1.1 INTRODUCTION

The idea of the convergence of a sequence of sets goes back to Archimedes. In 1910, Hausdorff introduced the idea of distance between two sets. The study of convergence of sequence of convex sets and sequence of convex functions and related operators has received increasing attention since the mid 1960s. In the year 1963, the study of convergence of sequences of convex sets was motivated by the question that arose in a proof of the optimum property of sequential probability ratio tests [30, 88].

Convergence plays a vital role in Convex Programming problems. The set convergences have applications in problems of stability, perturbation and approximation. Further, the set convergence permits us to define the notions of semi continuity and continuity for set valued mappings. The set convergence also has applications in Probability, Statistics and Granulometry.
Wijsman [88, 89] was the first to introduce a new type of convergence of sequence of convex functions, viz., "epigraph convergence" in finite dimensions. Umber to Mosco [68] studied the convergence of sequence of convex functions using the notion of epigraph convergence in the context of variational problems and convex optimization problems. The relationship among various kinds of convergence for sequence of convex functions including pointwise convergence and epigraph convergence were studied by Salinetti and Wets [80, 81].

The ordinary pointwise convergence of functions is not sufficient to deal with the study of stability of the optimal value, of the optimal solutions and of the approximate solutions under various perturbations of the objective function and/or the constrained set. In this context, many authors [2, 3, 8, 17, 20, 48, 79] have observed that epi—convergence (epigraph convergence) notion for functions is most suitable for stability analysis under perturbations of the objective function and/or the constrained set.

Sometimes, it may not be easy to solve a given optimization problem involving a function f. In that case, we can approximate the function f by a sequence of functions \( \{f_n\} \) and we can consider the optimization problem involving
for each \( n \). From this, we can find the solution of the given optimization problem involving \( f \). This kind of work has been studied by Kanniappan and Sundaram Sastry [49] using Uniform convergence of functions.

In finite dimensional spaces, Wets [87] proved that if a sequence \( \{f_n\} \) of proper lower semi continuous convex functions epi-converges to a proper lower semi continuous convex function \( f \) and \( f \) is inf-bounded, then the sequence of optimal values of \( f_n \) converges to the optimal value of \( f \). Uniform convergence of functions on bounded sets and epidistance convergence of functions were employed to extend this to infinite dimensional spaces [20].

Convergence of sets and functions can be applied to sequence of constrained optimization problems and also to semi infinite mathematical programming problems with a countable number of constraints [41].

Scalar convergence was found to be useful in studying random convex sets [41]. For example, in [71], Pucci and Vitillaro have obtained extensions of Fatou's lemma and Lebesque's dominated theorem for sequences of integrably bounded (which is a kind of uniform boundedness) random sets for scalar convergence. By using the scalar convergence as a tool, to reduce to the scalar case, Hiai [44] and Hess [43]
obtained the laws of large numbers for random independent and identically distributed closed convex sets. Also Scalar convergence was used by Van Cutsem [86] for studying sequences of random compact sets in \( \mathbb{R} \).

Convergence of \( \varepsilon \)-approximate optimal solutions has been studied under various types of convergence of \( \{f_n\} \). Using these results \( \varepsilon \)-subdifferentials are also investigated. We can investigate convergence of solutions of a sequence of approximating problems when the objective function as well as the constrained sets are perturbed under suitable qualification assumptions satisfied by the initial problem. We can investigate the convergence of \( f_n + \delta_{S_n} \) to \( f + \delta_{S} \) when \( f_n \) converges to \( f \) and \( S_n \) converges to \( S \) in some specified sense.

The plan of the chapter is as follows. In section 2, the various notions of convergence of sequence of sets so far found in the literature due to various authors are described and their inter-relationships are exhibited in the form of a network. In section 3, the various notions of convergence of sequence of functions are described. These convergence concepts of sets and functions have been used in various chapters of this thesis.
1.2 Various types of set convergence

We, now, describe the various notions of convergence of sequence of sets which can be found in [1, 3, 4, 5, 6, 7, 9, 13, 16, 17, 18, 19, 23, 24, 25, 28, 36, 42, 51, 53, 54, 66, 68, 81, 82, 84, 87, 88].

(i) Kuratowski Convergence

Definition 1.1

Let \( \{A_n, n \in \mathbb{N}\} \) be a sequence of sets such that \( A_n \in \mathcal{D}(X) \). Then, the sequence \( \{A_n\} \) is said to converge to \( A \in \mathcal{D}(X) \) in Kuratowski sense, denoted by \( A = K\text{-}\lim A_n \), if

\[
A = \bigcap_{n=1}^{\infty} \{ A_n : A_n \to A \}
\]

\[
\lim \sup A_n \subseteq A \subseteq \lim \inf A_n.
\]

Example 1.1

Let \( A_n = (-\infty, -1-(1/n)] \cup [2+(1/n), \infty) \) and

\[
A = (-\infty, -1] \cup [2, \infty) \text{ be subsets of } X.
\]

Then, \( A = K\text{-}\lim A_n \).

(ii) Hausdorff Convergence

Definition 1.2

A sequence \( \{A_n, n \in \mathbb{N}\}, A_n \in \mathcal{CL}(X) \) is said to converge to \( A \in \mathcal{CL}(X) \) in Hausdorff sense, denoted by
\[ A = \text{H-lim } A_n (A_n \xrightarrow{H} A), \text{ if } \lim_{n \to \infty} H(A_n, A) = 0. \]

In other words, if for every \( \varepsilon > 0 \), \( A_n \subset B_\varepsilon(A) \) eventually (that is, it is denoted by \( A_n \xrightarrow{H^+} A \)) and \( A \subset B_\varepsilon(A_n) \) eventually (that is, it is denoted by \( A_n \xrightarrow{H^-} A \)).

**Remark 1.1**

In the following definition and results of \( r \)-convergence, uniformly \( r \)-convergence, \( * \)-convergence, \( X \) denotes a finite dimensional normed linear space.

(iii) \( r \)-Convergence

**Definition 1.3**

A sequence \( \{A_n, \ n \in N\} \), \( A_n \in \text{CL}(X) \) is said to \( r \)-converge to a subset \( A \in \text{CL}(X) \) if there exists \( r_o > 0 \) such that for all \( r > r_o \),

\[ \lim_{n \to \infty} H_r(A_n, A) = 0 \]

and we denote it by \( A = r \)-lim \( A_n \).
(iv) **Uniformly $r$-Convergence**

**Definition 1.4**

The sequence $\{A_n, n \in \mathbb{N}\} \subset CC(X)$ of sets $r$-converges uniformly (or Uniformly $r$-Convergence) to $A \in CC(X)$ if there exists $r_0 > 0$ and for any $a > 0$ there exists $n_a$ such that for all $r > r_0$ and $n \geq n_a$, we have

$$H_r(A_n, A) \leq a.$$

The uniformity is with respect to $r$. That is, the choice of $n_a$ is to be independent of $r$ provided that $r$ is sufficiently large.

(v) **$\ast$-Convergence**

**Definition 1.5**

A sequence $\{A_n, n \in \mathbb{N}\} \subset CC(X)$ is said to $\ast$-Converge to $A \in CC(X)$ if their support functions converge pointwise. That is, $\{A_n\}$ converges to $A$ in the $\ast$ sense denoted by $A = \ast\lim A_n$, if the sequence of support functions $\{\psi_{A_n}\}$ converges to $\psi_A$ pointwise. That is, $\psi_A(x^*) = \lim_{n \to \infty} \psi_{A_n}(x^*)$, for all $x^* \in X^*$.

**Example 1.2**

Let $A_n = \{(x, y) : y \geq nx + n, x \leq 0\}$ and $A = \{(x, y) : x \leq 0\}$. Then, $A = \ast\lim A_n$. 
(vi) **Wijsman Convergence**

**Definition 1.6**

The sequence \( \{A_n, n \in \mathbb{N}\} \subset \text{CL}(X) \) converges in the sense of **Wijsman** to a set \( A \in \text{CL}(X) \) if for every \( x \in X \),

\[
d(x, A) = \lim_{n \to \infty} d(x, A_n)
\]

and we write \( A = W\text{-}\lim A_n (A_n \xrightarrow{W} A) \).

**Example 1.3**

Let \( A_n = \{(x, y) : x^2 + y^2 - 2ny = 0\} \) and \( A = \{(x, 0) : x \in \mathbb{R}\} \).

Then, \( A = W\text{-}\lim A_n \).

(vii) **Fisher Convergence**

**Definition 1.7**

The sequence of sets \( \{A_n, n \in \mathbb{N}\} \subset \text{D}(X) \) converges to a set \( A \in \text{D}(X) \) in the **Fisher sense** (*Z*-Convergence) for every \( \varepsilon > 0 \),

\[
A_n \subset B_\varepsilon(A) \text{ eventually and } A \subseteq \lim \inf A_n.
\]

It is denoted by \( A = F\text{-}\lim A_n (Z\text{-}\lim A_n \text{ or } A_n \xrightarrow{F} A) \). In other words,

\[
A = Z\text{-}\lim A_n \text{ if and only if } A \subseteq \lim \inf A_n \text{ and }
\lim_{a \to \infty} \left[ \sup_{a \in A_n} d(a, A) \right] = 0.
\]
(viii) **Mosco Convergence**

**Definition 1.8**

The sequence \( \{A_n, \ n \in \mathbb{N}\} \subseteq \text{CC}(X) \) in a reflexive Banach space is said to converge to \( A \in \text{CC}(X) \) in the Mosco sense denoted by \( A = \lim_{M} A_n \), if

(i) for each \( x \in A \), there exists a sequence \( \{x_n\} \) converging strongly to \( x \) with \( x_n \in A_n \), for each \( n \) and

(ii) \( x_k \in A_{n_k} \) for each positive integer \( k \) then weak convergence of the sequence \( \{x_k\} \) to \( x \in X \) implies \( x \in A \).

**Example 1.4**

Let \( A_n = \{(x, y) : y = x/n, \ x \in \mathbb{R}\} \) and

\[
A = \{(x, 0) : \ x \in \mathbb{R}\}.
\]

Then \( A = \lim_{M} A_n \).

(ix) **Attouch–Wets Convergence**

**Definition 1.9**

The sequence \( \{A, A_n, \ n \in \mathbb{N}\} \) of bounded sets \( X \) converges to \( A \) in the Attouch–Wets sense (also \( H_{\lambda} \)-convergence) if for some \( \rho_0 > 0 \) and for every \( A_n \cap \rho U \subseteq B_\varepsilon(A) \) eventually and \( A \subseteq B_\varepsilon(A_n) \cap \)
eventually for every $\rho \geq \rho_0$. We denote this by

$$A = \text{AW-lim } A_n (A_n^{ow} \to A).$$

**Example 1.5**

Let $X = \mathbb{R}^2$, $A_n = \{(s, s/n) : s \in \mathbb{R}\}$ and $A = \{(s, 0) : s \in \mathbb{R}\}$.

Then $A = \text{AW-lim } A_n$.

**Definition 1.10**

The sequence $\{A, A_n, n \in \mathbb{N}\} \subseteq \text{CL}(X)$ is said to converge to $A$ in the SP sense if for some $\rho_0 > 0$ and for every $\varepsilon > 0$, $A \subseteq \text{lim inf } A_n$ and $A_n \cap \rho \overline{U} \subseteq B_\varepsilon(A)$ eventually for every $\rho \geq \rho_0$. This can be written as

$$A = \text{SP-lim } A_n.$$ Also it is denoted by $A_n^{SP} \to A$.

**Example 1.6**

Let $X = \mathbb{R}^2$, $A_n = \{(t, (1/n)t) : t \in \mathbb{R} \text{ and } t \geq 0\}$ and $A = \{(t, 0) : t \in \mathbb{R} \text{ and } t \geq 0\}$. Then $A = \text{SP-lim } A_n$. 
(xi) **Slice Convergence**

**Definition 1.11**

A sequence \( \{A_n, n \in \mathbb{N}\} \subseteq \text{CC}(X) \) is said to **converge slice** to the set \( A \in \text{CC}(X) \) if for every bounded convex set \( B \), \( d(B, A_n) \) converges to \( d(B, A) \) where \( d(A, B) = \inf \{||a-b|| : a \in A \text{ and } b \in B\} \).

(xii) **Scalar Convergence**

**Definition 1.12**

A sequence \( \{A_n, n \in \mathbb{N}\} \subseteq \text{CC}(X) \) is said to **Scalar converge** to a set \( A \in \text{CC}(X) \), denoted by \( A = \text{SC-lim } A_n \), if for all \( x^* \in X^* \)

\[
\psi_A(x^*) = \lim_{n \to \infty} \psi_{A_n}(x^*). 
\]

**Remark 1.2**

The study of this notion in finite dimensions was made under the name \(*-convergence\).
(xiii) **Linear Convergence**

**Definition 1.13**

A sequence \( \{A_n, n \in \mathbb{N}\} \subset CC(X) \) is said to converge to a set \( A \in CC(X) \), called **linear convergence** if

\[
A = W\text{-lim} A_n \quad \text{and} \quad A = SC\text{-lim} A_n.
\]

(xiv) **Affine Convergence**

**Definition 1.14**

A sequence \( \{A_n, n \in \mathbb{N}\} \subset CC(X) \) is said to converge to a set \( A \in CC(X) \), called **affine convergence** if

\[
A = M\text{-lim} A_n \quad \text{and} \quad A = SC\text{-lim} A_n.
\]

(xv) **Bounded Scalar Convergence or \( \rho \)-Scalar Convergence**

**Definition 1.15**

A sequence \( \{A_n, n \in \mathbb{N}\} \subset CC(X) \) is said to converge to a set \( A \in CC(X) \), called **bounded scalar or \( \rho \)-scalar convergence**, denoted by \( A = BS\text{-lim} A_n \) if

\[
\psi \left( A_n \cap \rho \overline{U} (x^*) \right) \text{ converges to } \psi \left( A \cap \rho \overline{U} (x^*) \right),
\]

for all \( x^* \in X^* \) and for all \( \rho > 0 \) such that \( A \cap \text{int}(\rho \overline{U}) \neq \emptyset \).
The following results connecting the above fourteen convergences viz., Kuratowski convergence, Hausdorff convergence, \( r \)-convergence, uniformly \( r \)-convergence, \( * \)-convergence (Scalar convergence), Wijsman convergence, Fisher convergence, Mosco convergence, Attouch-Wets convergence, SP convergence, Linear convergence, Bounded Scalar convergence, affine convergence and Slice convergence have been pooled together which can be found in [7, 15, 42, 80, 81, 82]. The various connections are shown in the network diagram given at the end of this section. Some of the results are used in this thesis.

1. If \( \{A, A_n, n \in \mathbb{N}\} \subseteq \text{CL}(X) \), then Hausdorff convergence implies Kuratowski convergence.

2. If the sequence \( \{A_n, n \in \mathbb{N}\} \subseteq \text{CL}(X) \) is a sequence of connected subsets of \( X \) converging to a non-empty compact subset \( A \) of \( X \) in the Kuratowski sense then \( A = H\text{-lim } A_n \).

3. Let \( \{A, A_n, n \in \mathbb{N}\} \subseteq \text{CC}(X) \). Then \( A = K\text{-lim } A_n \) if and only if \( A = r\text{-lim } A_n \).

4. Let \( \{A, A_n, n \in \mathbb{N}\} \subseteq \text{CC}(X) \). Then the sequence \( \{A_n, n \in \mathbb{N}\} \) \( r \)-converges uniformly to the set \( A \) if and only if \( A = H\text{-lim } A_n \).
5. Let \( \{A_n, n \in N\} \subset CC(X) \). Then Hausdorff convergence and Kuratowski convergence are equivalent if and only if \( \{A_n, n \in N\} \) \( r \)-converges uniformly to \( A \).

6. Let \( \{A, A_n, n \in N\} \subset CC(X) \). Then \( A = H\)-lim \( A_n \), if and only if the sequence \( \{\psi_{A_n}, n \in N\} \) converges uniformly to \( \psi_A \) on the unit ball \( \overline{U}^* \), where \( \overline{U}^*_p \), the polar of \( U \).

7. Let \( \{A, A_n, n \in N\} \subset CC(X) \). Then \( A = H\)-lim \( A_n \) implies \( A = \ast\)-lim \( A_n \).

8. Let \( \{A, A_n, n \in N\} \subset CC(X) \) be such that \( \{\psi_A, \psi_{A_n}, n \in N\} \) be an equi-lower semi continuous collection. Then \( A = K\)-lim \( A_n \) if and only if \( A = \ast\)-lim \( A_n \).

9. Let \( \{A, A_n, n \in N\} \) be a family of non-empty compact convex subsets of \( X \). Then \( A = K\)-lim \( A_n \) if and only if \( A = \ast\)-lim \( A_n \).

10. \( A^H_n \rightarrow A \Rightarrow A^AW_n \rightarrow A \Rightarrow A^SP_n \rightarrow A \Rightarrow A^K_n \rightarrow A \).

11. If \( A \) is convex or weakly compact, then \( A^SP_n \rightarrow A \Rightarrow A^M_n \rightarrow A \).

12. Let \( A, A_n \in CC(X), n = 1, 2, \ldots, \) and \( A \) be bounded. Then
$\frac{SP}{F}$
$A_n \rightarrow A \Leftrightarrow A_n \rightarrow A.$

13. Let $A, A_n \in \text{CL}(X), n = 1, 2, \ldots$ and $X$ be finite dimensional. Then

$\frac{SP}{A_n \rightarrow A} \Leftrightarrow A_n \rightarrow A.$

14. If $A, A_n \in \text{CC}(X)$ then $A = \text{K-lim } A_n \Leftrightarrow A = \text{W-lim } A_n$

15. If $A, A_n \in \text{CC}(X)$ then $A = \text{H-lim } A_n \Leftrightarrow A = \text{W-lim } A_n$

16. If $A, A_n \in \text{D}(X)$ then $A = \text{M-lim } A_n \Leftrightarrow A = \text{K-lim } A_n$

17. If $A, A_n \in \text{D}(X)$ then $A = \text{H-lim } A_n \Leftrightarrow A = \text{M-lim } A_n$

18. Let $X$ be finite dimensional and $A, A_n \in \text{CL}(X)$ then

$A = \text{BS-lim } A_n \Leftrightarrow A = \text{K-lim } A_n$

19. Let $A, A_n \in \text{CC}(X)$ then $A = \text{SL-lim } A_n \Leftrightarrow A = \text{W-lim } A_n$
The connections between various types of set convergences in the case of sequence of non-empty closed convex sets in the finite dimensional space are given in the following network. We use the following abbreviations to understand the network.

1. ZC-Z-Convergence (Fisher)  
3. HC-Hausdorff Convergence  
5. UrC-Uniformly r-Convergence  
7. KC-Kuratowski Convergence  
9. SPC-SP Convergence  
11. AC-Affine Convergence  
13. AWC-Attouch-Wets Convergence

2. LC-Linear Convergence  
4. WC-Wijsman Convergence  
6. *C-* Convergence  
8. rC-r-Convergence  
10. MC-Mosco Convergence  
12. SL-Slice Convergence  
H.BSC-Bounded Scalar Convergence
(xvi) **Vietoris Convergence**

**Definition 1.16**

The sequence of sets \( \{A_n, n \in \mathbb{N}\} \subseteq D(X) \) is said to converge to a subset \( A \) of \( X \) in the **Vietoris sense** if

(i) \( A_n \xrightarrow{v^+} A \) in the sense of \( V^+ \), i.e., \( A_n \xrightarrow{V^+} A \) if whenever \( A \subseteq V \) for some open subset \( V \) of \( X \), then \( A_n \subseteq V \) eventually and

(ii) \( A_n \xrightarrow{v^-} A \) in the sense of \( V^- \), i.e., \( A_n \xrightarrow{V^-} A \) if \( A \subseteq \lim \inf A_n \).

(xvii) **Variational Convergence**

**Definition 1.17**

A sequence \( \{A_n, n \in \mathbb{N}\} \subseteq D(X) \) is said to **converge variationally** to a set \( A \in D(X) \) with respect to a proper function \( f \) on \( X \), denoted by \( A_n \xrightarrow{\text{var}} A \), if the following conditions are satisfied.

(i) \( \lim \sup A_n \subseteq A \) and

(ii) for all \( x_o \in A \) and for all \( \varepsilon > 0 \), there exists a sequence \( \{x_n\} \) such that \( x_n \in A_n \) for every \( n \) and \( f(x_o) < f(x_n) + \varepsilon \) eventually.
Example 1.8

Let \( A_n = [0, 1/n], n \neq 0, A = [0, 1] \) and let \( f(x) = x \).

Then, \( A_n \xrightarrow{\text{var}} A \) with respect to \( f \).

1.3 Various types of function convergence

We, now, describe various types of convergence of sequence of functions, which can be found in [13, 24, 28, 82, 83, 87, 88].

(i) Pointwise Convergence

Definition 1.18

A sequence of functions \( \{f_n\} \) defined on \( X \) is said to converge to a function \( f \) defined on \( X \) pointwise if the sequence of numbers \( \{f_n(x)\} \) converges to \( f(x) \) for all \( x \in X \). That is,

\[
  f(x) = \lim_{n \to \infty} f_n(x), \text{ for all } x.
\]

Example 1.9

Let \( f_n, f : \mathbb{R} \to \mathbb{R}, n = 1, 2, \ldots \), be defined by

\[
f_n(x) = \begin{cases}  
  |x| & \text{if } |x| > 1 \\
  0 & \text{if } |x| = 1/n \\
  1 & \text{otherwise}
\end{cases}
\]
and
\[ f(x) = \begin{cases} 
|x| & \text{if } |x| > 1 \\
1 & \text{otherwise}
\end{cases} \]

Then, \( f_n \to f \) pointwise.

(ii) Uniform Convergence

**Definition 1.19**

A sequence of functions \( \{f_n\} \) defined on \( X \) is said to converge uniformly on \( X \) to a function \( f \) defined on \( X \) if for every \( \varepsilon > 0 \), there is an integer \( N \) such that \( n \geq N \) implies \( \|f_n(x) - f(x)\| \leq \varepsilon \) for all \( x \in X \).

**Example 1.10**

Let \( f_n(x) = \{nx/(1+n^3x^2)\} \) for all real \( x \).

\( \{f_n\} \) converging uniformly to zero for \( 0 \leq x \leq 1 \).

(iii) Epi-Convergence

**Definition 1.20**

A sequence \( \{f_n\} \in \Gamma(X) \) is said to be epi-convergent (epigraph convergence) to \( f \in \Gamma(X) \) if the sequence of epigraphs \( \{\text{epi } f_n\} \) is Kuratowski convergent to \( \text{epi } f \) and it is denoted by \( f_n \overset{\varepsilon}{\to} f \).
In other words, for each \( x \in X \),

(i) whenever \( \{x_n\} \) is convergent to \( x \), then

\[
\liminf_{n \to \infty} f_n(x) \geq f(x) \text{ and }
\]

(ii) there exists a sequence \( \{x_n\} \) convergent to \( x \) such that

\[
\lim_{n \to \infty} f_n(x_n) = f(x).
\]

Example 1.11

Let \( f_n, f : \mathbb{R} \to \mathbb{R} \) be defined by \( f_n(x) = x^2/n \) and \( f(x) = 0 \).

Then, \( f_n \xrightarrow{\text{loc}} f \).

(iv) **Infimal Convergence**

**Definition 1.21**

Let \( \{f_n, n = 1,2,\ldots\} \) and \( f \) be functions defined on \( \mathbb{R}^n \). Define \( \rho f(x) = \inf \{f(y) : |y - x| < \rho\} \), for \( \rho > 0 \). We say that \( \{f_n\} \) converges infimally to \( f \), denoted by \( f_n \xrightarrow{\text{inf}} f \) if

\[
\lim_{\rho \to 0} \lim_{n \to \infty} \inf \rho f_n = \lim_{\rho \to 0} \lim_{n \to \infty} \sup \rho f_n = f.
\]
Example 1.12

Let $f_n$ and $f$ be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{n} & \text{if } x \geq 0
\end{cases}$$

Then, $\lim_{n \to \infty} f_n = f$.

(v) Mosco Convergence

Definition 1.22

A sequence $\{f_n\} \subset \Gamma(X)$ on a reflexive Banach space $X$ is said to converge in Mosco sense to $f \in \Gamma(X)$ if the sequence of sets $\{\text{epi } f_n\}$ converges to the set $\text{epi } f$ in the Mosco sense. It is denoted by $f_n \stackrel{M}{\to} f$. That is, for each $x \in X$,

(i) there exists a sequence $\{x_n\}$ which converges strongly to $x \in X$ for which $\lim_{n \to \infty} f_n(x) = f(x)$ and

(ii) whenever $\{x_n\}$ converges weakly to $x \in X$, then $\lim_{n \to \infty} \inf_{n \to \infty} f_n(x_n) \geq f(x)$. 
Example 1.13

Let \( f_n(x) \) = \[
\begin{cases}
\infty & \text{if } x < 0 \\
-x/n & \text{if } 0 \leq x \leq n \\
-1 & \text{if } x > n
\end{cases}
\]

and

\( f(x) = \[
\begin{cases}
\infty & \text{if } x < 0 \\
0 & \text{if } x \geq 0
\end{cases}
\]

Then, \( f_n \xrightarrow{M} f \).

(vi) Mosco Slice Convergence

Definition 1.23

A sequence \( \{f_n\} \subset \Gamma(X) \) on a reflexive space \( X \) is said to be \textbf{Mosco slice convergent} to \( f \in \Gamma(X) \) if \( epi f_n \rightarrow epi f \) in slice sense in \( X \times \mathbb{R} \).

Example 1.14

Let \( f \) be the indicator function for the origin and for each \( n \), let \( f_n: l_2 \rightarrow \mathbb{R} \) be defined by

\( f_n(x) = \[
\begin{cases}
\max \{(\alpha/n), -1\} & \text{if } x = \alpha e_n \text{ and } \alpha \geq 0 \\
\infty & \text{Otherwise}
\end{cases}
\]

Then, \( f_n \rightarrow f \) in Mosco slice sense.
(vii) Wijsman Convergence

Definition 1.24

A sequence $\{f_n\} \subseteq \Gamma(X)$ is said to be Wijsman convergent to $f \in \Gamma(X)$ if $\text{epi } f_n \rightarrow \text{epi } f$ in Wijsman sense in $X \times \mathbb{R}$.

(viii) Attouch–Wets Convergence

Definition 1.25

A sequence of lower semi continuous functions $f_n : X \rightarrow (-\infty, \infty]$, $n = 1, 2, \ldots$, is said to converge to a lower semi continuous function $f$ in the Attouch–Wets sense if $\text{epi } f_n \rightarrow \text{epi } f$ in the Attouch–Wets sense and denoted by $f_n \xrightarrow{\text{AW}} f$.

(ix) SP Convergence

Definition 1.26

A sequence of lower semi continuous functions $f_n : X \rightarrow (-\infty, \infty]$, $n = 1, 2, \ldots$, is said to converge to a lower semi continuous function $f$ in the SP sense if $\text{epi } f_n \rightarrow \text{epi } f$ in the SP sense and denoted by $f_n \xrightarrow{\text{SP}} f$. 
(x) **Variational Convergence**

**Definition 1.27**

Let $f_n : X \rightarrow (-\infty, \infty]$, $n = 1, 2, \ldots$ be proper functions. Then the sequence $\{f_n\}$ is said to converge *variationally* to a proper function $f$ on $X$, written $f_n \overset{\text{var}}{\rightarrow} f$ if and only if the following conditions are satisfied.

(i) $x_n \rightarrow x$ in $X \Rightarrow f(x) \leq \lim \inf_{n \rightarrow \infty} f_n(x_n)$

(ii) for all $x \in X$ and for all $\delta > 0$, there exists a sequence $\{x_n\}$ in $X$ such that $f_n(x_n) < f(x) + \delta$ eventually.

The following results about the convergence of the sequence of functions and its applications in optimization theory can be found in [29, 49, 71, 80, 82].

1. A sequence of continuous convex functions $\{f_n\}$ converges uniformly to a continuous convex function $f$.

2. A maximal class of closed proper convex functions defined on a finite dimensional space, pointwise convergence and epigraph convergence are equivalent.
3. In any reflexive space, Mosco convergence implies Wijsman convergence.

4. In any non reflexive space, Wijsman convergence is not guaranteed by Mosco convergence even for line segments.

5. If $X$ is a normed linear space and $f, f_n \in \Gamma(X)$, $n = 1, 2, \ldots$, and $f_n^{SP} \rightarrow f$, then for each $a > \alpha(= \inf_{x \in X} f(x))$, we have

\[ \lim_{n \to \infty} \sup_{x \in X} (f_n(x) - f(x)) = 0. \]

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