CHAPTER 5

FLUID QUEUE DRIVEN BY AN $M/M/1$ QUEUE
WITH WORKING VACATION

In many real time situations, the server may become unavailable for a random period of time to perform a secondary task, when there are no customer in the waiting line at the service completion epoch. Such period of server absence is termed as server vacation. Queueing models subject to single or multiple exponential vacation are apt to model many practical scenarios (Huo et al 2008, Narayanan et al 2008 and Jain & Upadhyaya 2009). However, a better modelling assumption would be to assume that the server works at a slower rate during vacation periods in comparison to that of regular working period. Such models are classified as queues subject to working vacation (Gao & Wang 2013, Gao & Yin 2013 and Vijaya Lakshmi et al 2013).

Server vacation may occur due to several factor like insufficient workload in human behaviour, failure of the server subject to repair period, preventive maintenance period in a production system, secondary tasks assigned to the server (which occurs in computer maintenance and testing), service rendered to arrivals as in priority queueing discipline and so on. For example, in a production inventory system, the facility produces products to fulfill the customers demands while simultaneously maintaining sufficient products in the inventory to meet the future demands. However, between any two production cycles, the idle time of the machines will be effectively used to perform maintenance action in the facility. This time period can be viewed as the period of vacation during which secondary task are performed at a slower rate. Another situation involving human behaviour is with regard to workers in the industry
who might go on a strike. However, to avoid loss to the company, it may hire workers from outside to complete the tasks. The hired workers might lack professionalism and hence does the job at slower rate compared to the skilled labourers. This scenario can once again be viewed as a queueing model subject to working vacation.

This chapter presents an analysis of fluid queue driven by an $M/M/1$ queue subject to working vacation in stationary regime. Closed form expressions for the joint state probabilities, buffer content distribution and the mean buffer content are obtained using continued fraction and generating function methodologies.

5.1 MODEL DESCRIPTION

Consider an $M/M/1$ queue subject to working vacation. The server begins a vacation of random length $V$ at the instant when the queue becomes empty and the vacation duration $V$ follows an exponential distribution with parameter $\theta$. During the vacation period, an arriving customer is assumed to be served at an exponential rate with parameter $\mu_\nu(< \mu)$. Define

$$J(t) = \begin{cases} 
1, & \text{if the server is busy at time } t. \\
0, & \text{if the server is on vacation at time } t. 
\end{cases}$$

It is well known that the process $\{(X(t), J(t)), t \geq 0\}$ is a Markov process with the state space

$$S = \{(0,0)\} \bigcup \{(n,j), n = 1, 2, \ldots, j = 0, 1\}.$$  

The state transition diagram of the background queueing model is given in
Figure 5.1. Let

\[ \pi_{n,j} = \lim_{t \to \infty} Pr\{X(t) = n, J(t) = j \} \text{ for } (n, j) \in S. \]

Then, the steady state probabilities of the background queueing model are given by

\[ \pi_{n,0} = \frac{\pi_{0,0}}{\tau_n^2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \]

\[ \pi_{n,1} = \left( \frac{\lambda}{\mu} \right)^n \left( 1 - \frac{1}{\tau_n^2} \right) \frac{\lambda - \mu \tau_n^{-1}}{\lambda - \mu \tau_2^{-1}} \pi_{0,0}, \quad n = 1, 2, \ldots, \]

where

\[ \pi_{0,0} = \frac{(1 - \left( \frac{\lambda}{\mu} \right))(1 - \tau_2^{-1})}{1 - \frac{\mu}{\mu \tau_2}}, \]

with \( \tau_1, \tau_2 = \frac{(\lambda + \mu + \theta) \pm \sqrt{(\lambda + \mu + \theta)^2 - 4\lambda \mu \theta}}{2\lambda} \). Refer Servi & Finn (2002). Further,

\[ \sum_{n=1}^{\infty} \pi_{n,0} = \frac{\pi_{0,0}}{1 - \tau_1^{-1}} \quad \text{and} \]

\[ \sum_{n=1}^{\infty} \pi_{n,1} = \left( \frac{\mu}{\mu - \lambda} - \frac{1}{1 - \tau_2^{-1}} \right) \left( \frac{\lambda - \mu \tau_2^{-1}}{\lambda - \mu \tau_1^{-1}} \right) \pi_{0,0}. \]
Let $C(t)$ be the content of the buffer at time $t$. It is assumed that the content of the buffer increases at the rate of $r$ when there are customers in the background queueing model, while the buffer content decreases at the rate $r_0$ when the system is empty. Therefore,

$$
\frac{dC(t)}{dt} = \begin{cases} 
0, & X(t) = 0, C(t) = 0 \\
 r_0, & X(t) = 0, C(t) > 0 \\
r, & X(t) > 0,
\end{cases}
$$

where $r_0 < 0$ and $r > 0$.

Clearly the 3-dimensional process $\{(X(t), J(t), C(t)), t \geq 0\}$ represent a fluid queue driven by an $M/M/1$ queue with working vacation subject to the stability conditions given by

$$
\frac{\lambda}{\mu} < 1 \quad \text{and} \quad d = r_0 \pi_{0,0} + r \sum_{n=1}^{\infty} \pi_{n,0} + r \sum_{n=1}^{\infty} \pi_{n,1} < 0.
$$

The environment process $\{(X(t), J(t)), t \geq 0\}$ is stable if and only if $\frac{\lambda}{\mu} < 1$. The quantity $d$ is called the mean drift of the process $\{C(t), t \geq 0\}$. When the buffer is infinite, the stochastic process $\{(X(t), J(t), C(t)), t \geq 0\}$ is stable if the mean drift $d < 0$ and $\frac{\lambda}{\mu} < 1$. Letting

$$
F_{n,j}(x) = \lim_{t \to \infty} Pr\{X(t) = n, J(t) = j, C(t) \leq x\} \quad \text{for } x > 0 \text{ and } (n, j) \in S,
$$

the stationary probability distribution of the buffer content is given by

$$
F(x) = \lim_{t \to \infty} P\{C(t) \leq x\} = F_{0,0}(x) + \sum_{n=1}^{\infty} F_{n,0}(x) + \sum_{n=1}^{\infty} F_{n,1}(x).
$$

Note that the long run probability for the content of the buffer to be empty is
given by

$$\lim_{t \to \infty} P\{C(t) = 0\} = F_{0,0}(0) + \sum_{n=1}^{\infty} \sum_{j=0}^{1} F_{n,j}(0). \tag{5.4}$$

By standard arguments, the system of differential difference equations that governs the fluid queueing model are given by

$$r_0 \frac{dF_{0,0}(x)}{dx} = \mu F_{1,1}(x) - \lambda F_{0,0}(x) + \mu_v F_{1,0}(x), \tag{5.5}$$

$$r \frac{dF_{n,0}(x)}{dx} = \lambda F_{n-1,0}(x) - (\lambda + \mu_v + \theta) F_{n,0}(x) + \mu_v F_{n+1,0}(x), \quad n \geq 1, \tag{5.6}$$

$$r \frac{dF_{1,1}(x)}{dx} = \theta F_{1,0}(x) - (\lambda + \mu) F_{1,1}(x) + \mu F_{2,1}(x), \tag{5.7}$$

$$r \frac{dF_{n,1}(x)}{dx} = \theta F_{n,0}(x) - (\lambda + \mu) F_{n,1}(x) + \mu F_{n+1,1}(x) + \lambda F_{n-1,1}(x), \quad n \geq 2, \tag{5.8}$$

subject to boundary conditions

$$F_{0,0}(0) = a \text{ and } F_{n,j}(0) = 0, (n, j) \in S \setminus \{0, 0\}. \tag{5.9}$$

To determine the constant $a$ which represents $F_{0,0}(0)$, adding the equations (5.5) to (5.8) yields

$$r_0 \frac{dF_{0,0}(x)}{dx} + r \sum_{n=1}^{\infty} \frac{dF_{n,0}(x)}{dx} + r \sum_{n=1}^{\infty} \frac{dF_{n,1}(x)}{dx} = 0. \tag{5.10}$$

Integrating equation (5.10) from zero to infinity gives

$$r_0 [F_{0,0}(\infty) - F_{0,0}(0)] + r \sum_{n=1}^{\infty} [F_{n,0}(\infty) - F_{n,0}(0)] + r \sum_{n=1}^{\infty} [F_{n,1}(\infty) - F_{n,1}(0)] = 0. \tag{5.11}$$

Using the boundary conditions represented by equation (5.9) in equation (5.11),
we obtain
\[ r_0[\pi_{0,0} - F_{0,0}(0)] + r \sum_{n=1}^{\infty} \pi_{n,0} + r \sum_{n=1}^{\infty} \pi_{n,1} = 0, \]

which on simplification yields
\[ F_{0,0}(0) = \frac{r_0 \pi_{0,0} + r \sum_{n=1}^{\infty} \pi_{n,0} + r \sum_{n=1}^{\infty} \pi_{n,1}}{r_0} \]  \hspace{1cm} (5.12)

Also, using the boundary conditions represented by equation (5.9) in equation (5.4) leads to
\[ \lim_{t \to \infty} P\{C(t) = 0\} = F_{0,0}(0) = \frac{d}{r_0} \]  \hspace{1cm} (from equation (5.12)).

Further, from the normalizing condition,
\[ \pi_{0,0} + \sum_{n=1}^{\infty} \pi_{n,0} + \sum_{n=1}^{\infty} \pi_{n,1} = 1, \]

it is seen that
\[ F_{0,0}(0) = \frac{r_0 \pi_{0,0} + r \left( \sum_{n=1}^{\infty} \pi_{n,0} + \sum_{n=1}^{\infty} \pi_{n,1} \right)}{r_0}, \]
\[ = \frac{r_0 \pi_{0,0} + r (1 - \pi_{0,0})}{r_0}, \]
\[ = \frac{(r_0 - r) \pi_{0,0} + r}{r_0}. \]

Therefore, the constant \( a \) is explicitly given by
\[ F_{0,0}(0) = a = \frac{(r_0 - r) \pi_{0,0} + r}{r_0}, \]  \hspace{1cm} (5.13)

where \( \pi_{0,0} \) is given by equation (5.3).
5.2 STATIONARY DISTRIBUTION

This section presents explicit expressions for the joint steady state probabilities of the background queuing model and the content of the buffer in terms of modified Bessel function of the first kind. The stationary probabilities are explicitly obtained using continued fraction, generating function and Laplace transform techniques. Let \( \hat{F}_{n,j}(s) \) denote the Laplace transform of \( F_{n,j}(x) \). The quantity \( F_{n,0}(x) \) for \( n = 1, 2, \ldots \) are first expressed in terms of \( F_{0,0}(x) \) using continued fraction approach for the Laplace transform of equation (5.6). We then define a generating function, \( G_2(z,x) = \sum_{n=1}^{\infty} F_{n,1}(x)z^n \) for the joint functional state probabilities represented by \( F_{n,1}(x) \) for \( n = 1, 2, \ldots \). From equations (5.7) and (5.8), the governing system of equations are reduced to a standard linear differential equation and hence solved. The joint state probabilities, \( F_{n,1}(x) \) are hence expressed in terms of \( F_{0,0}(x) \). Finally, \( F_{0,0}(x) \) is determined by taking Laplace transform of equation (5.5). The detailed mathematical analysis is presented below.

5.2.1 Evaluation of \( F_{n,0}(x) \)

Taking Laplace transform of equation (5.6) gives

\[
(r_s + \lambda + \mu_v + \theta)\hat{F}_{n,0}(s) - \mu_v \hat{F}_{n+1,0}(s) = \lambda \hat{F}_{n-1,0}(s), \quad n = 1, 2, 3, \ldots,
\]

which can be written as

\[
\frac{\hat{F}_{n,0}(s)}{\hat{F}_{n-1,0}(s)} = \frac{\lambda}{rs + \lambda + \mu_v + \theta - \mu_v \frac{\hat{F}_{n+1,0}(s)}{\hat{F}_{n,0}(s)}}.
\]

The above continued fraction is represented by

\[
\frac{\hat{F}_{n,0}(s)}{\hat{F}_{n-1,0}(s)} = \frac{\lambda}{rs + \lambda + \mu_v + \theta} - \frac{\lambda \mu_v}{rs + \lambda + \mu_v + \theta} - \frac{\lambda \mu_v}{rs + \lambda + \mu_v + \theta} - \ldots.
\]
Now, assume
\[ f_1(s) = \frac{\lambda \mu_v}{rs + \lambda + \mu_v + \theta} - \frac{\lambda \mu_v}{rs + \lambda + \mu_v + \theta} \ldots. \]

Then,
\[ \frac{\hat{F}_{n,0}(s)}{\hat{F}_{n-1,0}(s)} = \frac{1}{\mu_v} f_1(s), \]

and hence
\[ \hat{F}_{n,0}(s) = \frac{f_1(s)}{\mu_v} \hat{F}_{n-1,0}(s) = \left(\frac{f_1(s)}{\mu_v}\right)^n \hat{F}_{0,0}(s). \quad (5.14) \]

Also \( f_1(s) \) can be rewritten as
\[ f_1(s) = \frac{\lambda \mu_v}{rs + \lambda + \mu_v + \theta - f_1(s)} = \frac{\lambda \mu_v}{s + \frac{\lambda + \mu_v + \theta}{r} - f_1(s)}, \]

which leads to the quadratic equation given by
\[ \frac{f_1(s)^2}{r} - \left(s + \frac{\lambda + \mu_v + \theta}{r}\right)f_1(s) + \frac{\lambda \mu_v}{r} = 0. \]

Upon solving the above equation, we get
\[ f_1(s) = \frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{\frac{2}{r}}, \quad (5.15) \]

where \( p_2 = s + \frac{\lambda + \mu_v + \theta}{r} \) and \( \alpha_v = \frac{2\sqrt{\lambda \mu_v}}{r} \). Substituting equation (5.15) in equation (5.14) leads to
\[ \hat{F}_{n,0}(s) = \left(\frac{f_1(s)}{\mu_v}\right)^n \hat{F}_{0,0}(s), \]
\[ = \left(\frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{\frac{2\mu_v}{r}}\right)^n \hat{F}_{0,0}(s). \]
Therefore,

\[
\hat{F}_{n,0}(s) = \left( \frac{r}{2\mu_v} \right)^n \left[ p_2 - \sqrt{p_2^2 - \alpha_v^2} \right]^n \hat{F}_{0,0}(s),
\]

(5.16)

which on the inversion yields

\[
F_{n,0}(x) = \left( \frac{r}{2\mu_v} \right)^n \left( \frac{nI_n(\alpha_v x)\alpha_v^n}{x} \right) \exp \left[ - \left( \frac{\lambda + \mu_v + \theta}{r} \right) x \right] * F_{0,0}(x),
\]

(5.17)

Thus \( F_{n,0}(x) \) for \( n = 1, 2, 3, \ldots \) is expressed in terms of \( F_{0,0}(x) \).

### 5.2.2 Evaluation of \( F_{n,1}(x) \)

Define the generating function

\[
G_2(z, x) = \sum_{n=1}^{\infty} F_{n,1}(x)z^n.
\]

By an analysis similar to Section 4.2, the system of difference-differential equations represented by equations (5.7) and (5.8) leads to a linear differential equation given by

\[
\frac{\partial G_2(z, x)}{\partial x} = \left[ - \left( \frac{\lambda + \mu}{r} \right) + \frac{\lambda z}{r} + \frac{\mu}{rz} \right] G_2(z, x) + \frac{\theta}{r} \sum_{n=1}^{\infty} F_{n,0}(x)z^n - \frac{\mu}{r} F_{1,1}(x).
\]

(5.18)

Integrating the above equation yields

\[
G_2(z, x) = \frac{\theta}{r} \int_0^x \sum_{m=1}^{\infty} F_{m,0}(y)z^m e^{-\left( \frac{\lambda + \mu}{r} \right)(x-y)} e^{\left( \frac{\lambda y + \mu}{r} \right)(x-y)} dy
\]

\[
- \frac{\mu}{r} \int_0^x F_{1,1}(y) e^{-\left( \frac{\lambda + \mu}{r} \right)(x-y)} e^{\left( \frac{\lambda y + \mu}{r} \right)(x-y)} dy.
\]

(5.19)

Comparing the coefficients of \( z^n \) on both sides of equation (5.19), for \( n = 1, 2 \cdots \)
yields

\[ F_{n,1}(x) = \frac{\theta}{r} \int_{0}^{x} \sum_{m=1}^{\infty} F_{m,0}(y) \beta^{n-m} I_{n-m}(\alpha(x-y)) \exp \left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right] \, dy \]

\[-\frac{\mu}{r} \int_{0}^{x} F_{1,1}(y) \beta^{n} I_{n}(\alpha(x-y)) \exp \left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right] \, dy. \] (5.20)

The above equation holds for \( n = -1, -2, -3, \cdots \) with the left-hand side replaced by zero. Using \( I_{-n}(\alpha(x-y)) = I_{n}(\alpha(x-y)) \) for \( n = 1, 2, 3, \cdots \) yields

\[ 0 = \frac{\theta}{r} \int_{0}^{x} \sum_{m=1}^{\infty} F_{m,0}(y) \beta^{-n-m} I_{n+m}(\alpha(x-y)) \exp \left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right] \, dy \]

\[ + \frac{\mu}{r} \int_{0}^{x} F_{1,1}(y) \beta^{-n} I_{n}(\alpha(x-y)) \exp \left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right] \, dy. \] (5.21)

From the equations (5.20) and (5.21), we get

\[ F_{n,1}(x) = \frac{\theta}{r} \int_{0}^{x} e^{\left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right]} \sum_{m=1}^{\infty} \beta^{-n-m} F_{m,0}(y) \]

\[ [I_{n-m}(\alpha(x-y)) - I_{n+m}(\alpha(x-y))] \, dy, \] (5.22)

for \( n = 1, 2, 3, \cdots \). Thus, \( F_{n,1}(x) \) is expressed in terms of \( F_{n,0}(x) \) for \( n = 1, 2, 3, \cdots \). It is seen that equation (5.17) gives \( F_{n,0}(x) \) in terms of \( F_{0,0}(x) \). It still remains to determine \( F_{0,0}(x) \).

### 5.2.3 Evaluation of \( F_{0,0}(x) \)

Taking Laplace transform of equation (5.5) yields

\[ (sr_0 + \lambda) \hat{F}_{0,0}(s) = ar_0 + \mu \hat{F}_{1,1}(s) + \mu_0 \hat{F}_{1,0}(s). \]
Therefore,
\[
\hat{F}_{0,0}(s) = \frac{a}{s + \frac{\lambda}{r_0}} + \frac{\mu}{r_0(s + \frac{\lambda}{r_0})} \hat{F}_{1,1}(s) + \frac{\mu_v}{r_0(s + \frac{\lambda}{r_0})} \hat{F}_{1,0}(s),
\]  
\(5.23\)

From equation (5.22) for \(n = 1\), we get
\[
F_{1,1}(x) = \frac{\theta}{r} \int_0^x \exp \left[ -\left( \frac{\lambda + \mu}{r} \right) (x-y) \right] \sum_{m=1}^{\infty} \beta^{1-m} F_{m,0}(y) \left[ I_{m-1}(\alpha(x-y)) - I_{m+1}(\alpha(x-y)) \right] dy,
\]
and its Laplace transform is given by
\[
\hat{F}_{1,1}(s) = \frac{\theta}{\mu} \sum_{m=1}^{\infty} \left( \frac{\lambda}{\beta} \right)^m \left( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{\alpha} \right)^m \hat{F}_{0,0}(s),
\]  
\(5.24\)

where \(p_1 = s + \frac{\lambda + \mu}{r} \). Also, from equation (5.16) for \(n = 1\), we get
\[
\hat{F}_{1,0}(s) = \frac{1}{\mu_v} \left( \frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{\frac{2}{r}} \right) \hat{F}_{0,0}(s).
\]  
\(5.25\)

Substituting equations (5.24) and (5.25) in equation (5.23) leads to
\[
\hat{F}_{0,0}(s) = \frac{a}{s + \frac{\lambda}{r_0}} + \frac{\theta}{r_0(s + \frac{\lambda}{r_0})} \sum_{m=1}^{\infty} \left( \frac{\lambda}{\beta} \right)^m \left( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{\alpha} \right)^m \hat{F}_{0,0}(s) \]
\[- \frac{1}{r_0(s + \frac{\lambda}{r_0})} \left( \frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{\frac{2}{r}} \right) \hat{F}_{0,0}(s)
\]

which can be written as
\[
\hat{F}_{0,0}(s) \left[ 1 - \left( \frac{1}{r_0(s + \frac{\lambda}{r_0})} \right) \left\{ \theta \sum_{m=1}^{\infty} \left( \frac{\lambda}{\beta} \right)^m \left( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{\alpha} \right)^m \right. \right. \\
+ \left. \left. \left( \frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{\frac{2}{r}} \right) \right\} \right] = \frac{a}{s + \frac{\lambda}{r_0}}.
\]
On simplification, we get

\[
\hat{F}_{0,0}(s) = \frac{a}{s + \frac{\lambda}{r_0}} \times \frac{1}{1 - \left(\frac{1}{r_0(s + \frac{\lambda}{r_0})}\right) \left[\theta \sum_{m=1}^{\infty} \left(\frac{\lambda}{r_0(s + \frac{\lambda}{r_0})}\right)^{m} \left(p_1 - \sqrt{p_1 - \alpha^2} \right)^{m} \left(p_2 - \sqrt{p_2 - \alpha^2} \right)^{m}\right]}
\]

Therefore,

\[
\hat{F}_{0,0}(s) = a \sum_{j=0}^{\infty} \left(\frac{1}{r_0(s + \frac{\lambda}{r_0})}\right)^{j+1} \left[\theta \sum_{m=0}^{\infty} \left(\frac{\lambda}{r_0(s + \frac{\lambda}{r_0})}\right)^{m+1} \left(p_1 - \sqrt{p_1 - \alpha^2} \right)^{m+1} \left(p_2 - \sqrt{p_2 - \alpha^2} \right)^{m+1}\right]}
\]

Inversion of the above equation yields

\[
F_{0,0}(x) = a \sum_{j=0}^{\infty} \left(\frac{1}{r_0}\right)^{j} e^{-\left(\frac{\lambda}{r_0}\right)x} \times \left[\theta \sum_{m=0}^{\infty} \left(\frac{\lambda}{r_0}\right)^{m+1} \frac{x^m}{m!} \right] \exp \left[-\left(\frac{\lambda}{r} + \frac{\theta}{r}\right)x\right] \times \frac{(m+1)I_{m+1}(\alpha x)}{x} + \frac{r I_1(\alpha x)\alpha x}{2} \right]^{j}.
\]

Hence all the joint steady state probabilities of the fluid queueing model are explicitly determined.

5.3 BUFFER CONTENT DISTRIBUTION

The stationary buffer content distribution of the fluid model under consideration is given by

\[
F(x) = F_{0,0}(x) + \sum_{n=1}^{\infty} F_{n,0}(x) + \sum_{n=1}^{\infty} F_{n,1}(x).
\]
Taking Laplace transform of the above equation yields

$$\hat{F}(s) = \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \hat{F}_{n,0}(s) + \sum_{n=1}^{\infty} \hat{F}_{n,1}(s).$$

Toward this end, Laplace transform of equation (5.22) leads to

$$\hat{F}_{n,1}(s) = \frac{\theta}{r} \sum_{m=1}^{\infty} n^m \beta^{n-m} \hat{F}_{m,0}(s) \left[ \frac{(p_1 - \sqrt{p_1^2 - \alpha^2})^{n-m}}{\alpha^{n-m} \sqrt{p_1^2 - \alpha^2}} - \frac{(p_1 - \sqrt{p_1^2 - \alpha^2})^{n+m}}{\alpha^{n+m} \sqrt{p_1^2 - \alpha^2}} \right]$$

and hence, $\hat{F}_{n,1}(s) = \omega_1(s) \hat{F}_{0,0}(s)$, where

$$\omega_1(s) = \frac{\theta}{r} \sum_{m=1}^{\infty} n^m \beta^{n-m} \left( \frac{f_1(s)}{\mu_v} \right)^m \left[ \frac{(p_1 - \sqrt{p_1^2 - \alpha^2})^{n-m}}{\alpha^{n-m} \sqrt{p_1^2 - \alpha^2}} - \frac{(p_1 - \sqrt{p_1^2 - \alpha^2})^{n+m}}{\alpha^{n+m} \sqrt{p_1^2 - \alpha^2}} \right].$$

Therefore,

$$\hat{F}(s) = \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \left( \frac{f_1(s)}{\mu_v} \right)^n \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \omega_1(s) \hat{F}_{0,0}(s),$$

On inversion, we get

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{\mu_v^n} e^{-\left(\xi + \frac{\nu}{\alpha} + \frac{\theta}{\alpha}\right) x} \frac{mI_n(\alpha x) \alpha^n}{x} + \sum_{n=1}^{\infty} \left( \frac{\theta}{r} \sum_{m=1}^{\infty} \frac{\beta^{n-m}}{\mu_v^m} e^{-\left(\xi + \frac{\nu}{\alpha} + \frac{\theta}{\alpha}\right) x} \frac{mI_m(\alpha x) \alpha^m}{x} \right) * [I_{n-m}(\alpha x) - I_{n+m}(\alpha x)] \right] \ast F_{0,0}(x),$$

where $F_{0,0}(x)$ is given by equation (5.27).
5.4 MEAN BUFFER CONTENT

Consider

$$
\hat{F}(s) = \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \left( \frac{f_1(s)}{\mu_v} \right)^n \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \hat{F}_{n,1}(s),
$$

$$
= \hat{F}_{0,0}(s) + \frac{f_1(s)}{\mu_v - f_1(s)} \hat{F}_{0,0}(s) + \sum_{n=1}^{\infty} \hat{F}_{n,1}(s). \quad (5.29)
$$

Taking Laplace transform of equation (5.18) yields

$$
s\hat{G}_2(z, s) = \left[ -\left( \frac{\lambda + \mu}{r} \right) + \frac{\lambda z}{r} + \frac{\mu}{rz} \right] \hat{G}_2(z, s) + \frac{\theta}{r} \sum_{n=1}^{\infty} \hat{F}_{n,0}(s) z^n - \frac{\mu}{r} \hat{F}_{1,1}(s).
$$

At $z = 1$, and after simplification, we obtain

$$
\sum_{n=1}^{\infty} \hat{F}_{n,1}(s) = \frac{\theta}{sr} \sum_{n=1}^{\infty} \hat{F}_{n,0}(s) - \mu \hat{F}_{1,1}(s).
$$

Rewriting equation (5.24) as

$$
\hat{F}_{1,1}(s) = \omega(s) \hat{F}_{0,0}(s), \quad (5.30)
$$

where

$$
\omega(s) = \frac{\theta}{\mu} \sum_{m=1}^{\infty} \left( \frac{\lambda}{s + \frac{\lambda}{r} + \frac{\theta}{r}} \right)^m \left( p_1 - \sqrt{p_1^2 - \alpha^2} \right)^m,
$$

we get

$$
\sum_{n=1}^{\infty} \hat{F}_{n,1}(s) = \left[ \frac{\theta f_1(s)}{\mu_v - f_1(s)} - \mu \omega(s) \right] \hat{F}_{0,0}(s). \quad (5.31)
$$

Substituting the equation (5.31) in equation (5.29) leads to

$$
\hat{F}(s) = \frac{\mu_v}{\mu_v - f_1(s)} \hat{F}_{0,0}(s) + \frac{\hat{F}_{0,0}(s)}{sr} \left[ \frac{\theta f_1(s)}{\mu_v - f_1(s)} - \mu \omega(s) \right]
$$
and hence
\[ \hat{F}(s) = \frac{\hat{F}_{0,0}(s)}{sr(\mu_v - f_1(s))} \left[ sr\mu_v + \Theta f_1(s) - \mu_o(s)(\mu_v - f_1(s)) \right]. \] (5.32)

Using the equation (5.30) and equation (5.14), rewriting the equation (5.23) as
\[ (sr_0 + \lambda)\hat{F}_{0,0}(s) = ar_0 + \mu_o(s)\hat{F}_{0,0}(s) + f_1(s)\hat{F}_{0,0}(s), \]
leads to
\[ \hat{F}_{0,0}(s) = \frac{ar_0}{sr_0 + \lambda - \mu_o(s) - f_1(s)}. \] (5.33)

Substituting the equation (5.33) in equation (5.32) gives
\[ \hat{F}(s) = \left( \frac{ar_0}{sr} \right) \frac{[sr\mu_v + \Theta f_1(s) - \mu_o(s)(\mu_v - f_1(s))]}{\left[ \mu_v - f_1(s) \right] \left[ sr_0 + \lambda - \mu_o(s) - f_1(s) \right]^r}. \] (5.34)

Let \( F^\prime(s) \) denote the Laplace-stieltjes transform of the buffer content distribution, which is defined as
\[ F^\prime(s) = \int_0^\infty e^{-sx} dF(x) = s\hat{F}(s). \]

From equation (5.34), we obtain
\[ F^\prime(s) = \left( \frac{ar_0}{r} \right) \frac{[sr\mu_v + \Theta f_1(s) - \mu_o(s)(\mu_v - f_1(s))]}{\left[ \mu_v - f_1(s) \right] \left[ sr_0 + \lambda - \mu_o(s) - f_1(s) \right]^r}. \] (5.35)

Note that the mean buffer content is given by
\[ E(C) = \left. \frac{dF^\prime(s)}{ds} \right|_{s=0}. \]

Therefore, taking the derivatives on both sides of equation (5.35) with respect
to $s$, letting $s$ to zero yields

$$E(C) = \frac{ar_0}{r} \frac{1}{f_1(0)\mu_v - f_1(0)} \left[ -r\mu_v - \theta \frac{df_1(s)}{ds} \bigg|_{s=0} + r_0\mu_v - \mu_v \frac{df_1(s)}{ds} \bigg|_{s=0} ight] - r_0f_1(0) + 2f_1(0) \frac{df_1(s)}{ds} \bigg|_{s=0} - \lambda \frac{df_1(s)}{ds} \bigg|_{s=0}.$$ (5.36)

Consider

$$f_1(s) = \frac{p_2 - \sqrt{p_2^2 - \alpha_v^2}}{2}.$$ 

With $p_2 = s + \frac{\lambda + \mu_v + \theta}{r}$ and $\alpha_v = \frac{2\sqrt{\lambda\mu_v}}{r}$, we obtain

$$f_1(s) = \frac{rs + \lambda + \mu_v + \theta - \sqrt{(rs + \lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}}{2}.$$ 

At $s = 0$, we get

$$f_1(0) = \frac{\lambda + \mu_v + \theta - \sqrt{(\lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}}{2} = \lambda\tau_1.$$ (5.37)

and differentiating $f_1(s)$ with respect to $s$ yields

$$\frac{df_1(s)}{ds} = \frac{r}{2} \left[ 1 - \frac{rs + \lambda + \mu_v + \theta}{\sqrt{(rs + \lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}} \right].$$

At $s = 0$, we obtain

$$\frac{df_1(s)}{ds} \bigg|_{s=0} = \frac{r}{2} \left[ 1 - \frac{\lambda + \mu_v + \theta}{\sqrt{(\lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}} \right],$$

$$= \frac{r}{2} \left[ 1 - \frac{\tau_1 + \tau_2}{\tau_2 - \tau_1} \right].$$

Hence,

$$\frac{df_1(s)}{ds} \bigg|_{s=0} = \frac{r\tau_1}{(\tau_1 - \tau_2)}.$$ (5.38)
Substituting equation (5.37) and equation (5.38) in equation (5.36) leads to

\[ E(C) = \frac{ar_0}{r} \frac{1}{\lambda \tau_1 (\mu_v - \lambda \tau_1)} \left[ -r \mu_v + r_0 \mu_v - r_0 \lambda \tau_1 + (2 \lambda \tau_1 - \lambda - \theta - \mu_v) \frac{r \tau_1}{(\tau_1 - \tau_2)} \right]. \]

Therefore,

\[ E(C) = \frac{ar_0}{\lambda r \tau_1 (\lambda \tau_1 - \mu_v)} \left[ \mu_v (r - r_0) + r_0 \lambda \tau_1 + (\lambda + \theta + \mu_v - 2 \lambda \tau_1) \frac{r \tau_1}{(\tau_1 - \tau_2)} \right]. \]  

(5.39)

5.5 OBSERVATIONS

Remark 1: When the working vacation parameter \( \mu_v = 0 \), the model under consideration reduces to a fluid queue driven by an \( M/M/1 \) queue subject to multiple exponential vacation. Under this assumption, equation (5.26) becomes

\[ \hat{F}_{0,0}(s) = a \sum_{j=0}^{\infty} \frac{\theta^j}{r_0^j (s + \frac{\lambda}{r})^{j+1}} \left[ \sum_{m=0}^{\infty} \left( \frac{1}{s + \frac{\lambda}{r} + \frac{\theta}{r}} \right)^{m+1} \left( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{2} \right)^m \right]^j. \]

The above result is seen to coincide with that of Sherif I.Ammar (2014).

Remark 2: Let \( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{2} = \frac{\lambda z_0(s)}{r} \), where \( z_0(s) = \frac{sr + \lambda + \mu - \sqrt{(sr + \lambda + \mu)^2 - 4kr^2}}{2r} \), then \( \hat{F}_{0,0}(s) \) can be rewritten as

\[ \hat{F}_{0,0}(s) = a \sum_{j=0}^{\infty} \frac{\theta^j}{r_0^j (s + \frac{\lambda}{r})^{j+1}} \left[ \sum_{m=0}^{\infty} \left( \frac{r}{sr + \lambda + \theta} \right)^{m+1} \left( \frac{\lambda z_0(s)}{r} \right)^m \right]^j, \]

\[ = \frac{ar_0}{(sr_0 + \lambda)} \sum_{j=0}^{\infty} \frac{\theta^j (z_0(s))^j (sr + \lambda)^{j+1} (sr + \lambda + \theta - \lambda z_0(s))^{j}}{(sr_0 + \lambda)^j [sr + \lambda + \theta - \lambda z_0(s)]^j}. \]
Therefore,

$$
\hat{F}_{0,0}(s) = \frac{ar_0}{(sr_0 + \lambda)} \left[ \frac{1}{1 - \frac{\theta \lambda z_0(s)}{(sr_0 + \lambda)(sr_0 + \lambda(1 - z_0(s)) + \theta)}} \right],
$$

and hence

$$
\hat{F}_{0,0}(s) = \frac{ar_0 \left[ sr + \lambda(1 - z_0(s)) + \Theta \right]}{(sr_0 + \lambda)(sr_0 + \lambda(1 - z_0(s)) + \Theta) - \lambda \theta z_0(s)}.
$$

The above result coincides with equation (16) of Wang et al (2010). Further, with \( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{2} = \frac{\lambda z_0(s)}{\mu} \), \( \omega(s) \) becomes

$$
\omega(s) = \frac{\theta \lambda z_0(s)}{\mu \left[ rs + \lambda(1 - z_0(s)) + \Theta \right]}. 
$$

When \( \mu_v = 0 \), \( f_1(s) \) vanishes and applying L’Hospital rule to \( \lim_{\mu_v \to 0} \hat{F}(s) \) leads to

$$
\hat{f}(s) = \frac{ar_0}{sr} \left[ sr + \frac{\lambda \theta}{sr + \lambda + \Theta} - \frac{\theta \lambda z_0(s)}{(sr + \lambda + \Theta)(sr + \lambda(1 - z_0(s)) + \Theta)} \right],
$$

$$
= \frac{ar_0}{sr} \left[ sr + \lambda - \frac{\theta \lambda z_0(s)}{(sr + \lambda(1 - z_0(s)) + \Theta)} \right],
$$

$$
= \frac{ar_0}{sr} \left( sr + \lambda \right) \left[ rs + \lambda(1 - z_0(s)) + \Theta - \theta \lambda z_0(s) \right],
$$

The above result coincides with equation (17) of Wang et al (2010).

**Remark 3:** When \( \mu_v = 0 \), the model also reduces to fluid queue driven by an \( M/M/1 \) with multiple exponential vacation and \( N \)-policy with \( N = 1 \). With \( \frac{p_1 - \sqrt{p_1^2 - \alpha^2}}{2} = \frac{\mu r_0(s)}{r} \), where \( r_0(s) = \frac{\lambda + \mu - \sqrt{(\lambda + \mu)^2 - 4\lambda \mu}}{2\mu} \).
equation (5.40) leads to

\[
\hat{F}_{0,0}(s) = a \sum_{j=0}^{\infty} r_0^j \left( \sum_{m=0}^{\infty} \left( \frac{r}{sr + \lambda + \theta} \right)^{m+1} \left( \frac{\mu r_0(s)}{r} \right)^{m+1} \right)^j,
\]

\[
= \frac{ar_0}{(sr_0 + \lambda)} \sum_{j=0}^{\infty} \frac{\theta^j \lambda^j (r_0(s))^j}{(sr_0 + \lambda)^j [sr + \lambda + \theta - \mu r_0(s)]^j},
\]

\[
= \frac{ar_0}{(sr_0 + \lambda)} \left[ \frac{1}{1 - \frac{\theta r_0(s)}{(sr_0 + \lambda)[sr + \lambda + \theta - \mu r_0(s)]}} \right].
\]

Hence

\[
\hat{F}_{0,0}(s) = \frac{ar_0 [sr + \lambda + \theta - \mu r_0(s)]}{(sr_0 + \lambda)[sr + \lambda + \theta - \mu r_0(s)] - \mu \theta r_0(s)}.
\]

It is observed that with \( r_0 = \sigma_0 \) and \( r = \sigma \), the above result coincides with \( \hat{F}_0(s) \) presented in Remark 1 of page 128 in Mao et al (2012).

Further, with \( \rho_i - \sqrt{\rho_i^2 - \sigma_i^2} = \frac{\mu r_0(s)}{r} \), \( \omega(s) \) becomes

\[
\omega(s) = \frac{\theta r_0(s)}{\mu [rs + \lambda + \theta - \mu r_0(s)]}.
\]

When \( \mu_I = 0 \), \( f_1(s) \) vanishes and applying L’Hospital rule to \( \lim_{\mu_I \to 0} \hat{F}(s) \) leads to

\[
\hat{F}(s) = \frac{ar_0}{sr} \left[ \frac{sr + \lambda + \theta - \frac{\theta r_0(s)}{[rs + \lambda + \theta - \mu r_0(s)]}}{sr_0 + \lambda - \frac{\theta r_0(s)}{[rs + \lambda + \theta - \mu r_0(s)]}} \right],
\]

\[
= \frac{ar_0}{sr} \left[ \frac{sr + \lambda - \frac{\theta r_0(s)}{[rs + \lambda + \theta - \mu r_0(s)]}}{sr_0 + \lambda - \frac{\theta r_0(s)}{[rs + \lambda + \theta - \mu r_0(s)]}} \right].
\]
and hence

\[ \hat{F}(s) = \frac{ar_0 (sr + \lambda) [rs + \lambda + \theta - \mu r_0(s)] - \theta \lambda r_0(s)}{sr (sr_0 + \lambda) [rs + \lambda + \theta - \mu r_0(s)] - \theta \lambda r_0(s)}. \]

The above result coincides with \( \hat{F}(s) \) presented in equation (13) with \( N = 1 \) in Mao et al (2012).

### 5.6 NUMERICAL ILLUSTRATIONS

This section illustrates the variations of joint steady state probabilities of the buffer content and the state of the background queueing model for varying values of the parameters.

Figure 5.2 depicts the variations of \( F_{n,0}(x) \) for \( n = 0, 1, 2, 3 \) against the buffer content, \( x \) for \( \lambda = 1, \mu = 2, \mu_v = 1.1, \ r_0 = -2.5, \ r = 1 \) and \( \theta = 0.5 \). Observe that both the stability conditions, \( \lambda < \mu \) and \( d = -0.2310 < 0 \) are satisfied for this choice of the parameter values. As discussed earlier, the boundary conditions are given by \( F_{0,0}(0) = a = 0.0925 \) and \( F_{n,0}(0) = 0 \). Further as \( x \) tends to infinity, \( F_{n,0}(x) \) will converge to the corresponding steady state probabilities of the background queueing model, \( \pi_{n,0} \) (given by equation (5.1)). Therefore \( F_{0,0}(x) \) starts with \( a = 0.0925 \) and converge to \( \pi_{0,0} = 0.3518 \) (from equation (5.3)) as \( x \) increases. Similarly \( F_{n,0}(x) \) \( (n = 1, 2, 3) \) starts with zero, increases with increase in \( x \) and converges to \( \pi_{n,0} \) as \( x \) tends to infinity. The values of \( \pi_{n,0} \) \( (n = 1, 2, 3) \) for different \( 'n' \) are depicted in the graph as converging values.

Figure 5.3 presents the corresponding behavior of \( F_{n,1}(x) \) against \( x \) for the same set of parameter values. As before, \( F_{n,1}(x) \) for \( n = 1, 2, 3, 4 \), increases with increase in \( x \) and tends to the value of \( \pi_{n,1} \) (given by equation (5.2)) as \( x \) tends to infinity. The values of \( \pi_{n,1} \) for different \( 'n' \) are
depicted in the graph as converging values.

Figure 5.4 and 5.5 depicts the behavior of the buffer content distribution, $F(x)$ against $x$ for the same set of parameters and varying values of $\theta$ and $(r_0, r)$ respectively. It is seen that $F(x)$ increases with increase in the value of the parameter $\theta$ and converges to 1 as $x$ tends to infinity. Also, observe that $F(x)$ increases with decrease in the value of $r_0$ and while simultaneously increasing the value of $r$ in order to satisfy the stability condition.

Figure 5.6 depicts the behavior of the mean buffer content, $E(C)$ against $\frac{\lambda}{\mu}$ for $\mu_v = 0.05$ $r_0 = -2$, $r = 1$ and varying values of $\theta$. It is seen that $E(C)$ increases with decrease in the value of the parameter $\theta$. Also for a specific choice of the parameter $\theta$, $E(C)$ is seen to increase with increase $\frac{\lambda}{\mu}$.

Thus all the joint state probabilities and related performance measures are explicitly obtained under steady state and their corresponding behaviour is illustrated numerically for varying values of the parameters.
Figure 5.2: Variations of $F_{n,0}(x)$ against $x$ for $n = 0$, $n = 1$, $n = 2$ and $n = 3$. 
Figure 5.3: Variations of $F_{n,1}(x)$ against $x$ for $n = 1, n = 2, n = 3$ and $n = 4$
Figure 5.4: Variations of the buffer content distribution against $x$ for different values of $\theta$
Figure 5.5: Variations of the buffer content distribution against $x$ for different pairs of $r_0$ and $r$
Figure 5.6: Mean buffer content against $\frac{\lambda}{\mu}$ for different values of $\theta$