.

Figure 7.2 depicts the behavior of the buffer content distribution, \( F(x) \) against \( x \) for the above choice of the parameter values with \( r_0 = -450.9 \). For this choice of the parameters, it is seen that \( d = -140.86 < 0 \). Therefore, the stability condition is satisfied. It is seen that \( F(x) \) increases with increase in the value of \( x \) and converges to 1 as \( x \) tends to infinity. Furthermore, as the value of \( r_0 \) greatly affects buffer content distribution, its variation against \( x \) for different value of \( r_0 \) is presented in Table 7.1.

Thus all the joint steady state probabilities and the buffer content distribution of the fluid queue driven by an \( M/M/1 \) queue subject to Bernoulli Schedule Controlled Vacation and Vacation Interruption are explicitly obtained under steady state and their corresponding behaviour is illustrated numerically for varying values of the parameters.
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Table 7.1: Convergence of stationary buffer content distribution for varying values of $r_0$
Figure 7.2: Variations of the buffer content distribution against $x$