CHAPTER 6

MORE SPACES

In this chapter we consider X with three generalized topologies and study their properties. Here we define generalized tri topological space, also we introduce two classes of open sets and study their properties. We extend these ideas to fuzzy topology also.

6.1 Generalized Tri Topological Spaces

Let X be a non empty set. Let G be a subset of P(X). G is called a generalized topology on X if the following conditions are satisfied. 1. \( \phi \in G \). 2. Arbitrary union of elements of G is in G.

**Definition 6.1.1:**

Let X be a non empty set. Let \( G_1, G_2, G_3 \) be three generalized topologies on X. Then \((X, G_1, G_2, G_3)\) is called a tri-generalized topological space. (or generalized tri topological space)

**Definition 6.1.2:**

Let \((X, G_1, G_2, G_3)\) be a tri-generalized topological space. Let \( A \subseteq X \).

A is called tri-g open set if \( A \in G_i \) for each \( i = 1, 2, 3 \). A is called tri-g closed set if the complement of A is tri-g open.

**Example 6.1.3:**

Let X = \{1, 2\}

Let \( G_1 = \{ \phi, \{1\} \} \)

\( G_2 = \{ \phi, \{1\}, \{2\}, X \} \)

\( G_3 = \{ \phi, \{1\}, X \} \)

\( \phi \) and \( \{1\} \) are tri-g open sets.
Note 6.1.4:

φ is always tri-g open and X need not be tri-g open.

Definition 6.1.5:

Let \((X,G_1,G_2,G_3)\) be a tri-generalized topological space. Let \(A \subseteq X\).

\(A\) is called tri \(\alpha g\) open if \(A \subseteq G_1 \text{int} G_2 \text{cl} G_3 \text{int} A\).

Definition 6.1.6:

Let \((X,G_1,G_2,G_3)\) be a tri-generalized topological space. Let \(A \subseteq X\).

\(A\) is called tri \(\alpha g\) closed in \(X\) if \(A^c\) is tri \(\alpha g\) open.

Example 6.1.7:

Let \(X = \{1,2,3\}\)

Let \(G_1 = \{\phi, \{1\}\}\)

\(G_2 = \{\phi, \{2\}\}\)

\(G_3 = \{\phi, \{1\}, \{2\}, \{1,2\}\}\)

Obviously \(\phi\) is tri \(\alpha g\) open.

\(G_1 \text{int} G_2 \text{cl} G_3 \text{int} \{1\} = G_1 \text{int} G_2 \text{cl} \{1\}\)

\(= G_1 \text{int} \{1,3\}\)

\(= \{1\}\)

\(\{1\} \subseteq G_1 \text{int} G_2 \text{cl} G_3 \text{int} \{1\}\).

\(\{1\}\) is tri \(\alpha g\) open.

\(G_1 \text{int} G_2 \text{cl} G_3 \text{int} \{2\} = G_1 \text{int} G_2 \text{cl} \{2\}\)

\(= G_1 \text{int} (X) = \{1\}\)

\(\{2\}\) is not subset of \(G_1 \text{int} G_2 \text{cl} G_3 \text{int} \{2\}\) and \(\{2\}\) is not tri \(\alpha g\) open.

\(G_1 \text{int} G_2 \text{cl} G_3 \text{int} \{3\} = G_1 \text{int} G_2 \text{cl} \phi\)

\(= G_1 \text{int} \{1,3\} = \{1\}\)
\{3\} is not a subset of \(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{3\}\) and \(\{3\}\) is not tri \(\alpha\) g open.

\[G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{1,2\} = G_1 \text{ int } G_2 \text{ cl } \{1,2\}\]
\[= G_1 \text{ int } X = \{1\}\]

\(\{1,2\}\) is not a subset of \(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{1,2\}\).

\(\{1,2\}\) is not tri \(\alpha\) g open.

\[G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{1,3\} = G_1 \text{ int } G_2 \text{ cl } \{1\}\]
\[= G_1 \text{ int } \{1,3\} = \{1\}\]

\(\{1,3\}\) is not a subset of \(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{1,3\}\).

\(\{1,3\}\) is not tri \(\alpha\) g open.

\[G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{2,3\} = G_1 \text{ int } G_2 \text{ cl } \{2\}\]
\[= G_1 \text{ int } X = \{1\}\]

\(\{2,3\}\) is not a subset of \(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } \{2,3\}\)

\(\{2,3\}\) is not tri \(\alpha\) g open.

\[G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } X = G_1 \text{ int } G_2 \text{ cl } \{1,2\}\]
\[= G_1 \text{ int } X = \{1\}\]

\(X\) is not a subset of \(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } X\)

Hence \(X\) is not tri \(\alpha\) g open.

The only tri \(\alpha\) g open sets are \(\emptyset\) and \(\{1\}\).

**Theorem 6.1.8:**

Let \((X,G_1,G_2,G_3)\) be a tri-generalized topological space then each tri g open set is tri \(\alpha\) g open.

**Proof:**

Let \(A \subseteq X\) be tri g open then,

\[A = G_i \text{ int } A \text{ for all } i = 1 \text{ to } 3.\]

\[A \subseteq G_2 \text{ cl } A\]
\[ G_1 \text{ int } A \subset G_1 \text{ int } G_2 \text{ cl } A = G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A. \]

\[ A = G_1 \text{ int } A \subset G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A. \]

Hence \( A \) is tri \( \alpha \) g open.

**Result 6.1.9:**

Converse of the above theorem is not true.

**Example 6.1.10:**

In the previous example 6.1.7. \( \{ 1 \} \) is tri \( \alpha \) g open. But it is not tri g open.

### 6.2 Induced Tri Topological spaces.

**Definition 6.2.1:**

Let \( G \) be a generalized topology on \( X \). The intersection of all topologies on \( X \) containing \( G \) is the unique smallest topology containing \( G \).

**Definition 6.2.2:**

Let \((X,G_1,G_2,G_3)\) be a tri generalized topological space. Let \( T_i \) be the smallest topology containing \( G_i \) for all \( i = 1,2,3 \). Then \((X,T_1,T_2,T_3)\) is called the induced tri topological space.

**Theorem 6.2.3:**

Let \((X,G_1,G_2,G_3)\) be a tri generalized topological space. Let \((X,T_1,T_2,T_3)\) be the induced tri topological space. Then each tri g open set of \((X,G_1,G_2,G_3)\) is tri open in \((X,T_1,T_2,T_3)\).

**Proof:**

Let \( A \) be tri g open in \((X,G_1,G_2,G_3)\).

(ie) \( A = G_i \text{ int } A \) for all \( i = 1,2,3 \).

Since \( G_i \subset T_i \) for all \( i \), \( G_i \text{ int } A \subset T_i \text{ int } A \) for all \( i = 1,2,3 \)

\( A = G_i \text{ int } A \subset T_i \text{ int } A \) for all \( i = 1,2,3 \).

Always \( T_i \text{ int } A \subset A \) for all \( i = 1,2,3 \).
\[ A = T_i \text{ int } A \text{ for all } i = 1,2,3. \]

A is tri open set in \( (X,T_1,T_2,T_3) \).

**Result 6.2.4:**

Converse is not true.

**Example 6.2.5:**

Let \( X = \{1,2\} \).

\[
\begin{align*}
G_1 &= \{\phi, \{1\}\} \\
G_2 &= \{\phi, \{2\}, X\} \\
G_3 &= \{\phi, \{2\}\} \\
T_1 &= \{\phi, \{1\}, X\} \\
T_2 &= \{\phi, \{2\}, X\} \\
T_3 &= \{\phi, \{2\}, X\}
\end{align*}
\]

X is tri open in \( (X,T_1,T_2,T_3) \).

But \( X \) is not tri g open in \( (X,G_1,G_2,G_3) \).

**Result 6.2.6:**

Let \( (X,G_1,G_2,G_3) \) be a tri generalized topological space. Let \( (X,T_1,T_2,T_3) \) be the induced tri topological space. Each tri g open set is tri open and each tri open set is tri \( \alpha \) open set. Hence each tri g open is tri \( \alpha \) open set. But a tri \( \alpha \) open set need not be tri g open. \( X \) is always tri \( \alpha \) open but \( X \) need not be tri g open.

**Result 6.2.7:**

A tri \( \alpha \) open set need not be tri \( \alpha g \) open and tri \( \alpha g \) open set need not be tri \( \alpha \) open.

**Example 6.2.8:**

Let \( X = \{1,2\} \).

\[
\begin{align*}
G_1 &= \{\phi, \{1\}\} \\
G_2 &= \{\phi, \{2\}\} \text{ and } G_3 = \{\phi, \{1\}, \{2\}, \{1,2\}\} \\
T_1 &= \{\phi, \{1\}, \{1,2\}\} \text{, } T_2 = \{\phi, \{2\}, \{1,2\}\} \\
T_3 &= \{\phi, \{1\}, \{2\}, \{1,2\}\}
\end{align*}
\]

Tri \( \alpha g \) open sets are \( \phi \) and \{1\}.

Tri \( \alpha \) open sets are \( \phi, \{1\}, \{2\} \) and \( X \).
{2} and X are tri α open. But they are not tri αg open sets.

Hence a tri α open set need not be tri αg open.

**Example 6.2.9:**

Let X = \{1,2,3\}

\(G_1 = \{\phi, X\}\), \(G_2 = \{\phi, \{1,2\}, \{2,3\}, \{1,3\}, X\}\) and \(G_3 = P(X)\)

\(T_1 = \{\phi, X\}\), \(T_2 = P(X)\) and \(T_3 = P(X)\)

Let \(A = \{1,2\}\)

\(G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A = G_1 \text{ int } G_2 \text{ cl } A\)

\[= G_1 \text{ int } X = X.\]

\(A \subset X = G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A.\)

A is tri αg open.

\(T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A = T_1 \text{ int } T_2 \text{ cl } A.\)

\[= T_1 \text{ int } A. = \phi.\]

A is not a subset of \(T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A.\)

A is not tri α open.

Hence a tri αg open set need not be tri α open.

**Theorem 6.2.10:**

Arbitrary union of tri αg open sets is tri αg open in a tri generalized topological space.

**Proof :**

In a tri generalized topological space \((X, G_1, G_2, G_3)\)

Let \(A_\alpha\) be tri αg open set for each \(\alpha \in I.\)

\(A_\alpha \subset G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A_\alpha\) for each \(\alpha \in I.\)

\(\cup A_\alpha \subset \cup [G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A_\alpha]\) where \(\alpha \in I\)
∪ A_α ⊂ G_1 \text{ int } [ ∪ G_2 \text{ cl } G_3 \text{ int } A_α ] \quad \text{Since } \text{Int}(A∪B) \supset (\text{Int } A)∪(\text{Int } B)

⊂ G_1 \text{ int } G_2 \text{ cl } [ ∪ G_3 \text{ int } A_α] \quad \text{Since } \text{cl } (A∪B) \supset (\text{cl } A)∪(\text{cl } B)

⊂ G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } [ ∪ A_α] \quad \text{where } \alpha ∈ I

Hence ∪ A_α is tri αg open set where α ∈ I.

**Theorem 6.2.11:**

Arbitrary intersection of tri αg closed sets is tri αg closed in a tri generalized topological space.

**Proof:**

In a tri generalized topological space (X, G_1,G_2,G_3), let A_α be tri αg closed for each α ∈ I.

Let B_α= A_α^c, then B_α is tri αg open for each α ∈ I.

∪ B_α is tri αg open. Hence (∪ B_α)^c is tri αg closed.

Hence ∩ A_α is tri αg closed.

### 6.3. TRI αg CLOSURE AND TRI αg INTERIOR OF A SET

**Definition 6.3.1:**

Let (X, G_1,G_2,G_3) be a tri generalized topological space.

Let A⊂X. Tri αg closure of A is defined as the intersection of all tri αg closed sets containing A.

**Result 6.3.2:**

Let (X,G_1,G_2,G_3) be a tri generalized topological space. Let A⊂X.

Tri αg closure of A is the smallest tri αg closed set containing A.

**Example 6.3.3:**

Let X = {1, 2}, G_1 = {φ, {1}, {2}, X}, G_2 = {φ, {1}}, G_3 = {φ, {2}}
set of all tri $\alpha g$ open sets is $\{\phi , \{2\}\}$

Tri $\alpha g$ closed sets are $\{1\}$ and $X$.

Let $A = \phi$  $\triangledown\alpha g$ cl $\phi = \{1\}$

Let $A = \{1\}$  $\triangledown\alpha g$ cl $\{1\} = \{1\}$

Let $A = \{2\}$  $\triangledown\alpha g$ cl $\{2\} = X$.

Let $A = X$  $\triangledown\alpha g$ cl $X = X$.

**Note 6.3.4:**

Tri $\alpha g$ closure of a set is tri $\alpha g$ closed, since arbitrary intersection of tri $\alpha g$ closed set is tri $\alpha g$ closed.

**Theorem 6.3.5:**

Let $(X,G_1, G_2, G_3)$ be a tri generalized topological space. Let $A \subset X$.

$A$ is tri $\alpha g$ closed iff $\triangledown\alpha g$ cl $A = A$.

**Proof:**

Suppose $A$ is tri $\alpha g$ closed. $\triangledown\alpha g$ cl $A = \cap \{ A_{\alpha} / A_{\alpha}$ is tri $\alpha g$ closed and $A_{\alpha} \supseteq A\}$. Since $A$ is tri $\alpha g$ closed. $A = A_{\alpha}$ for some $\alpha \in I$. Hence $\triangledown\alpha g$ cl $A = A$.

Conversely, Suppose $A = \triangledown\alpha g$ cl $A$ then,

$A = \cap \{ A_{\alpha} / A_{\alpha}$ is tri $\alpha g$ closed and $A_{\alpha} \supseteq A\}$ where $\alpha \in I$

Since arbitrary intersection of tri $\alpha g$ closed sets is tri $\alpha g$ closed,

$A$ is tri $\alpha g$ closed.

**Definition 6.3.6:**

Let $(X,G_1,G_2,G_3)$ be a tri generalized topological space. Let $A \subset X$. Let $x \in A$.

Then $x$ is called tri $\alpha g$ interior point of $A$ if $\exists$ a tri $\alpha g$ open set $V$ such that $x \in V \subset A$.

**Example 6.3.7:**

Let $X = \{1,2,3\}$. Let $G_1 = \{\phi , \{1\} , \{1,2\}\}$
$G_2 = \{ \emptyset, \{2\}, \{2,3\} \}, \quad G_3 = \{ \emptyset, \{3\}, \{1,3\} \}$

Tri αg open sets are φ and \{1\}.

Let $A = \{1\}$, \quad 1 is tri αg interior point of \{1\}.

Let $A = \{1,3\}$.

1 is tri αg interior point of \{1,3\}. Since $1 \in \{1\} \subset \{1,3\}$.

3 is not tri αg interior point of \{1,3\}

Let $A = \{2,3\}$. No point of $A$ is tri αg interior point of $A$.

**Definition 6.3.8:**

The set of all tri αg interior points of $A$ is called tri αg int $A$.

**Example 6.3.9:**

In the previous example 6.3.7,

If $A = \{1\}$; \quad tri αg int $A = \{1\}$

If $A = \{1,3\}$; \quad tri αg int $A = \{1\}$

If $A = \{2,3\}$; \quad tri αg int $A = \emptyset$

If $A = \{2\}$; \quad tri αg int $A = \emptyset$

If $A = \{1,2\}$; \quad tri αg int $A = \{1\}$

If $A = X$; \quad tri αg int $A = \{1\}$

**Theorem 6.3.10:**

Let $(X,G_1,G_2,G_3)$ be a tri generalized topological space. Let $A \subset X$.

Tri αg int $A$ is the union of all tri αg open subsets of $A$.

**Proof:**

Let $B =$ Union of all tri αg open subsets of $A$.

Let $x \in B \Rightarrow x \in V$ where $V$ is a tri αg open set and $V \subset A$.

$\Rightarrow x \in V \subset A$ where $V$ is tri αg open
\[ \Rightarrow x \text{ is a tri } \alpha \text{g interior point of } A. \]
\[ \Rightarrow x \in \text{tri } \alpha \text{g int } A \]

Hence \( B \subseteq \text{tri } \alpha \text{g int } A \) ...........1

Let \( x \in \text{tri } \alpha \text{g int } A \)

\( \Rightarrow x \text{ is tri } \alpha \text{g interior point of } A \)

\( \Rightarrow \exists \text{ a tri } \alpha \text{g open set } V \text{ such that } x \in V \subseteq A \)

\( \Rightarrow x \in \text{Union of all tri } \alpha \text{g open subsets of } A \)

\( \Rightarrow x \in B \)

\( \text{tri } \alpha \text{g int } A \subseteq B \) ...........2

1 and 2 imply \( B = \text{tri } \alpha \text{g int } A \)

**Note 6.3.11:**

Let \((X,G_1,G_2,G_3)\) be a tri generalized topological space. Let \( A \subseteq X \).

\( A \text{ is tri } \alpha \text{g open } \iff A = \text{tri } \alpha \text{g int } A \text{ since union of all tri } \alpha \text{g open sets is tri } \alpha \text{g open.} \)

**Theorem 6.3.12:**

\((\text{Tri } \alpha \text{g int } A) \cup (\text{Tri } \alpha \text{g int } B) \subseteq \text{Tri } \alpha \text{g int } (A \cup B)\)

**Proof:**

\( \text{Tri } \alpha \text{g int } A \text{ is a tri } \alpha \text{g open subset of } A \)

\( \text{Tri } \alpha \text{g int } B \text{ is a tri } \alpha \text{g open subset of } B \)

\((\text{tri } \alpha \text{g int } A) \cup (\text{tri } \alpha \text{g int } B) \text{ is a union of tri } \alpha \text{g open subsets of } A \cup B.\)

But \( \text{tri } \alpha \text{g int } (A \cup B) \text{ is union of all tri } \alpha \text{g open subsets of } A \cup B. \)

Hence \((\text{Tri } \alpha \text{g int } A) \cup (\text{Tri } \alpha \text{g int } B) \subseteq \text{Tri } \alpha \text{g int } (A \cup B).\)

**Theorem 6.3.13:**

Let \((X,G_1,G_2,G_3)\) be a tri generalized topological space. Let \( A \subseteq X. \)

\( A \text{ is tri } \alpha \text{g closed iff } A \supseteq G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A. \)
**Proof:**

Let \( A^c = B \)

\( A \) is tri \( \alpha \gamma \) closed \( \iff \) \( B \) is tri \( \alpha \gamma \) open.

\( \iff B \subseteq G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } B \)

\( \iff B^c \supseteq G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } B^c \)

\( \iff A \supseteq G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A \).

**Theorem 6.3.14:**

Let \((X, G_1, G_2, G_3)\) be a tri generalized topological space. For any \( A \subseteq X \)

\[ A \cup G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A \subseteq \text{ tri } \alpha \gamma \text{ cl } A. \]

**Proof:**

\( A \subseteq \text{ tri } \alpha \gamma \text{ cl } A \ldots \ldots 1 \)

\( G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A \subseteq G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } (\text{ tri } \alpha \gamma \text{ cl } A) \ldots \ldots 2 \)

Since \( \text{ tri } \alpha \gamma \text{ cl } A \) is tri \( \alpha \gamma \) closed.

\( \text{ tri } \alpha \gamma \text{ cl } A \supseteq G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } (\text{ tri } \alpha \gamma \text{ cl } A) \ldots \ldots 3 \)

From 2 and 3 we get,

\( G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A \subseteq \text{ tri } \alpha \gamma \text{ cl } A \ldots \ldots 4 \)

From 1 and 4 we get,

\[ A \cup G_1 \text{ cl } G_2 \text{ int } G_3 \text{ cl } A \subseteq \text{ tri } \alpha \gamma \text{ cl } A. \]

**Definition 6.3.15:**

Let \((X, G_1, G_2, G_3)\) be a tri generalized topological space. Let \( A \subseteq X \). Let \( x \in X \).

Then \( x \) is called a tri \( \alpha \gamma \) limit point of \( A \) if every tri \( \alpha \gamma \) open set containing \( x \) intersects \( A \setminus \{x\} \). (ie). \( x \) is a limit point of \( A \) if every tri \( \alpha \gamma \) open set containing \( x \) contains a point of \( A \) other than \( x \).
Example 6.3.16:

X = {a,b,c}, G_1 = {\phi, \{a,b\}, X}, G_2 = {\phi, \{a\}, X}, G_3 = {\phi, \{a,c\}, X}

Tri \alpha g open sets are \phi, \{a,c\}, X

Tri \alpha g closed sets are \phi, \{b\}, X

A = \{a,c\}. Consider the element b. Tri \alpha open set containing b is X.

X \cap A-\{b\} \neq \phi. Hence b is a tri \alpha g limit point of A.

6.4 T_1T_2T_3 Open Sets

Definition 6.4.1:

Let (X,T_1,T_2,T_3) be a tri topological space. A subset A is called T_1T_2T_3 open if \ A \in T_1 \cup T_2 \cup T_3. (i.e.) A is open in atleast one topology.

Definition 6.4.2: Let (X,T_1,T_2,T_3) be a tri topological space. Let A \subset X.

A is T_1T_2T_3closed if A^c is T_1T_2T_3open.

Definition 6.4.3: Let (X,T_1,T_2,T_3) be a tri topological space. Let A \subset X.

T_1T_2T_3intA = union of all T_1T_2T_3open sets contained in A.

Definition 6.4.4: Let (X,T_1,T_2,T_3) be a tri topological space. Let A \subset X.

T_1T_2T_3clA = Intersection of all T_1T_2T_3closed sets containing A.

Result 6.4.5: Let (X,T_1,T_2,T_3) be a tri topological space. Let A \subset X.

[T_1T_2T_3intA]^c = T_1T_2T_3clA^c.

Theorem 6.4.6: Let (X,T_1,T_2,T_3) be a tri topological space. Let A \subset X.

A is T_1T_2T_3open \Rightarrow T_1T_2T_3intA = A.

Proof:

T_1T_2T_3intA = Union of all T_1T_2T_3open sets contained in A.
Since A is $T_1T_2T_3$ open, union of all $T_1T_2T_3$ open sets contained in A is A.

Hence $T_1T_2T_3 = A$.

**Result 6.4.7:**

Converse is not true.

**Example 6.4.8:**

Let $X = \{a,b,c\}$

$T_1 = \{\phi, \{a\},X\}$, $T_2 = \{\phi, \{b\},X\}$, $T_3 = \{\phi, \{c\},X\}$

Let $A = \{a,b\} = \{a\} \cup \{b\}$

$\{a\}$ and $\{b\}$ are $T_1T_2T_3$ open sets contained in $A$

Hence $A = T_1T_2T_3\text{int}A$

But $\{a,b\} = A$ is not $T_1T_2T_3$ open.

**Result 6.4.9:**

Union of two $T_1T_2T_3$ open sets need not be $T_1T_2T_3$ open in a tri topological space.

**Example 6.4.10:**

Let $X = \{a,b,c\}$

$T_1 = \{\phi, \{a\},X\}$, $T_2 = \{\phi, \{b\},X\}$, $T_3 = \{\phi, \{c\},X\}$

$\{a\}$ and $\{b\}$ are $T_1T_2T_3$ open sets.

Their union $\{a,b\}$ is not $T_1T_2T_3$ open.

**Result 6.4.11:**

Intersection of two $T_1T_2T_3$ closed sets need not be $T_1T_2T_3$ closed.

**Example 6.4.12:**

Let $X = \{a,b,c\}$

$T_1 = \{\phi, \{a\},X\}$, $T_2 = \{\phi, \{b\}, \{b,c\},X\}$, $T_3 = \{\phi, \{b\}, \{a,c\},X\}$

$\{b,c\}$ and $\{a,c\}$ are $T_1T_2T_3$ open

$\{b,c\} \cap \{a,c\} = \{c\}$ is not $T_1T_2T_3$ open.
Result 6.4.13:
Intersection of two $T_1T_2T_3$ open sets need not be $T_1T_2T_3$ open.

Result 6.4.14:
Union of two $T_1T_2T_3$ closed sets need not be $T_1T_2T_3$ closed.

6.5 $T_{1,2,3}$ Open Sets

Definition 6.5.1:
Let $(X,T_1,T_2,T_3)$ be a tri topological space $S \subset X$ is $T_{1,2,3}$ open if
$S = A \cup B \cup C$ where $A \in T_1, B \in T_2, C \in T_3$.

Result 6.5.2:
$\emptyset$ and $X$ are $T_{1,2,3}$ open.

Theorem 6.5.3:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $S_1, S_2 \subset X$.
If $S_1$ and $S_2$ are $T_{1,2,3}$ open then $S_1 \cup S_2$ is $T_{1,2,3}$ open.

Proof:
$S_1$ and $S_2$ are $T_{1,2,3}$ open.
$S_1 = A_1 \cup B_1 \cup C_1$
$S_2 = A_2 \cup B_2 \cup C_2$ where $A_1, A_2 \in T_1$, $B_1, B_2 \in T_2$ and $C_1, C_2 \in T_3$

$S_1 \cup S_2 = (A_1 \cup A_2) \cup (B_1 \cup B_2) \cup (C_1 \cup C_2)$ is $T_{1,2,3}$ open.

Theorem 6.5.4:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $S_\alpha \in X$ where $\alpha \in I$.
If $S_\alpha$ is $T_{1,2,3}$ open for each $\alpha$ then $\cup S_\alpha$ is $T_{1,2,3}$ open.

Proof:
$S_\alpha = A_\alpha \cup B_\alpha \cup C_\alpha$ where $A_\alpha \in T_1$, $B_\alpha \in T_2$ and $C_\alpha \in T_3$

$\cup_{\alpha \in I} S_\alpha = \cup [A_\alpha \cup B_\alpha \cup C_\alpha]$


\[ (\cup A_\alpha) \cup (\cup B_\alpha) \cup (\cup C_\alpha) \]

\[ \cup A_\alpha \in T_1, \cup B_\alpha \in T_2 \text{ and } \cup C_\alpha \in T_3 \]

Hence \( \cup S_\alpha \) is \( T_{1,2,3} \) open.

**Definition 6.5.5:** Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \( A \subset X \).

If \( A \) is \( T_{1,2,3} \) open then the complement of \( A \) is said to be \( T_{1,2,3} \) closed.

**Definition 6.5.6:** Let \((X, T_1, T_2, T_3)\) be a tri topological space.

\( A \subset X \). \( T_{1,2,3} \)\( \text{clA} = \cap \{ F / F \supset A \text{ and } F \text{ is } T_{1,2,3} \text{ closed} \} \)

\( A \subset X \). \( T_{1,2,3} \)\( \text{intA} = \cup \{ G / G \subset A \text{ and } G \text{ is } T_{1,2,3} \text{ open} \} \)

**Theorem 6.5.7:**

Arbitrary intersection of \( T_{1,2,3} \) closed sets is \( T_{1,2,3} \) closed in a tri topological space.

**Proof:**

Follows from Theorem 6.5.4

**Theorem 6.5.8:**

Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \( A \subset X \).

\( A \) is \( T_{1,2,3} \) open iff \( T_{1,2,3} \)\( \text{intA} = A \)

**Proof:**

Follows from the fact that arbitrary union of \( T_{1,2,3} \) open sets is \( T_{1,2,3} \) open.

**Theorem 6.5.9:**

Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \( A \subset X \).

\( A \) is \( T_{1,2,3} \) closed iff \( T_{1,2,3} \)\( \text{clA} = A \)

**Proof:**

Follows from the fact that arbitrary intersection of \( T_{1,2,3} \) closed sets is \( T_{1,2,3} \) closed.

**Theorem 6.5.10:**

Let \((X, T_1, T_2, T_3)\) be a tri topological space.

For any \( A \subset X \), \( (T_{1,2,3} \text{intA})^C = T_{1,2,3} \text{clA}^C \)
Proof:

\[(T_{1,2,3}{\text{int}}A)^C = [\cup \{G/G \subseteq A \text{ & } G \text{ is } T_{1,2,3} \text{ open}\}]^C\]

\[= \cap \{G^C/G^C \supseteq A^C \text{ & } G^C \text{ is } T_{1,2,3} \text{ closed}\}\]

\[= \cap \{F=G^C/F \supseteq A^C \text{ & } F \text{ is } T_{1,2,3} \text{ closed}\} \text{ where } F = G^C\]

\[= T_{1,2,3}{\text{cl}}A^C\]

Result 6.5.11:

\(\phi\) and \(X\) are \(T_{1,2,3}\) closed.

Result 6.5.12:

The intersection of two \(T_{1,2,3}\) open sets need not be \(T_{1,2,3}\) open.

Example 6.5.13:

\(X=\{a,b,c\}\).

\(T_1=\{\phi,\{a\},\{a,b\},X\}, \ T_2=\{\phi,\{b,c\},X\}, \ T_3=\{\phi,\{c\},X\}\).

\(A=\{a,b\}=\{a,b\} \cup \phi \cup \phi \Rightarrow A\) is \(T_{1,2,3}\) open.

\(B=\{b,c\}=\phi \cup \{b,c\} \cup \{c\} \Rightarrow B\) is \(T_{1,2,3}\) open.

But \(A \cap B=\{a,b\} \cap \{b,c\}=\{b\}\) is not \(T_{1,2,3}\) open.

\(\{b\}\) cannot be written as \(B_1 \cup B_2 \cup B_3\), where \(B_i\) is \(T_i\) open \(\forall i=1\) to 3

Result 6.5.14:

The union of two \(T_{1,2,3}\) closed sets need not be \(T_{1,2,3}\) closed.

Example 6.5.15:

In the Example 6.5.13 A and B are \(T_{1,2,3}\) open

\[\Rightarrow A^C\) and \(B^C\) are \(T_{1,2,3}\) closed\]

\(A^C = \{c\}\) and \(B^C = \{a\}\)

\(A^C \cup B^C = \{a,c\}\) is not \(T_{1,2,3}\) closed, since \(\{a,b\}^C = \{b\}\) is not \(T_{1,2,3}\) open.

Hence \(\{a,c\}\) is not \(T_{1,2,3}\) closed.
**Result 6.5.16:**
The set of all $T_{1,2,3}$ open sets contains $T_1 \cup T_2 \cup T_3$

**Theorem 6.5.17:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space. The set of all $T_{1,2,3}$ open sets is a generalized topology on $X$.

**Proof:**
Follows from Result 6.5.2, Theorem 6.5.4, Result 6.5.12,

**Definition 6.5.18:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space.

$A \subset X$ is called semi$(1,2,3)$ open if $A \subset T_{1,2,3}\text{cl}T_{1,2,3}\text{int}A$.

**Definition 6.5.19:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$. $A$ is called semi $(1,2,3)$ closed if $A^c$ is semi $(1,2,3)$ open.

**Definition 6.5.20:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space.

$A \subset X$ is called pre$(1,2,3)$ open if $A \subset T_{1,2,3}\text{int}T_{1,2,3}\text{cl}A$.

**Definition 6.5.21:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.

$A$ is called pre$(1,2,3)$ closed if $A^c$ is pre$(1,2,3)$ open.

**Definition 6.5.22:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space.

$A \subset X$ is called $\alpha(1,2,3)$ open if $A \subset T_{1,2,3}\text{int}T_{1,2,3}\text{cl}T_{1,2,3}\text{int}A$.

**Definition 6.5.23:**
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.

$A$ is called $\alpha(1,2,3)$ closed if $A^c$ is $\alpha(1,2,3)$ open.
Definition 6.5.24:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.
$A$ is called $\beta(1,2,3)$ open if $A \subset T_{1,2,3}\text{cl}T_{1,2,3}\text{int}T_{1,2,3}\text{cl}A$.

Definition 6.5.25:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.
$A$ is called $\beta(1,2,3)$ closed if $A^c$ is $\beta(1,2,3)$ open.

Definition 6.5.26:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.
$A$ is called regular $(1,2,3)$ open if $A = T_{1,2,3}\text{int}T_{1,2,3}\text{cl}A$.

Definition 6.5.27:
Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.
$A$ is called regular $(1,2,3)$ closed if $A = T_{1,2,3}\text{cl}T_{1,2,3}\text{int}A$.

Example 6.5.28:
$X = \{a,b,c,d\}$, $T_1 = \{\phi, \{a,b\}, X\}$
$T_2 = \{\phi, \{b,c\}, X\}$, $T_3 = \{\phi, \{a,b\}, X\}$

$T_{1,2,3}$ open sets are $\phi, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c,d\}, X$

$T_{1,2,3}$ closed sets are $\phi, \{c,d\}, \{a,d\}, \{b,d\}, \{d\}, X$

Take $A = \{a,b,d\}$

$T_{1,2,3}\text{int}A = \{a,b\}$ $T_{1,2,3}\text{cl}T_{1,2,3}\text{int}A = X$

$A \subset T_{1,2,3}\text{cl}T_{1,2,3}\text{int}A$. Hence $A$ is semi $(1,2,3)$ open.

Hence $\{c\}$ is semi $(1,2,3)$ closed.

$T_{1,2,3}\text{cl}A = X$, $T_{1,2,3}\text{int}T_{1,2,3}\text{cl}A = X$.

$A \subset T_{1,2,3}\text{int}T_{1,2,3}\text{cl}A$. Hence $A$ is pre $(1,2,3)$ open.

$\{c\}$ is pre $(1,2,3)$ closed.
$T_{1,2,3}$ int $T_{1,2,3}$ cl $T_{1,2,3}$ int $A = X$

$A \subset T_{1,2,3}$ int $T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

Hence $A$ is $\alpha(1,2,3)$ open.

$A$ is $\beta(1,2,3)$ open.

**Theorem 6.5.29:**

Let $(X,T_1,T_2,T_3)$ be a tri topological space. Let $A \subset X$.

$A$ is semi $(1,2,3)$ open iff $\exists$ a $T_{1,2,3}$ open set $O$ such that $O \subset A \subset T_{1,2,3}$ cl $O$.

**Proof:**

Let $A$ be semi $(1,2,3)$ open. Then $A \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

Now $T_{1,2,3}$ int $A \subset A \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$. Take $O = T_{1,2,3}$ int $A$.

Then $O$ is $T_{1,2,3}$ open and $O \subset A \subset T_{1,2,3}$ cl $O$.

Conversely suppose $\exists$ $T_{1,2,3}$ open set $O$ with $O \subset A \subset T_{1,2,3}$ cl $O$.

Now $O$ is $T_{1,2,3}$ open and $O \subset A$ and hence $O \subset T_{1,2,3}$ int $A$.

Therefore $T_{1,2,3}$ cl $O \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

Now $A \subset T_{1,2,3}$ cl $O$ and hence $A \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

**Theorem 6.5.30:**

Let $(X,T_1,T_2,T_3)$ be a tri topological space. $A \subset X$ is semi $(1,2,3)$ open iff $T_{1,2,3}$ cl $A = T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

**Proof:**

Suppose $A$ is semi $(1,2,3)$ open. Then $A \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$.

Always $T_{1,2,3}$ int $A \subset A$.

$\Rightarrow T_{1,2,3}$ cl $[T_{1,2,3}$ int $A] \subset T_{1,2,3}$ cl $A$ --------1

Also $T_{1,2,3}$ cl $A \subset T_{1,2,3}$ cl $T_{1,2,3}$ int $A$ --------2

1 and 2 $\Rightarrow T_{1,2,3}$ cl $A = T_{1,2,3}$ cl $T_{1,2,3}$ int $A$. 
Conversely,

Suppose \( T_{1,2,3} \text{cl } A = T_{1,2,3} \text{ int } A \).

Claim: \( A \) is semi open.

\( A \subset T_{1,2,3} \text{cl } A = T_{1,2,3} \text{ cl } T_{1,2,3} \text{ int } A. \)

\( \Rightarrow A \subset T_{1,2,3} \text{ cl } T_{1,2,3} \text{ int } A. \)

Therefore \( A \) is semi \((1,2,3)\) open.

**Theorem 6.5.31:**

Let \((X,T_{1},T_{2},T_{3})\) be a tri topological space. \( \text{A \subset X} \) is regular \((1,2,3)\) open iff \( A^c \) is regular \((1,2,3)\) closed.

**Proof:**

Suppose \( A \) is regular \((1,2,3)\) open.

\( \Leftrightarrow A = T_{1,2,3} \text{ int } T_{1,2,3} \text{ cl } A \)

\( \Leftrightarrow A^c = [T_{1,2,3} \text{ int } T_{1,2,3} \text{ cl } A]^c \)

\( \Leftrightarrow A^c = T_{1,2,3} \text{ cl } T_{1,2,3} \text{ int } A^c. \)

\( \Leftrightarrow A^c \) is regular \((1,2,3)\) closed.

**Theorem 6.5.32:**

Let \((X,T_{1},T_{2},T_{3})\) be a tri topological space. Let \( A \subset X. \)

\[ \text{Semi } (1,2,3) \text{ int } A^c = \text{ Semi } (1,2,3) \text{ cl } A^c \]

**Proof:**

\[ \text{Semi } (1,2,3) \text{ int } A^c = \bigcup \{ G / G \subset A \text{ and } G \text{ is semi } (1,2,3) \text{ open} \}^c \]

\[ = \bigcap \{ G^c / G^c \supset A^c \text{ and } G^c \text{ is semi } (1,2,3) \text{ closed} \} \]

\[ = \bigcap \{ F / F \supset A^c \text{ and } F \text{ is semi } (1,2,3) \text{ closed} \} \text{ where } F = G^c \]

\[ = \text{ Semi } (1,2,3) \text{ cl } A^c \]

Hence \[ \text{Semi } (1,2,3) \text{ int } A^c = \text{ Semi } (1,2,3) \text{ cl } A^c. \]
Theorem 6.5.33:
Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \(A \subset X\).

\(A\) is semi\((1, 2, 3)\) closed iff \(A \supset T_{1,2,3} \text{int} T_{1,2,3} \text{cl} A\).

Proof:
\(A\) is semi\((1, 2, 3)\) closed
\(\iff\) \(A^C\) is semi\((1, 2, 3)\) open.
\(\iff\) \(A^C \subset T_{1,2,3} \text{ cl} T_{1,2,3} \text{int} A^C\)
\(\iff\) \(A \supset (T_{1,2,3} \text{ cl} T_{1,2,3} \text{ int} A)^C\)
\(\iff\) \(A \supset T_{1,2,3} \text{ int} (T_{1,2,3} \text{ int} A)^C\)
\(\iff\) \(A \supset T_{1,2,3} \text{ int} T_{1,2,3} \text{ cl} A\).

Theorem 6.5.34:
Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \(A \subset X\).

\(A\) is pre\((1, 2, 3)\) closed iff \(A \supset T_{1,2,3} \text{ cl} T_{1,2,3} \text{int} A\).

Proof:
\(A \supset T_{1,2,3} \text{ cl} T_{1,2,3} \text{int} A\)

\(A\) is pre\((1, 2, 3)\) closed \(\iff\) \(A^C\) is pre\((1, 2, 3)\) open
\(\iff\) \(A^C \subset T_{1,2,3} \text{ int} T_{1,2,3} \text{cl} A^C\)
\(\iff\) \((A^C)^C \supset [T_{1,2,3} \text{ int} T_{1,2,3} \text{cl} A]^C\)
\(\iff\) \(A \supset T_{1,2,3} \text{ cl} T_{1,2,3} \text{int} A\).

Theorem 6.5.35:
Let \((X, T_1, T_2, T_3)\) be a tri topological space. Let \(A \subset X\).

\(A\) is \(\alpha\)(1,2,3) closed iff \(A \supset T_{1,2,3} \text{ cl} T_{1,2,3} \text{int} T_{1,2,3} \text{cl} A\).

Proof:
\(A\) is \(\alpha\)(1,2,3) closed
⇔ \( A^c \) is \( \alpha(1,2,3) \) open

⇔ \( A^c \subset T_{1,2,3} \text{int} T_{1,2,3} \text{cl} T_{1,2,3} \text{int} A^c \)

⇔ \( (A^c)^c \supset [T_{1,2,3} \text{int} T_{1,2,3} \text{cl} T_{1,2,3} \text{int} A^c]^c \)

⇔ \( A \supset T_{1,2,3} \text{cl} T_{1,2,3} \text{int} T_{1,2,3} \text{cl} A \).

**Theorem 6.5.36:**

Let \((X,T_1,T_2,T_3)\) be a tri topological space. Let \( A \subset X \).

\( A \) is \( \beta(1,2,3) \) closed iff \( A \supset T_{1,2,3} \text{int} T_{1,2,3} \text{cl} T_{1,2,3} \text{int} A \).

**Proof:**

\( A \) is \( \beta(1,2,3) \) closed

⇔ \( A^c \) is \( \beta(1,2,3) \) open

⇔ \( A^c \subset T_{1,2,3} \text{cl} T_{1,2,3} \text{int} T_{1,2,3} \text{cl} A^c \)

⇔ \( (A^c)^c \supset [T_{1,2,3} \text{cl} T_{1,2,3} \text{int} T_{1,2,3} \text{cl} A^c]^c \)

⇔ \( A \supset T_{1,2,3} \text{int} T_{1,2,3} \text{cl} T_{1,2,3} \text{int} A \)

**Theorem 6.5.37:**

Let \( T = \text{Set of all } T_1 T_2 T_3 \text{ open sets} \) and \( T^* = \text{Set of all } T_{1,2,3} \text{ open sets} \).

Then \( T \subset T^* \).

**Proof:**

Let \( A \in T \Rightarrow A \) is \( T_1 T_2 T_3 \text{ open} \).

\[ \Rightarrow A \in T_i \text{ for some } i = 1,2,3 \]

\[ \Rightarrow A = A \cup \emptyset \text{ where } A \in T_i \text{ and } \emptyset \in T_j \text{ } 1 \leq j \leq 3 \text{ and } j \neq i \]

\[ \Rightarrow A \text{ is } T_{1,2,3} \text{ open } \Rightarrow A \in T^* . \text{ Hence } T \subset T^* \]

**Theorem 6.5.38:**

Let \((X,T_1,T_2,T_3)\) be a tri topological space. For any \( A \subset X \)

\( T_1 T_2 T_3 \text{int} A \subset T_{123} \text{int} A \subset A \subset T_{123} \text{cl} A \subset T_1 T_2 T_3 \text{cl} A \).
Proof:

Every $T_1 T_2 T_3$ open set is $T_{1,2,3}$ open.

Set of all $T_1, T_2, T_3$ open sets is a subset of the set of all $T_{1,2,3}$ open sets.

Hence $T_1 T_2 T_3 \text{ int } A \subset T_{1,2,3} \text{ int } A$.

Set of all $T_1 T_2 T_3$ closed sets is a subset of the set of all $T_{1,2,3}$ closed sets.

Hence $T_{1,2,3} \text{ cl } A \subset T_1 T_2 T_3 \text{ cl } A$.

6.6 FUZZY TRI TOPOLOGICAL SPACES

We recall the definition of fuzzy topology.

Let $X$ be a non empty set. Let $\mathcal{F}$ be the collection of some fuzzy sets on $X$. $\mathcal{F}$ is called a fuzzy topology on $X$ if the following conditions are satisfied.

1. $0_X, 1_X \in \mathcal{F}$

2. $A_i \in \mathcal{F}$ for $i \in I \Rightarrow \bigcup A_i \in \mathcal{F}$ where $I$ is any index set.

3. $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$.

Definition 6.6.1:

Let $X$ be a non empty set. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be fuzzy topologies on $X$. Then $(X, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is called a fuzzy tri topological space.

Example 6.6.2: $X = \{a, b, c\}$

$A_1 : X \rightarrow [0,1]$ is defined as $A_2 : X \rightarrow [0,1]$ is defined as

$a \rightarrow .4$ $a \rightarrow .1$

$b \rightarrow .2$ $b \rightarrow .1$

$c \rightarrow .2$ $c \rightarrow .3$
A_3 : X \rightarrow [0,1] is defined as
\begin{align*}
a &\rightarrow .4 \\
b &\rightarrow .2 \\
c &\rightarrow .3
\end{align*}

A_4 : X \rightarrow [0,1] is defined as
\begin{align*}
a &\rightarrow .1 \\
b &\rightarrow .1 \\
c &\rightarrow .2
\end{align*}

F_1 = \{0_X, A_1, A_2, A_3, A_4, 1_X\} is a fuzzy topology.

F_2 = \{0_X, A_1, 1_X\} is a fuzzy topology.

F_3 = \{0_X, A_2, 1_X\} is a fuzzy topology.

(X, F_1, F_2, F_3) is called a fuzzy tri topological space

**Definition 6.6.3:**

Let (X, F_1, F_2, F_3) be a fuzzy tri topological space. A fuzzy set A on X is called a fuzzy tri α open set if

\[ A \subseteq \text{int } F_1 \text{ int } F_2 \text{ cl } F_3 \text{ int } A. \]

**Note 6.6.4:**

Let A be a non empty set in X. Then the characteristic function of A on X denoted as \( \chi_A \) is defined as

\[ \chi_A(x) = 1 \text{ if } x \in A \text{ and } 0 \text{ otherwise. } \]

**6.6.5 Properties of characteristic function:**

1. \( \chi_{A \cup B} = \chi_A \cup \chi_B \)

2. \( \chi_{A \cap B} = \chi_A \cap \chi_B \)

3. A = B iff \( \chi_A = \chi_B \)

4. A \subseteq B \iff \chi_A \subseteq \chi_B

**Theorem 6.6.6:**

Let (X, T_1, T_2, T_3) be a tri topological space. Let \( \chi_{T_i} = \{ \chi_A / A \in T_i \} \). Then (X, \( \chi_{T_1}, \chi_{T_2}, \chi_{T_3} \)) is a fuzzy tri topological space.
Proof:

$T_1$ is a topology on $X$. $\chi_{T_1} = \{ \chi_A / A \in T_1 \}$

From the properties of the characteristic function we see that $\chi_{T_1}$ is a fuzzy topology on $X$. Similarly $\chi_{T_2}, \chi_{T_3}$ are fuzzy topologies on $X$.

Hence $(X, \chi_{T_1}, \chi_{T_2}, \chi_{T_3})$ is a fuzzy tri topological space.

Note 6.6.7:

Let $(X,T)$ be a topological space and let $(X, F)$ be the corresponding fuzzy topological space. $F = \{ \chi_A / A \in T \}$. Then for any $A \subset X$.

1. $\chi_{T \text{ int } A} = F \text{ int } \chi_A$.
2. $\chi_{T \text{ cl } A} = F \text{ cl } \chi_A$.

Theorem 6.6.8:

Let $(X,T_1,T_2,T_3)$ be a tri topological space. $(X, F_1, F_2, F_3)$ be the corresponding fuzzy tri topological space where $F_i = \{ \chi_A / A \in T_i \} \ \forall \ i = 1,2,3$. If $A \subset X$ is tri $\alpha$ open then $\chi_A$ is fuzzy tri $\alpha$ open.

Proof:

$A$ is tri $\alpha$ open $\Rightarrow A \subset T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A$.

$$\Rightarrow \chi_A \subset \chi_{T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A}$$

$$\Rightarrow \chi_A \subset \chi_{T_1 \text{ int } B} \quad \text{where } B = T_2 \text{ cl } T_3 \text{ int } A.$$  

$$\Rightarrow \chi_A \subset F_1 \text{ int } \chi_B$$

$$\Rightarrow \chi_A \subset F_1 \text{ int } \chi_{T_2 \text{ cl } T_3 \text{ int } A}$$

$$\Rightarrow \chi_A \subset F_1 \text{ int } \chi_{T_2 \text{ cl } C} \quad \text{where } C = T_3 \text{ int } A.$$  

$$\Rightarrow \chi_A \subset F_1 \text{ int } F_2 \text{ cl } \chi_C$$
$\Rightarrow \chi_A \subset F_1 \text{int } F_2 \text{cl } \chi_{T_3 \text{int } A}$

$\Rightarrow \chi_A \subset F_1 \text{int } F_2 \text{cl } F_3 \text{int } \chi_A$

$\Rightarrow \chi_A$ is $F_{\text{tri } \alpha}$ open

Now $\chi_A \subset F_1 \text{int } F_2 \text{ cl } F_3 \text{ int } \chi_A$. Hence $\chi_A$ is fuzzy tri $\alpha$ open.

**Theorem 6.6.9:**

Let $(X, T_1, T_2, T_3)$ be a tri topological space. Let $(X, F_1, F_2, F_3)$ be the corresponding fuzzy tri topological space. If $\chi_A$ is fuzzy tri $\alpha$ open then $A$ is tri $\alpha$ open.

**Proof:**

Retracing the steps of proof of Theorem 6.6.8 we get the proof.

**Definition 6.6.10:**

Let $(X, F_1, F_2, F_3)$ be a fuzzy tri topological space. A fuzzy set on $X$ is called a fuzzy tri $\beta$ open set if $A \subset F_1 \text{ cl } F_2 \text{ int } F_3 \text{ cl } A$.

**Theorem 6.6.11:**

Let $(X, F_1, F_2, F_3)$ be a fuzzy tri topological space. Let $A \subseteq X$. $A$ is tri $\beta$ open iff $\chi_A$ is fuzzy tri $\beta$ open

**Proof:**

Proof is similar to Theorem 6.6.8

**6.7 $\alpha$ - Perfect Spaces:**

**Definition 6.7.1:**

A tri topological space $(X, T_1, T_2, T_3)$ is called $\alpha$ – perfect space if $A$ is tri open set iff $A$ is tri $\alpha$ open set for every $A \subseteq X$. 
Theorem 6.7.2:

Let \((X,T_1,T_2,T_3)\) is a tri topological space.

If \(T_3\) is indiscrete then \((X, T_1, T_2, T_3)\) is \(\alpha\) perfect.

Proof:

Take any \(A \subset X\) where \(A \neq \emptyset\) and \(A \neq X\).

Since \(A \neq X\) and \(T_3\) is indiscrete, \(T_3 \text{ int } A = \emptyset\).

\(T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A = \emptyset\).

\(A\) is not a subset of \(T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A\).

\(A\) is not tri \(\alpha\) open. The tri \(\alpha\) open sets are \(\emptyset\) and \(X\).

Also the tri open sets are \(\emptyset\) and \(X\), since \(T_3\) is indiscrete. Hence \(X\) is \(\alpha\) perfect.

Result 6.7.3:

Converse is not true.

Example 6.7.4:

Let \(X = \{a,b,c\}\)

\(T_1 = \{\emptyset, \{a\}, \{a,b\}, X\}, \quad T_2 = \mathcal{P}(X) \quad \text{and} \quad T_3 = \{\emptyset, \{a\}, X\}\)

The tri open sets are \(\emptyset\), \(\{a\}\) and \(X\).

The tri \(\alpha\) open sets also \(\emptyset\), \(\{a\}\) and \(X\).

Hence \(X\) is \(\alpha\) perfect but \(T_3\) is not indiscrete.

Theorem 6.7.5:

Let \((X,T_1,T_2,T_3)\) be a tri topological space.

If \(T_2\) is discrete then \(X\) is \(\alpha\) perfect.

Proof:

\(A\) is tri \(\alpha\) open \(\Rightarrow A \subset T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A\)

\(\Rightarrow A \subset T_1 \text{ int } T_3 \text{ int } A\)
\[ \Rightarrow A = T_1 \text{ int } T_3 \text{ int } A \]
\[ \Rightarrow A \text{ is } T_1 \text{open} & A \text{ is } T_3 \text{open}. \]
\[ \Rightarrow A \text{ is tri open.} \]

Also \( A \text{ is tri open} \Rightarrow A \text{ is tri } \alpha \text{ open.} \)

Hence \( A \text{ is tri } \alpha \text{ open iff } A \text{ is tri open.} \)

Hence \( X \text{ is } \alpha \text{ perfect.} \)

**Result 6.7.6:**

Converse is not true.

**Example 6.7.7:**

Let \( X = \{a,b\}. \) Let \( T_1 = \{\phi, \{a\}, X\}, \quad T_2 = \{\phi, \{b\}, X\}, \quad T_3 = \{\phi, X\} \)

Since \( T_3 \) is indiscrete tri open sets are \( \phi \) and \( X \) only. The tri \( \alpha \) open sets are \( \phi \) and \( X \) only.

Hence \( X \) is \( \alpha \)-perfect but \( T_2 \) is not discrete.

**6.8 \( \alpha \)g Perfect Spaces :-**

**Definition 6.8.1:**

Let \( (X,G_1,G_2,G_3) \) be a tri generalized topological space and \( (X,T_1,T_2,T_3) \) be the induced tri topological space. If each tri \( \alpha \) open set of \( (X,T_1,T_2,T_3) \) is tri \( \alpha \)g open set of \( (X,G_1,G_2,G_3) \) then \( X \) is called \( \alpha \)g-Perfect Space.

**Example 6.8.2:**

Let \( X = \{a,b\}. \) Let \( G_1 = \{\phi, \{a\}, \{b\}, X\} \)
\[ G_2 = \{\phi, \{a\}\}, \quad G_3 = \{\phi, \{b\}, X\} \]

Let \( T_1 = G_1 \quad \text{and} \quad T_2 = \{\phi, \{a\}, X\} \)
\[ T_3 = \{\phi, \{b\}, X\} \]

Tri \( \alpha \)g open sets are \( \phi, \{b\}, X. \)
Tri $\alpha$ open sets are $\emptyset$, \{b\}, X.

Hence X is $\alpha$g Perfect Space

**Theorem 6.8.3:**

Let $(X,G_1,G_2,G_3)$ be a tri generalized topological space. Let $(X,T_1,T_2,T_3)$ be the induced tri topological space. Then X is $\alpha$g-Perfect space if $G_2$ is a Topology.

**Proof:**

Let $A$ be tri $\alpha$g open.

Then $A \subset G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A$

$\subset G_1 \text{ int } G_2 \text{ cl } T_3 \text{ int } A$, since $G_3 \subset T_3$

$\subset G_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A$, since $G_2 = T_2$

$\subset T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A$, since $G_1 \subset T_1$

Therefore A is tri $\alpha$ open.

Hence X is $\alpha$g-Perfect.

**Result 6.8.4:** Converse is not true.

**Example 6.8.5:**

In the previous example 6.8.2 X is $\alpha$g-Perfect Space, but $G_2$ is not a topology.

**Theorem 6.8.6:**

Let $(X,G_1,G_2,G_3)$ be a tri generalized topological space. Let $(X,T_1,T_2,T_3)$ be the induced tri topological space. Then X is $\alpha$g-Perfect if $G_1$ and $G_3$ are Topologies.

**Proof:**

Let $A \subset X$ be tri $\alpha$ open.

$A \subset T_1 \text{ int } T_2 \text{ cl } T_3 \text{ int } A = G_1 \text{ int } T_2 \text{ cl } G_3 \text{ int } A$ since $G_1 = T_1$ and $G_3 = T_3$

Hence $A \subset G_1 \text{ int } G_2 \text{ cl } G_3 \text{ int } A$ since $T_2 \text{ cl } B \subset G_2 \text{ cl } B$ for any $B \subset X$.

Hence A is tri $\alpha$g open. Hence X is $\alpha$g-Perfect.