Effect of Variation of the Background Profile on the Reflection Statistics of Random Laser

The Fokker-Planck equation for the probability distribution of the reflection coefficient of the light incident on a one-dimensional active random medium in which the background profile varies with length is derived. The effective amplification of the system of finite length depends on the gradient of the variation of the background profile.

4.1 Introduction

Anderson localization [1] produces states characterized by an exponential decrease of the wave profile. But it was shown that [2,3,4,5] for electrons in a disordered media with a linear potential in the background the exponential nature of the localization changes to polynomial one. Thus the localized states get comparatively extended in the presence of a background potential. The transport properties of electron also change. The ensemble average resistances
and its moments change over from exponential length dependence to polynomial ones [6].

For photon transport in disordered media the variation of background profile with respect to sample length will thus change the exponential nature of localized waves to non-exponential ones. Now since a photon is localized in more region in an active media the probability of amplification/absorption will get modified. In order to investigate this situation we consider a one-dimensional random medium with a complex refractive index and with the real part having a non-stochastic spatially varying part \( b(L) \).

### 4.2 Model

We consider a one dimensional active disordered medium of length \( L \) with a random complex refractive index \( \eta \). By assuming light to be a scalar wave we have neglected the polarization effects. We also emphasize that our treatment is for the possibility of super reflection \( (r \| l) \) i.e., for an amplifier and not an oscillator. The complex wave amplitude \( E(x) \) obeys the Helmholtz equation,

\[
\frac{d}{dx} \left( \frac{E(x)}{i} \right) + k_0^2 \left[ 1 + \eta(x) \right] E(x) = 0, \tag{4.1}
\]

where \( k_0 \) is the wave vector in the medium with \( k_0^2 = \frac{\omega^2}{c^2} \varepsilon_0 \),

\[
\eta(x) = [\eta_1(x) + b(x)] + i[\eta_2(x) + a(x)]. \tag{4.2}
\]

\( \eta_1 \) and \( \eta_2 \) are Gaussian random functions with mean zero and variance given by...
\begin{equation}
\langle \eta_1(x) \eta_1(x') \rangle = g_1 \delta(x - x')
\end{equation}

and \( \langle \eta_2(x) \eta_2(x') \rangle = g_2 \delta(x - x') \)

\( b(x) \) is the background refractive index over which \( \eta_1(x) \) are the random fluctuations, \( a(x) \) is the background amplification (or absorption) over which \( \eta_2(x) \) is the random amplification (or absorption).

Using the invariant imbedding method [7,8] we transform Eq. 4.1 to give the equation for the complex reflection coefficient,

\[
\frac{dR(L)}{dL} = \left[ \frac{1}{i} \frac{k_1^2 - k_2^2}{2k_1} \right] R_1^2 - \left[ \frac{1}{i} \left( \frac{k_1^2 + k_2^2}{k_1} \right) + \frac{k_1^2}{k_0 + k_1} \right] R_1^2 + \left[ \frac{1}{i} \left( \frac{k_1^2 - k_2^2}{2k_1} \right) + \frac{k_1^2}{k_0 + k_1} \right]
\]\n
(4.4)

where \( k_1^2 = k_0^2 \left[ 1 + b(L) \right] \) and \( k_2^2 = k_0^2 \left[ 1 + \{ \eta_1 + b(L) \} + i \eta_2 + a(L) \right] \).

From the stochastic equation we derive the corresponding Fokker-Planck equation for the probability distribution \( W(r,L) \) for the reflection coefficient \( r \left( = |R|^2 \right) \) of sample length \( L \) using Novikov's[9] theorem, we get

\[
\frac{2(1 + b) \partial W}{k_0^2} = \left[ \left( 5r^2 - 6r + 1 \right) g_1 + \left( 5r^2 + 30r + 1 \right) g_2 \right] \frac{\partial^2 W}{\partial r^2} + \left( \left( 5r^2 - 6r + 1 \right) g_1 + \left( 5r^2 + 30r + 1 \right) g_2 \right) \frac{\partial W}{\partial r} + \left[ 2(2r - 1) g_1 + 2(2r + 5) g_2 \right] W + \left\{ \frac{4a \sqrt{1 + b}}{k_0} + \frac{2b'}{k_0 \left( 1 + \sqrt{1 + b} \right)} \right\} \frac{\partial (rW)}{\partial r}
\]\n
(4.3)
The above equation can be rewritten as

\[
\frac{2(1+b)}{k_0^2} \frac{\partial W}{\partial L} = L_1 W(r, L) + L_2 W(r, L) + \left[ \frac{4a\sqrt{1+b}}{k_0} + \frac{2b'}{k_0(1+\sqrt{1+b})} \right] \frac{\partial(rW)}{\partial r} \tag{4.4}
\]

where

\[
L_1 W(r, L) = g_1 \left[ (r^3 - 2r^2 + r) \frac{\partial^3 W}{\partial r^3} + (5r^2 - 6r + 1) \frac{\partial W}{\partial r} + 2(2r - 1)W \right] \quad \text{and}
\]

\[
L_2 W(r, L) = g_1 \left[ (r^3 + 10r^2 + r) \frac{\partial^3 W}{\partial r^3} + (5r^2 + 30r + 1) \frac{\partial W}{\partial r} + 2(2r + 5)W \right] \tag{4.5}
\]

We have not been able to solve analytically the second order partial differential equation. But we have analyzed the solution of this equation by considering the effect of interplay between the variation of the background profile and amplification/absorption parameter \(a\) (average of the imaginary part of the refractive index). The coefficient of \(\frac{\partial(rW)}{\partial r}\) in Eq.4.3 is equated to be a constant \(B\) and analyzed for the solution. For \(B\) constant the solutions have already been derived in Chapter 3. So we analyze whether any realistic positive profile will give such \(\varepsilon B\).

Putting

\[
\frac{4a\sqrt{1+b}}{k_0} + \frac{b'}{k_0(1+\sqrt{1+b})} = B \tag{4.6}
\]

for \(a = 0\), we get
\[ \frac{b'}{k_0[1 + \sqrt{1+b}]} = B, \]  
which can be integrated upon to get

\[ BL = 2[1 + \sqrt{1+b}] - 210[1 + \sqrt{1+b}] \]  
(4.7)

Since \( 1 + \sqrt{1+b} \) increase faster than \( \ln[1 + \sqrt{1+b}] \). The r.h.s of the above equation is always positive. Further \( L \) is positive and hence \( B \) has to be positive always. This clearly shows that even if \( L \) is varied within a finite region we do not get any possibility of amplification from the medium, instead we have absorption.

For large \( L \), \( b(L) \to B_0 \) (a constant), and let \( \frac{db}{dL} \to -B_1 \).

For consistency \( B_1 \) has to be zero and hence the \( b' \) term in l.h.s in Eq.4.6 goes to zero.

Thus we find that in the large \( L \) limit this term has no contribution to either absorption or amplification.

In the previous case we have excluded the presence of the amplifying/absorbing term \( a \). Now we include that also. Then

\[ \frac{4a\sqrt{1+b}}{k_0} + \frac{b'}{k_0[1 + \sqrt{1+b}]} = B \]

which integrated upon gives

\[ L = 2 \int \frac{udu}{-2au^2 + (\frac{k_0B}{2} - 2a)u + \frac{k_0B}{2}} \]
where \( u = \sqrt{1 + b} \)

Let
\[
-4ak_B - \left( \frac{k_B}{2^2} - 2a^2 \right)^2 = \Delta
\]

The above integral has three solutions depending upon the value of \( \Delta \).

For \( \Delta = 0 \) we have the solution as,
\[
L = \lambda + \frac{2a - k_B}{a \left( \frac{k_B}{2} - 2a - 4a\sqrt{1 + b} \right)}
\]

For \( \Delta < 0 \)
\[
L = \lambda + \frac{-2 \left( \frac{k_B}{2} - 2a \right)}{a \sqrt{4ak_B + \left( \frac{k_B}{2} - 2a \right)^2}} \frac{k_B}{2} - 2a - 4a\sqrt{1 + b} \]

and for \( \Delta > 0 \)
\[
L = \lambda + \frac{2 \left( \frac{k_B}{2} - 2a \right)}{a \sqrt{4ak_B + \left( \frac{k_B}{2} - 2a \right)^2}} \text{artg} \frac{k_B}{2} - 2a - 4a\sqrt{1 + b} \]

where
\[
\lambda = -\frac{1}{2a} \ln \left[ \frac{k_B}{2} + \left( \frac{k_B}{2} - 2a \right) \sqrt{1 + b - 2a(1 + b)} \right]
\]

We see that the first term \( \lambda \) of all three solutions is common.
This implicit equation for $b$ when solved gives $b$ as a function of $L$. We examine the solution space of $b(L)$ in the parameter space $(a, B)$. The $\Lambda$ part determines whether such real $b(L)$ exists for $L \geq 0$.

1. $a < 0$, $B < 0$

In this case the argument of the logarithm in $\Lambda$ is positive implies that the inequality

$$
\left(2|a| \frac{k_0|B|}{2}\right) \sqrt{1 + b} + 2|a|(1 + b) > \frac{k_0|B|}{2}
$$

has to be satisfied by $a, B$ and $b$, for some bounded function $b(L)$ we see that it is possible to satisfy this inequality. Physically it means that the effective behaviour of the system will be as a homogeneous amplifier with amplification parameter $B$ when $a$ is amplifying.

2. $a < 0$, $B > 0$

The argument of logarithm in $\Lambda$ is always positive. Thus solution exists for positive $b$, when the following inequality

$$
\left(2|a| \frac{k_0|B|}{2}\right) \sqrt{1 + b} + 2|a|(1 + b) + \frac{k_0|B|}{2} > 0
$$

is satisfied. Thus an amplifying parameter $a$ can give rise to an effective homogeneously absorbing medium with parameter $B$.

3. $a > 0$, $B > 0$

For the real $\Lambda$ to exist the inequality
\[
\frac{k_0|B|}{2} + \left( \frac{k_0|B|}{2} - 2|\alpha| \right) \sqrt{1 + b} > 2|\alpha|(1 + b)
\]

has to be satisfied. One can always find \( b \) satisfying this interplay. An absorbing parameter \( \alpha \) can give rise to an effective homogeneous absorption.

4. \( \alpha > 0, \ b < 0 \)

In this case it can be easily seen that the argument of the logarithm in \( \Lambda \) can never be positive. So there exists no real solution. This means an absorbing parameter can never give rise to homogeneous amplification.

The above asymptotic case is relevant when \( L \) is much larger than the length scales in the problem and the gradient of \( b(L) \) becomes small compared to \( \frac{a(1+b)}{k_0} \). For \( l \) comparable to the length scales the full stochastic Eq.4.3 has to be solved.

4.3 Conclusion

In conclusion our analysis shows that if we consider a finite active media to be described by an effective amplification/absorption parameter then the effective amplification can be greater than the one given with no variation in the background profile. The gradient of the background profile can also change the system to an effectively absorbing one. But in no case an absorbing medium can become an effectively amplifying medium by adjusting the gradient of the background profile.
References