A Scaling Theory for Transport Across Dirty Luttinger Liquids

By considering a network of dirty Luttinger liquid wires and following Midgal-Kadanoff type bond moving procedure we derive the renormalization group equations for the characteristic function of the full probability distribution of conductance of two and three dimensional Luttinger liquid networks. In two dimension we get a new phase with the average conductance independent of sample size (L). The second cumulant $a_2$ has $1/L$ dependance and saturates to a disorder dependant constant $\gamma$. At unit length scale $a_2$ becomes sample size independent. In three dimension we get two different phases one with $L$ independent average and another with second cumulant decreasing as $1/L$ and saturating to a non zero value. The second phase has an average conductance which for large $L$ decreases as $L^{-2}$ and the second cumulant for large $L$ increases as $L$. All the other higher cumulants in both dimensions exponentially decrease with $L$ [1].
In one or two-dimensional metals the presence of disorder entails a localization of all electronic states and therefore a vanishing static conductivity at zero temperature \([2,3]\). This has been clearly demonstrated by exact calculations for one-dimensional metals \([4,5,6]\). A factor which changes this behaviour qualitatively is the interaction among electrons \([7,8,9,10]\). For sufficiently attractive interactions there is competition between superconducting fluctuations and disorder which can lead to delocalization. Electrons in one dimension have novel features compared to higher dimensions and form a new phase characterized by Luttinger liquid \([11]\). The edge states in quantum Hall effect, quasi one-dimensional wires, carbon nanotubes etc. form experimental realization of Luttinger liquid. P.W. Anderson has related the higher temperature superconductivity to the property of Luttinger liquid in one dimension \([12]\).

A renormalization approach was developed by Giamarchi and Schultz \([13]\) to study the localization/delocalization transition exhibited by a one-dimensional interacting electron gas in a random potential. They obtained the phase diagram and the exponents of the correlation functions in the delocalized regime. In this chapter in contrast to the work in \([13]\) we explore the possible aspects of Luttinger liquid behavior carrying over from one dimension to higher dimensions of a model system, by studying the scaling theory of the transport coefficients, in a different approach as used previously \([14]\). Our model is a ‘d’ dimensional substance idealized as a ‘d’ dimensional cubic lattice system with electrons being transported at the edges of the cubic cells.
This may be approximately realized by forming a cubic network with quasi one-dimensional wires. For example with gold wires where Luttinger liquid behaviour is suspected [15]. For weak Coulomb interaction the crossing of Luttinger liquid wires does not affect the transport properties as given by individual wires [16,17].

In section 1 we consider the one-dimensional dirty Luttinger liquid with a random potential present and derive the probability distribution for the conductance and its moments. In section 2 we generalize to ‘d’ dimensional case, calculate the cumulants in two and three dimensions and find the fixed point for the beta function for the conductance in two and three dimensions. In section 3 we find the cumulants of resistance and get the equation of its beta function. The zero of the beta function is solved for two dimensions and is shown to give rise to a new fixed point consistent with the previous section. The Luttinger liquid theory has strong electron-electron interactions incorporated into it. In the presence of strong electron-electron interactions localization effects are suppressed [18]. So we do not have Anderson localization [19] effects prevailing here.

5.1 Luttinger Liquid in one Dimension

We consider an infinitely long Luttinger liquid separated into three parts. A region of length \( L \) being the wire and the left and right parts of the wire being the leads. An applied electric field \( E \) is assumed to be zero outside the wire. The wire is considered to be weakly disordered, modeled by a random
potential \( V(x) \). There is no disorder in the leads. The current in the leads is
\[
i = \delta \rho v_L,
\]
where \( \delta \rho = -\frac{\partial \phi}{2\pi} \) [20]. We take the effective conductance as
\[
G = \frac{2\pi i}{v_L E} = -\frac{\partial \phi}{E} v_L \quad \text{and} \quad K_L \quad \text{are the velocity and the interaction dependant parameter in the leads characterizing the Luttinger liquid.}
\]
The action of a spinless Luttinger liquid is,
\[
S = \int_0^\beta dx \tau \left\{ \frac{1}{2K(x)} \left( \frac{v(x)}{K(x)} \partial_x \phi \right)^2 + V(x)(\partial_x \phi)^2 \right\} + \frac{2}{a} \int dx \int_0^\beta d\tau V(x) \cos(2K_F x + 2\sqrt{\pi} \phi) \tag{5.1}
\]
Assuming there is no \( \tau \) dependence for \( \phi \), \( \partial_x \phi = I, V(x) \sin(2K_F x + 2\sqrt{\pi} \phi) = U \) and assuming \( v(x) \) and \( K(x) \) to be constants, Eq. (5.2) becomes,
\[
\frac{v}{K} \partial_x G - \frac{4\sqrt{\pi}}{a} U(L) = \frac{eE}{2a} \tag{5.3}
\]
Assuming \( U(L) \) is gaussian random function with mean zero and
\[
\langle U(L)U(L') \rangle = \delta(L - L')U_o^2 \tag{5.4}
\]
the Fokker-Planck equation for the probability distribution of \( G \), \( W_G \) is
The $n^{th}$ moment of $G_n = \int W G^n dG$ satisfies the equation,

$$\frac{\partial G_n}{\partial L} = \alpha^2 U_0^2 n(n-1)G_{n-2} + \varepsilon nG_{n-1}$$

(5.7)

Taking $G_0=1$, $G_1$ and $G_2$ can be found as

$$G_1 = \varepsilon L$$
$$G_2 = \alpha^2 U_0^2 L + \varepsilon^2 L^2$$

(5.8)

All the moments can be explicitly calculated. The mean current is that of an ordinary Luttinger liquid.

The characteristic function $\chi_G$ is defined as

$$\chi_G(x, L) = \int W_G(x, L) dG$$

(5.9)

Using (5.5) $\chi_G$ satisfies the equation

$$\frac{\partial \chi_G(x, L)}{\partial L} = \chi_G(\alpha^2 U_0^2 x^2 - \varepsilon x),$$

(5.10)

whose solution is

$$\chi_G(x, L) = e^{(\alpha^2 U_0^2 x^2 - \varepsilon x)L},$$

(5.11)

The moment generating function $\ln \chi_G = K_G$ satisfies the equation,
Thus the dirty Luttinger liquid in one dimension has all the cumulants except the first two being non zero; and the probability distribution for conductance in one dimension is Gaussian with mean \( g_2^{(i)} = \varepsilon L \) and variance \( a_2^{(i)} = \alpha^2 U_0^2 L \).

### 5.2 d-Dimensional Luttinger Liquid

Now we study d-dimensional dirty Luttinger liquid using the same method as in [14]. Consider a d-dimensional hypercubic lattice with 2d incoming and outgoing channels. We partition it into hypercubic blocks of edge 'b'. The edges of the cubes are considered to be Luttinger liquid wires. These wires cross at the vertices of the hypercubic blocks. For weak coulomb interaction in the wires individual Luttinger liquid behavior is not changed [16]. Following Migdal-Kadanoff procedure [21] we single out arbitrarily the direction of current flow and cut the bonds in the d-1 transverse directions. Thus each block now consists of \( b^{d-1} \) chains in parallel. For a given realisation of randomness each chain has different current in general.

Defining the characteristic function for \( G \) in d-dimension as

\[
\chi_G^{(d)}(x, L) = \int e^{-xt^{d-1}} G^{(i)} W_G^{(d)}(J, L) dG ,
\]

we get using the same arguments as in [14].

\[
\frac{\partial K_G(x, L)}{\partial L} = \alpha^2 U_0^2 x^2 - \varepsilon x
\]

(5.12)

\[
K_G(x, L) = (\alpha^2 U_0^2 x - \varepsilon x)L
\]

(5.13)
The equation for the cumulant generating function $K_G^{(d)}(x,L)$ is

$$L \frac{\partial K_G^{(d)}(x,L)}{\partial L} = (\alpha^2 U_0^2 x^2 L^{2(d-1)} - \epsilon L^{d-1}) K_G^{(d)} + (d-1) x \frac{\partial K_G^{(d)}}{\partial x} \quad (5.15)$$

The equation for the cumulant generating function $K_G^{(d)}(x,L)$ is

$$L \frac{\partial K_G^{(d)}(x,L)}{\partial L} = (d-1) x \frac{\partial K_G^{(d)}}{\partial x} + L(\alpha^2 U_0^2 x^2 L^{2(d-1)} - \epsilon x L^{d-1}) \quad (5.16)$$

From Eq.(16) we get the equation for the cumulants as

$$\frac{\partial \ln g^{(d)}}{\partial \ln L} = (1-d) + \epsilon^{(1-d)}[g^{(d)}]^{d-1}, \quad (5.17)$$

$$\frac{\partial \ln a_2^{(d)}}{\partial \ln L} = (1-d) + \frac{\alpha^2 U_0^2 (g^{(d)})^{2d-1}}{a_2^{(d)}}, \quad (5.18)$$

$$\frac{\partial a_n^{(d)}}{\partial \ln L} = (1-d) n a_n \quad \text{for } n \geq 3. \quad (5.19)$$

The $\beta$ function $\left( \frac{\partial \ln g^{(d)}}{\partial \ln L} \right)$ has an $L$ dependance on the R.H.S which we remove by noting that in one dimension $L = \frac{g^{(l)}}{\epsilon}$. Assuming the $L$ independent form of $\beta$ function is the same in all dimensions we substitute $L = \frac{g^{(d)}}{\epsilon}$. The $\beta$ function has a fixed point (zero of the $\beta$ function) given by

$$a_1^{(d)} = (d-1)^{d-1} \frac{1}{\epsilon}.$$

In two and three dimensions the fixed points of the $\beta$ function for conductance is given by $\epsilon$ and $\sqrt{2\epsilon}$ respectively, indicating metallic behaviour. The $d$-dimensional Luttinger liquid has a fixed point
having non zero mean conductance, variance and with higher cumulants zero. This shows that the higher dimensional Luttinger liquid has a new phase whose conductance is independent of the sample length and depends only on the applied field and the amount of disorder.

In two dimension we have average conductance independent of sample size, depending only on applied field and interactions within the Luttinger liquid. The exact solution of Eq.18 for the second cumulant in two dimension is given by,

\[ a_{2}^{(2)} = \gamma - \frac{L_0}{L} \left( \gamma - a_{20}^{(2)} \right) \]  \hspace{1cm} (5.20)

\( a_{20} \) is the value of cumulant at \( L = L_0 \) and is arbitrary. In this case \( \gamma (=\alpha^2 \nu_0^2) \) is a fixed point (length independent) for the second cumulant. Thus if the initial value of the second cumulant at the unit length scale has value \( \gamma \) the second cumulant becomes independent of length, otherwise it saturates to the fixed point \( \gamma \).

In three dimensions Eq.(17) has two solutions given by

\[ g^{(3)}(L) = \sqrt{2\epsilon} \]  \hspace{1cm} (5.21)

or

\[ g^{(3)}(L) = \frac{\sqrt{2\epsilon}}{\sqrt{1 + \left( \frac{L}{L_0} \right)^4}} \]  \hspace{1cm} (5.22)
The first solution denotes conducting phase with sample size independent conductance while the second is that of an insulating phase which for large lengths tends to a zero conducting resistor.

The second cumulant \( a_2^{(3)} \) in three dimension has two solutions depending on \( g^{(3)}(L) \). For sample size independent \( g^{(3)}(L) = \sqrt{2} \epsilon \)

\[
a_2^{(3)}(L) = 2^{(3/2)} \gamma \left[ 1 - \left( \frac{L_0}{L} \right)^2 \left[ 1 - \frac{a_0^{(3)}}{2^{(3/2)} \gamma} \right] \right] \tag{5.23}
\]

and for sample size dependent average conductance given by Eq.(22)

\[
a_2^{(3)}(L) = 2^{(3/2)} \gamma \mu_0 \left[ \frac{2}{3} \left( \frac{L}{L_0} \right)^2 \left[ 3 + 2 \left( \frac{L}{L_0} \right)^2 \right] \right] \tag{5.24}
\]

The higher cumulants \((n \geq 3)\) in d-dimensions have exponential decay with respective to length.

\[
a_n^{(d)} = a_n^{(d)} \text{Exp}[-(d-1)nL] \tag{5.25}
\]

If the initial value of the cumulants are taken to be zero then they remain zero for arbitrary length.

In two dimension we see that the Luttinger liquid has a phase with sample size independent average conductance. The second cumulant decreases to zero as \( L^{-2} \), but if the Luttinger liquid at the unit length scale has the second cumulant \( \gamma \) then \( a_2 \) becomes independent of sample size. If we consider the two
dimensional Luttinger liquid as constructed from one dimensional Luttinger liquid and take as the initial value the value of higher cumulants of one dimensional Luttinger liquid then all the higher cumulants vanish.

In three dimensions the Luttinger liquid has two different phases. One phase has sample size independent conductance and second cumulant saturating to a constant. The higher cumulants can be considered to be zero or exponentially decreasing. The second phase has a sample size dependent conductance vanishing for large $L$. The higher cumulant behavior is the same in both.

5.3 Distribution of Resistance for $d$-Dimensional Luttinger Liquid

In order to find out the distribution of the resistance $R = G^{-1}$ we repeat the same procedure as for $G$. The Fokker-Planck equation for the probability distribution for $R$, $W_R$ is

$$\frac{\partial W_R}{\partial L} = \frac{\partial}{\partial R} \left[ R^2 \left\{ \alpha^2 U_0^2 R^2 \frac{\partial W_R}{\partial R} + \left( 2\alpha^2 U_0^2 R + \varepsilon \right) W_R \right\} \right]$$

(5.26)

The equation for $K_R = \ln \chi_R$ is
where $\beta = \alpha^2 U_0^2$. Substituting $K = \sum a_n x^n$, the coefficients of $x$ and $x^2$ satisfy the equations

$$\frac{\partial a_1}{\partial L} = \varepsilon a_1^2 + 2 \beta a_1^3 + 2(\varepsilon + 6 \beta a_2) a_2 + 12 \beta a_3$$

$$\frac{\partial a_2}{\partial L} = \beta a_1^4 + 4 \varepsilon a_1 a_2 + 12 \beta a_1^2 a_2 + 2 \alpha (6 \beta a_1^2 + 18 \beta a_2)$$

$$+ 24 \beta a_1 a_3 + 6(\varepsilon + 6 \beta a_2) a_3 + 72 \beta a_4$$

The above two equations involve higher moments and are difficult to solve.

The equation for $K_{R}^{(d)}$ can be derived from Eq. 24 as,

$$\frac{\partial K_{R}^{(d)}}{\partial L} = \left[ \frac{\partial^4 K_{R}^{(d)}}{\partial x^4} + 4 \frac{\partial^3 K_{R}^{(d)}}{\partial x^3} \frac{\partial K_{R}^{(d)}}{\partial x} + 3 \left( \frac{\partial^2 K_{R}^{(d)}}{\partial x^2} \right)^2 + 6 \frac{\partial K_{R}^{(d)}}{\partial x} \left( \frac{\partial K_{R}^{(d)}}{\partial x} \right)^2 \right]$$

$$+ \left( \frac{\partial K_{R}^{(d)}}{\partial x} \right)^4$$

$$\frac{2x \beta}{L^{2(1-d)}} \left[ \frac{\partial^3 K_{R}^{(d)}}{\partial x^3} + \frac{\partial^2 K_{R}^{(d)}}{\partial x^2} \left( \frac{\partial K_{R}^{(d)}}{\partial x} \right) + \left( \frac{\partial K_{R}^{(d)}}{\partial x} \right)^2 \right]$$

$$+ \frac{2 \partial K_{R}^{(d)}}{\partial x} \frac{\partial^2 K_{R}^{(d)}}{\partial x^2} \right]$$

$$+ \frac{\varepsilon x}{L^{(1-d)}} \left[ \frac{\partial^2 K_{R}^{(d)}}{\partial x^2} + \left( \frac{\partial K_{R}^{(d)}}{\partial x} \right)^2 \right] + \frac{x(1-d) \partial K_{R}^{(d)}}{L} \frac{\partial K_{R}^{(d)}}{\partial x}$$

(5.30)
Expanding \( K_R^{(d)} = \sum_{n=0}^{\infty} b_n x^n \) and substituting in the above equation we get the equations for \( b_1 \) and \( b_2 \) as

\[
\frac{db_1}{dL} = L^{2(d-1)} \left[ 2\beta \left( b_1^3 + 6b_1 b_2 + 6b_3 \right) \right] + \varepsilon L^{(d-1)} \left( b_1^2 + 2b_2 \right) + (1-d)L^{-1} b_1 \tag{5.31}
\]

\[
\frac{db_2}{dL} = L^{2(d-1)} \left[ \beta \left( b_1^4 + 12b_1^2 b_2 + 12b_2^2 + 24b_1 b_3 + 24b_4 \right) + \right]
\]

\[
\left[ 2\beta \left( 6b_1^2 b_2 + 4b_2^3 + 6b_1 b_3 \right) + 2 \left( 4b_2^2 + 6b_3 b_2 + 24b_4 \right) \right]
\tag{5.32}
\]

From Eq. 5.28 the \( \beta \) function can be written as

\[
\frac{\partial \ln b_1}{\partial \ln L} = (1-d) + \varepsilon L^d \left( b_1 + 2 \frac{b_2}{b_1} \right) + 2\beta L^{2(d-1)} \left( b_1^2 + 6b_2 + 6 \frac{b_3}{b_1} \right) \tag{5.33}
\]

The \( L \) dependance in \( \beta \) function can be removed by substituting for \( L = \frac{-a_1}{\varepsilon} \)

to get

\[
\frac{\partial \ln b_1}{\partial \ln L} = (1-d) + (-a_1)^d \varepsilon^{(1-d)} \left( b_1 + 2 \frac{b_2}{b_1} \right) + 2\beta \varepsilon(1-2d)(-a_1)^{2d-1} \left( b_1^2 + 6b_2 + 6 \frac{b_3}{b_1} \right) \tag{5.34}
\]

For \( d=2 \)

\[
\frac{\partial \ln b_1}{\partial L} = -1 + a_1^2 \varepsilon^{-1} \left( b_1 + 2 \frac{b_2}{b_1} \right) + 2\beta \varepsilon^{-3} \left( b_1^2 + 6b_2 + 6 \frac{b_3}{b_1} \right) a_1^3 \tag{5.35}
\]
To solve Eq.5.33 let us assume \( n_1 \approx \frac{1}{a_i} \) substituting \( \frac{1}{b_i} \) for \( a_i \), Eq.5.33 becomes,

\[
\frac{\partial \ln b_1}{\partial L} = -1 + \epsilon^{-1} \left( \frac{1}{b_1} + \frac{2b_2}{b_1^3} \right) - 2\beta \epsilon^{-3} \left( \frac{1}{b_1} + \frac{6b_2}{b_1^3} + \frac{6b_3}{b_1^4} \right)
\]

(5.36)

\[
\frac{\partial \ln b_1}{\partial L} = -1 + \frac{1}{b_1 \epsilon} \left( 1 - \frac{2\beta}{\epsilon^2} \right) + \frac{2b_2}{b_1} \epsilon \left( 1 - \frac{6\beta}{\epsilon^2} \right) - \frac{12\beta b_3}{\epsilon^3 b_1^4}
\]

(5.37)

To find the fixed points of the \( \beta \) function we put r.h.s of Eq.5.37 to zero, simplifying the resulting equation by setting \( b_3 \) and \( b_2 = 0 \) and set for \( b_1 \),

\[
b_1 = \frac{1}{\epsilon} \left( 1 - \frac{2\beta}{\epsilon^2} \right)
\]

(5.38)

From this we see that \( b_1 = \frac{1}{\epsilon} \left( 1 - \frac{2\beta}{\epsilon^2} \right), \ b_2 = 0, \ b_3 = 0 \) etc. is a consistent fixed point for the \( \beta \) function.

Thus in two dimension Luttinger liquid has a new phase whose effective resistance \( R \) has a finite mean and zero fluctuations. So \( G \) and \( R \) distributions give consistent fixed point for the \( \beta \) function.

### 5.4 Results and Discussion

We have studied dirty Luttinger liquid in one, two and three dimensions using scaling theory. In one dimension the dirty Luttinger liquid has a Gaussian probability distribution for its conductance. In two dimension the Luttinger
liquid has a phase with sample size independent average conductance and the second cumulant being a constant or vanishing as $L^{-2}$. In three dimension we find the Luttinger liquid has two phases. One phase has sample size independent conductance and the other phase with conductance decreasing as $L^{-2}$. The second cumulant decreases as $L^{-2}$. All higher cumulants in both dimensions are zero or exponentially decay. Thus Luttinger liquid gives new transport behaviour in two and three dimensions. The insulating type phase in three dimensions may be relevant to the normal state transport of high temperature superconductors. The sample size independent conductance is reminiscent of a super conductor but the $L^{-2}$ of the second cumulant characterises it as a different phase. The presence of two phases in three dimension, insulating and conducting implies the existance of a phase transition which only a more sophisticated treatment can unravel. We have approximated the effective non-linear potential to be Gaussian and averaged over it. In light of these our scaling theory does not reduce to the standard Fermi liquid theory in the ($k = 1$) limit. It may be possible to verify our Gaussian approximation by a monte-carlo simulation of the non-linear stochastic equation.
References