Chapter 3

CHANGES IN DIAGONAL ELEMENTS

3.1 INTRODUCTION

In the Hopfield network [Hopfield82] a neuron cannot give direct self-feedback. The absence of direct self-feedback is based on the concept of stability, capacity and local minima of the energy function [Hopfield82, Hopfield84, Amit89]. Although majority of researchers consider Hopfield network having no direct self-feedback, there is no biological evidence supporting this hypothesis. In fact, in certain biological studies it is observed that a neuron takes a feedback from itself directly [Carpenter90]. Chapter 2 reports results of some research articles [Houselander90, Prados89, Braham88, Sezan90, Gindi88, Yanai90] which consider Hopfield network with direct self-feedback. Moreover, some related concept of changes in threshold elements [Der92] has also been reported in Chapter 2.

In this chapter some experimental observations and theoretical conclusions on the study of direct self-feedback in Hopfield network is reported. Section 3.2 deals with the motivation for the study of changes in diagonal elements. Section 3.3, using an example introduces the concept of diagonal element changes and its effect on dynamics of Hopfield network. In Section 3.4 critical values for diagonal elements which provide the condition for changes of state of bipolar neurons are proposed. Theoretical basis of increase in value of diagonal elements leading to the neuron attaining no-change-state is given Section 3.5. Section 3.6 deals with the stability of a neuron given fixed state of one or more other neurons. A geometrical interpretation of neuron state changes is given in Section 3.7.
In Section 3.8 the critical values for changes in threshold are obtained and the effect of such changes is **analysed**. Section 3.9 illustrates the difference between the threshold changes and diagonal changes. In this section it is shown that these types of changes are not truly complementary. Section 3.10 deals with the study of energy function for changes in the diagonal elements. Section 3.3 to Section 3.7 and Section 3.10 reports the study of diagonal element changes in Hopfield network with bipolar neurons. Section 3.11 deals with changes in diagonal elements of Hopfield network with binary neurons. The conclusions of study of diagonal element changes are given in Section 3.12.

### 3.2 MOTIVATION FOR STUDY OF DIAGONAL CHANGES

Any learning rule to construct **synaptic** matrix makes use of the specified set of candidate state vectors. But the matrix so constructed need not have all these candidate state vectors as its stable states. **Moreover**, the introduction of new stable state vectors or the deletion of an existing stable state is done by making necessary changes in the synaptic matrix. This usually changes the stable status of other state vectors. Further, any study of dynamics of neural networks not only concentrates on the stable states but also concerns with equally important issues like basins of attraction, minimization of energy function etc. Hence, the learning rule which aims only at having a set of stable states may not provide proper (adequate) insight to the study of dynamics of neural networks. It may be required to have a separate study.

In [Gindi88] the Hopfield network with **non-zero** diagonal elements (with $J_{ii} = N$) is considered. It is shown that by allowing non-zero diagonal terms in the synaptic matrix the stable states of the network need not change. On the other hand, the **non-zero** diagonal network is shown to outperform the original Hopfield network [Gindi88]. The non-zero diagonal affects dynamics and can be effectively used to improve the recalling
ability of the Hopfield network [Yanai90]. It is proposed that the changes in the diagonal elements may be useful in obtaining a particular neural dynamics, stability of new candidate state vectors of a particular kind with little affect on the existing stable vectors, and removing the stable status of a particular kind of stable state vectors.

3.3 AN EXAMPLE

In this section the concept of diagonal element changes in Hopfield network is being introduced with the help of an example. Consider two Hopfield networks \((A, \theta)\) and \((B, \theta)\). The synaptic matrices \(A\) and \(B\), and threshold vector \(\theta\) are given below.

\[
A = \begin{pmatrix}
110 & 89 & 54 & -76 & -76 \\
89 & 116 & 12 & -25 & 19 \\
54 & 12 & 110 & -45 & -17 \\
-76 & -25 & -45 & 64 & -15 \\
-76 & 19 & -17 & -15 & 0 \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
110 & 89 & 54 & -76 & -76 \\
89 & 116 & 12 & -25 & 19 \\
54 & 12 & 0 & -45 & -17 \\
-76 & -25 & -45 & 64 & -15 \\
-76 & 19 & -17 & -15 & 110 \\
\end{pmatrix}
\]

\[
\theta = \begin{pmatrix}
40 \\
30 \\
20 \\
93 \\
56 \\
\end{pmatrix}
\]
Chapter 3. CHANGES IN DIAGONAL ELEMENTS

The matrix \( A \) differs from matrix \( B \) in only two diagonal terms \( J_{33} \) and \( J_{55} \). For matrix \( A \), \( J_{33} = 110 \) and \( J_{55} = 0 \) while for matrix \( B \), \( J_{33} = 0 \) and \( J_{55} = 110 \).

The neurons are considered to be bipolar neurons and asynchronous mode is used for updating the Hopfield network. Changes in neuron states are observed by considering the neuron state in initial vector \((IV)\) and neuron state in the corresponding stable state vector \((SV)\). Four possibilities for a neuron state changes are

- If \( IV_i \) is 1 and corresponding \( SV_i \) is 1 then value associated is 1.
- If \( IV_i \) is 1 and corresponding \( SV_i \) is -1 then value associated is 2.
- If \( IV_i \) is -1 and corresponding \( SV_i \) is 1 then value associated is 3.
- If \( IV_i \) is -1 and corresponding \( SV_i \) is -1 then value associates is 4

By observing each input vector and the corresponding stable state, a value 1, 2, 3 or 4 is associated with each neuron of the Hopfield network. For the Hopfield network \((A, \theta)\) and \((B, \theta)\) the observation of changes in input vector and corresponding stable vector are listed in Table 3.1. Summary of observations of neuron state changes in Hopfield network \((A, \theta)\) and \((B, \theta)\) is given in Table 3.2.

It can be observed from Table 3.1 and 3.2 that by changing the diagonal element \( J_{33} \) from 110 in \((A, \theta)\) to 0 in \((B, \theta)\) the number of changes in the neuron state of neuron 3 has increased from 2 out of 31 to 13 out of 31. It can also be observed that by changing the diagonal element \( J_{55} \) from 0 in \((A, \theta)\) to 110 in \((B, \theta)\) the number of changes in the neuron state of neuron 5 has decreased from 13 out of 31 to 6 out of 31.

Hence it is observed that by increasing the diagonal element value the corresponding neuron state is subjected to less changes. By decreasing the diagonal element value the corresponding neuron state is subjected to more changes.
### TABLE 3.1: OBSERVATION OF CHANGES IN NEURON STATES IN INPUT VECTOR AND CORRESPONDING STABLE STATE VECTOR

#### HOPFIELD NETWORK

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3.4 CRITICAL VALUE OF $J_a$ FOR BIPOLAR NEURONS

It can be observed from the example given in Section 3.3 that the higher the value of any diagonal element the lesser is the tendency of the corresponding neuron to change state. Thus, a critical value can be determined for a diagonal element $J_a$ so that beyond this value, the state of the corresponding bipolar neuron does not change. This critical value can be termed as no-change-critical-value and denoted as $J_a^{nc}$ for neuron $i$. This critical value depends on threshold associated with the neuron and the synaptic weights with other neurons.

Similarly it can also be observed that decrease in the value of a diagonal element increases the likelihood of change in the state of corresponding neuron. Hence, a critical
value can be reached so that, below this critical value the state of the neuron definitely changes. This critical value can be termed as sure-change-critical-value and is denoted for neuron \( t \) as \( J_t^{sc} \). These critical values \( J_t^{sc} \) and \( J_t^{nc} \) are estimated using sufficient conditions. But however, these are not necessary conditions.

The following Theorem 3.1 gives the estimate of \( J_t^{nc} \) and Theorem 3.2 gives estimates of \( J_t^{nc} \).

**THEOREM 3.1** In a Hopfield network with bipolar neurons, if the synaptic matrix satisfies \( \sum_{j=1,j\neq i}^N |J_{ij}| + \theta_i > 0 \), then neuron \( i \) does not change its state. The critical value \( J_t^{nc} \) is given by

\[
J_t^{nc} = \sum_{j=1,j\neq i}^N |J_{ij}| + |\theta_i|
\]

**Proof:** In order that neuron \( i \) does not change its state from \( \sigma_i = +1 \), it is necessary that

\[
J_{ii} + \sum_{j=1,j\neq i}^N J_{ij}\sigma_j - \theta_i \geq 0
\]

For the above expression to be satisfied it is necessary that

\[
J_{ii} \geq -\left( \sum_{j=1,j\neq i}^N J_{ij}\sigma_j - \theta_i \right)
\]  \( \text{(3.1)} \)

Similarly, in order that neuron \( i \) does not change its state from \( \sigma_i = -1 \) it is necessary that

\[
-J_{ii} + \sum_{j=1,j\neq i}^N J_{ij}\sigma_j - \theta_i < 0
\]

For the above expression to be satisfied it is necessary that

\[
J_{ii} > \sum_{j=1,j\neq i}^N J_{ij}\sigma_j - \theta_i
\]  \( \text{(3.2)} \)

It can be observed that

\[-|J_{ij}| < J_{ij}\sigma_j < |J_{ij}|, \quad \text{for any } \sigma_j \]
and

-1 ft 1 < -ft < 1 ft 1

Hence

\[- \sum_{j=1, j \neq i}^{N} |J_{ij}| - |\theta_i| \leq \frac{1}{E} \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i \leq \frac{1}{E} \sum_{j=1, j \neq i}^{N} |J_{ij}| + |\theta_i| \quad (3.3)\]

If

\[J_{ii} > \frac{1}{E} \sum_{j=1, j \neq i}^{N} |J_{ij}| + |\theta_i|\]

then

\[J_{ii} \geq \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i\]

This satisfies Condition 3.2.

If

\[J_{ii} > \frac{1}{E} \sum_{j=1, j \neq i}^{N} |J_{ij}| + |\theta_i|\]

then

\[-J_{ii} < -\sum_{j=1, j \neq i}^{N} |J_{ij}| - |\theta_i|\]

i.e.,

\[-J_{ii} < \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i\]

Hence

\[J_{ii} > -\left( \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i \right)\]

This satisfies Condition 3.1.

Hence the theorem is proved.

\[\square\]

**Theorem 3.2** In a Hopfield network with bipolar neurons, if the synaptic matrix satisfies the condition, \(J_{ii} < -\left( \sum_{j=1, j \neq i}^{N} J_{ij} I + 1 \text{ ft } 1 \right)\) then neuron s does change its state when updated.
The sure-change-critical-value is given by

\[ J_{ii} = -\left( \sum_{j=1,j \neq i}^{N} |J_{ij}| + |\theta_i| \right) \]

**Proof:** In order that neuron \( i \) changes its state from \( \sigma_i = +1 \), it is necessary that

\[ J_{ii} + \sum_{j=1,j \neq i}^{N} J_{ij}\sigma_j - \theta_i < 0 \]

For the above expression to be satisfied it is necessary that

\[ J_{ii} < -\left( \sum_{j=1,j \neq i}^{N} J_{ij}\sigma_j - \theta_i \right) \quad (3.4) \]

Similarly, in order that neuron \( i \) changes its state from \( \sigma_i = -1 \) it is necessary that

\[ -J_{ii} + \sum_{j=1,j \neq i}^{N} J_{ij}\sigma_j - \theta_i \geq 0 \]

For the above expression to be satisfied it is necessary that

\[ J_{ii} \leq \sum_{j=1,j \neq i}^{N} J_{ij}\sigma_j - \theta_i \quad (3.5) \]

Using similar argument as discussed in Theorem 3.1, it can be observed that,

\[ -1 \cdot J_{ij} < J_{ij}\sigma_j < 1 \cdot J_{ij} \cdot, \text{ for any } j \]

and

\[ -1 \cdot \theta_i < -\theta_i < 1 \cdot |\theta_i| \]

Hence

\[ -\sum_{j=1,j \neq i}^{N} |J_{ij}| - |\theta_i| \leq \sum_{j=1,j \neq i}^{N} J_{ij}\sigma_j - \theta_i \leq \sum_{j=1,j \neq i}^{N} |J_{ij}| + |\theta_i| \quad (3.6) \]

If

\[ J_{ii} < -\left( \sum_{j=1,j \neq i}^{N} |J_{ij}| + |\theta_i| \right) \]
then
\[ J_{ii} < \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i \]

This satisfies Condition 3.5.

If
\[ J_{ii} < -( \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j + \theta_i \) \]

i.e.

\[ -J_{ii} > \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i \]

then from inequality 3.6

\[ -J_{ii} > \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i \]

i.e.

\[ J_{ii} < -( \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j - \theta_i ) \]

This satisfies Condition 3.4.

Hence the Theorem is proved.

\[ \square \]

Based on the discussion in Theorem 3.1, it can be observed that each diagonal element reaches its critical value independent of other diagonal elements when the off-diagonal elements are kept unchanged. With all the diagonal elements greater than the respective no-change-critical-values, it can be ensured that all the state vectors are stable in the network.

Similarly based on the discussion in Theorem 3.2, it is observed that by having all the diagonal elements less than the respective sure-change-critical-value, it can be ensured that no state vector is stable in the network.
3.5 IMPACT OF DIAGONAL CHANGES ON NETWORK DYNAMICS

In this section an attempt is made to study the changes in dynamics of Hopfield network with changes in diagonal elements. The next state of a neuron being updated depends on present state of the neuron, corresponding diagonal element, the synaptic weights with other neurons, and corresponding threshold element. These factors determine the change or no-change status of a neuron in a particular time unit. A neuron of a network can belong to one of the following categories.

1. Sure-change state: The neuron changes its state whenever it is updated.

2. Flexible state: The neuron is in a change or no-change state in different time units depending on the factors listed above.

3. No-change state: The state of a neuron is not subjected to change when updated.

The diagonal element changes keeping other elements of synaptic matrix and threshold vector constant can be used to place a neuron in any one of these categories.

Consider a Hopfield network $(D,0)$ with all diagonal element values less than the corresponding $J_{ii}^*$. This network does not have any stable state. Any input vector to the network $(D,0)$ will oscillate between the vector and its complement. The network can be considered to have bidirectionally stable states. Bidirectionally stable states can be considered equivalent to the concept of stable states if such a situation occurs for some vectors in a network. But, a network like $(D,0)$ with all bidirectionally stable states, does not seem to have much practical value. The network with even one neuron in sure-change state will not have any direct stable state.

Let a Hopfield network $(E,9)$ with all neurons in flexible state is obtained by increasing the values of diagonal elements of the network $(D,6)$. The network $(E,9)$ has some stable states. By increasing the values of diagonal elements the network will have some more
stable states. Continuing this process further the **Hopfield** network (F, 0) can be attained. In network (F, 0) all diagonal elements are greater than corresponding $J_{ji}^{nc}$ and hence all possible input state vectors are stable. Any further increase in the values of diagonal elements will have no impact on the performance of the network. The class of Hopfield network between (E, 0) and (F, 0) are of special interest. This phenomenon of diagonal changes and changes in the stable states of the network can be used as a learning process. If at a stage of dynamics it is required that a neuron belongs to a particular category then the corresponding diagonal values can be accordingly changed.

For a 2-neuron Hopfield network this phenomenon is illustrated in Figure 3.1. At the stage I only one (-1,-1) of the four possible input vectors is stable. Neuron 1 is selected and the value of the corresponding diagonal element is increased. When the value of diagonal element corresponding to neuron 1 becomes more than the no-change-critical-value, (1,-1) also becomes a stable state of the network. This is due to the fact that the neuron 1 is in a no-change state. Whatever state that is associated with neuron 1 from the input vector, the neuron remains in the same state which is also reflected in the stable state. This is depicted as stage II of the network.

Similarly starting from stage I and increasing the value of the diagonal corresponding to neuron 2 it can be seen that the state (-1,1) also becomes a stable state of the network. This is represented as stage III in Figure 3.1. By having the diagonal elements corresponding to both the neurons, having values more than its no-change-critical-values all the input states become the stable states of the network. This situation is represented as stage IV in Figure 3.1.
Figure 3.1: Various stages of dynamics of two-neuron Hopfield network with changes in diagonal elements.
3.8 CONDITIONAL STABILITY

Increasing the value of a specific set of diagonal elements in a network with large number of neurons it is possible to ensure that the state of these neurons is not changed. However, the more desirable feature of such study is to make specific set of state vectors stable and not to make specific set of neurons stable independent of other neurons. For example in a 3-neuron Hopfield network, the first diagonal element can be increased so that first neuron does not change its state irrespective of the states of other neurons. \textbf{Whereas}, if the intention is to make a specific state vector (+1 -1 -1) to be stable and not (+1 +1 +1) then it is necessary to study a kind of \textit{conditional stability} of individual neurons and not the independent stability as discussed in earlier sections of this chapter. Conditional stability of a neuron \( t \) can be defined as the stability of a neuron \( t \) given a fixed state of one or more other neurons. However, such a study is much more complex than the one discussed for the independent stability. Another simple case namely \textit{pairwise conditional stability} of neurons is considered here. Critical value of the \( i^{th} \) diagonal element \( J_{ii} \) for a given state of neuron \( j \) is denoted as \( (J_{ii} \setminus j) \). The critical value of \( (J_{ii} | j) \) is characterised in the following theorem.

\textbf{THEOREM 3.3} The critical value of \( (J_{ii} \setminus j) \) such that neuron \( i \) will not change its state for a given state \( \sigma_j \) of neuron \( j \) but independent of other neurons is given by

\[
J_{ii} > \sum_{k=1, k \neq i,j}^{N} |J_{ik}| + |\theta_i| - \sigma_j J_{ij}
\]

The critical value \( (J_{ii}^c \setminus j) \) is given by

\[
(J_{ii}^c | j) = \sum_{k=1, k \neq i,j}^{N} |J_{ik}| + |\theta_i| - \sigma_j J_{ij}
\]

\textbf{Proof}:- In order that neuron \( t \) \textit{does} not change its state from \( \sigma_i = +1 \), it is necessary that

\[
J_{ii} + \sum_{k=1, k \neq i}^{N} J_{ik} \sigma_k - \theta_i \geq 0
\]
For the above expression to be true it is necessary that

\[ J_{ii} \geq - \left( \sum_{k=1, k \neq i}^{N} J_{ik} \sigma_k - \theta_i \right) \]

This implies

\[ J_{ii} > - \left( \sum_{k=1, k \neq i,j}^{N} J_{ik} \sigma_k - \theta_i \right) \cdot J_{ij} \sigma_j \quad (3.7) \]

Similarly, in order that neuron \( i \) does not change its state from \( \sigma_i = -1 \) it is necessary that

\[ -J_{ii} + \sum_{k=1, k \neq i}^{N} J_{ik} \sigma_k - \theta_i < 0 \]

For the above expression to be satisfied it is necessary that

\[ J_{ii} > \sum_{k \neq i,j}^{N} J_{ik} \sigma_k - \theta_i + J_{ij} \sigma_j \quad (3.8) \]

It can be observed that

\[ -|J_{ik}| < J_{ik} \sigma_k < |J_{ik}|, \quad \text{for any } \sigma_k \]

and

\[ -|\theta_i| \leq -\theta_i \leq |\theta_i| \]

Hence

\[ - \sum_{k=1, k \neq i,j}^{N} J_{ik} |\sigma_k| - |\theta_i| \leq \sum_{k=1, k \neq i,j}^{N} J_{ik} \sigma_k - \theta_i \leq \sum_{k=1, k \neq i,j}^{N} J_{ik} |\sigma_k| + |\theta_i| \quad (3.9) \]

if

\[ J_{ii} > \sum_{k=1, k \neq i,j}^{N} J_{ik} + |\theta_i| - \sigma_j J_{ij} \]

then

\[ -J_{ii} < - \sum_{k \neq i,j}^{N} J_{ik} |\sigma_k| - |\theta_i| + \sigma_j J_{ij} \]

i.e.,

\[ -J_{ii} < \sum_{k=1, k \neq i,j}^{N} J_{ik} \sigma_k - \theta_i + J_{ij} \sigma_j \]
This satisfies condition 3.7.

Similarly it can be shown that for \( J_{ii} > \sum_{k=1,k\neq i}^{N} | J_{ik} | + | \theta_i | - \sigma_i J_{ij}, \) condition 3.8 is satisfied.

**Hence** the theorem is proved.

\[
\begin{align*}
\text{In the same way as Theorem 3.3 a conditional, lower critical value } (J_{ii}^n \setminus j) \text{ for neuron } i \text{ can be determined so that } i^{th} \text{ neuron will surely change its state for all possible states of other neurons except neuron } j, \text{ which is given to be fixed at } \sigma_j. \\
\text{It is observed that pairwise conditional no-change-critical-value is smaller than the independent no-change-critical-value. It is also observed that pairwise conditional sure-change-critical-value is greater than the independent sure-change-critical-value. This is shown below.}
\end{align*}
\]

\[
| J_{ij} | \leq J_{ij} \sigma_j \quad \text{or} \quad | J_{ij} | \geq -J_{ij} \sigma_j
\]

**Hence**

\[
(J_{ii}^n \setminus i) < J_{ii}^n
\]

Similarly

\[
(J_{ii}^n \setminus j) > J_{ii}^n
\]

### 3.7 GEOMETRICAL INTERPRETATION

In this section the **no-change-state** of a neuron is interpreted geometrically. It is **observed** in Theorem 3.1 **that**, when \( J_{ii} > J_{ii}^n \) then due to **no-change-condition** we have,

\[
J_{ii} + \sum_{j=1,j\neq i}^{N} J_{ij} \sigma_j - \theta_i \geq 0
\]
irrespective of the value of $\sigma_j, j = 1, 2, \ldots, N$ and $j \neq i$. The inequality can be rewritten as
\[ \sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j \geq k \quad \text{where} \quad k = -J_{ii} + 0; \]
This inequality is satisfied by all values of $\sigma_j$ when $J_{ii} > J_{ii}^c$. The set of all possible values of $\sigma_j$ defines a $N - 1$ dimensional hypercube having vertices defined by $\sigma_j = \pm 1, j \neq i, j = 1, 2, \ldots, N$. In this space $\sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j = k$ defines a hyperplane and hence the inequality $\sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j > k$ defines a half-space in which the complete hypercube is contained. As the value of $k$ increases, the hyperplane approaches the hypercube and hence some vertices of the hypercube tend to violate the constraint $\sum_{j=1, j \neq i}^{N} J_{ij} \sigma_j \geq k$. With the increase of $k$ more and more vertices of the hypercube cross over to the other half-space. It is evident that increase in $k$ is also accomplished by decrease in $J_{ii}$. Thus in other words decrease in the value of $J_{ii}$ below $J_{ii}^c$ makes more and more neurons to deviate from no-change-state. Hence, more and more changes are observed in this process. This justifies the observation in the example given in section 3.3. The concept is illustrated in Figure 5.2.

3.8 THRESHOLD CHANGES AND CRITICAL VALUES

This section deals with the study of changes in threshold elements and its impact on the performance of Hopfield network. The observations and the conclusions in this section are restricted to a Hopfield network with all neurons belonging to the flexible category. Like critical values of $J_{ii}$'s, there also exist critical values of $\theta_i$'s. The critical values that can be associated with threshold vectors are defined and an estimate is derived for these critical values. Consider a Hopfield network with all neurons belonging to flexible
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The vertices in halfspace above the hyperplane \( k \)'s represent neurons in the no-change state.

\[
\sum_{j=1}^{N} J_{ji} \sigma_j = k
\]

Figure 3.2: Geometrical interpretation of Hopfield dynamics for \( N = 4 \) neurons.
category. If the threshold $\theta_i$ of neuron $i$ is increased to a very high value then,

$$ J_{ii} + \sum_{j=1,j\neq i}^{N} J_{ij} \sigma_j < 0 $$

Hence, the neuron $t$ changes its state only if $\sigma_i = +1$ and it will not change its state when $\sigma_i = -1$.

Similarly, if $\theta_i$ takes on very low value, then

$$ J_{ii} + \sum_{j=1,j\neq i}^{N} J_{ij} \sigma_j \geq \theta_i $$

Hence the neuron $t$ changes its state only if $\sigma_i = -1$ and remains in a no-change-state for $\sigma_i = 1$.

Thus, the change in the threshold values (keeping the elements of the synaptic matrix constant) ensures changes in the state of neuron only for one of the possible two states. That is, with large values of threshold the corresponding neuron will attain a $+1$ state irrespective of the state that is associated with it from the input vector. Similarly with a very low value of threshold the corresponding neuron will attain a $-1$ state irrespective of the state associated with the neuron from the input vector. The definitions of critical values and its estimates are given below.

**Plus one ($+1$) critical value of a threshold of a bipolar neuron ($\theta_i^+$):** The value of threshold of a neuron $t$ such that all values of threshold below this value will definitely guarantee that the neuron attains a $+1$ state. $\theta_i^+$ is estimated using a sufficient condition, but however, it is not a necessary condition that $\theta_i$ should be less than $\theta_i^+$ to ensure that the neuron $i$ definitely attains a $+1$ state. This critical value is given by

$$ \theta_i^+ = -\sum_{j=1}^{N} |J_{ij}| $$

**Minus one ($-1$) critical value of threshold of a bipolar neuron ($\theta_i^-$):** The value of threshold of a neuron $t$ such that all values of threshold above this value will definitely
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guarantee that the neuron attains a -1 state. \( \theta_i^M \) is estimated using a sufficient condition, but however, it is not a necessary condition that \( \theta_i \) should be greater than \( \theta_i^* \) to ensure that the neuron \( i \) definitely attains a -1 state. This critical value is given by

\[
\theta_i^M = \frac{1}{N} \sum_{j=1}^{N} |J_{ij}|
\]

By having threshold values of all neurons greater than \( \theta_i^p \) it is possible to make the vector with all elements as +1, as the only stable state of the network. Similarly by having threshold values of all neurons less than \( \theta_i^m \) it is possible to make the vector with all elements as -1, as the only stable state of the network.

3.9 DIAGONAL CHANGES vs THRESHOLD CHANGES

In literature the diagonal element value and threshold value of a neuron are considered to be complementary. In [Hecht-Nielsen91] it has been mentioned that instead of having an explicit threshold, a zero threshold value can be used by changing the diagonal elements of synaptic matrix. It is proposed here that the diagonal element values and the threshold values are not truly complementary.

As mentioned in earlier sections, the changes in the diagonal elements can lead to a situation where all input vectors are stable or an unstable situation where the network has no stable state. These situations are attained just by diagonal element changes keeping all other aspects of the Hopfield network constant. This is different from the effect on dynamics of Hopfield network due to the changes in the threshold elements keeping all other aspects of the network constant. The changes in threshold elements can affect only one of the values associated with the neuron. A threshold value less than \( \theta_i^p \) ensures that the neuron \( i \) ultimately attains a +1 state irrespective of initial state. Similarly threshold values greater than \( \theta_i^m \) ensures that the neuron \( i \) ultimately attains a -1 state irrespective of initial state. With changes in threshold element it is possible to
that one type of change of neuron state i.e., +1 to -1 or -1 to +1 does not occur whereas the changes of other type occurs with high probability. Thus it can be concluded that changes in diagonal element are different from changes in the threshold elements for a neuron. These two changes are not truly complementary.

Adjusting diagonal element values and threshold values in a Hopfield network may help in obtaining the required dynamics. These changes can play an important role in obtaining the required computational performance of the Hopfield network.

### 3.10 DIAGONAL CHANGES AND ENERGY FUNCTION

The Hopfield network, in fact, performs a local search where the neighbourhood of the search is the immediate neighbourhood of a vertex on the $N$ dimensional hypercube. The asynchronous mode of computation at any node $i$ can be viewed as the comparison of the energy $E$ at the current vertex and at the adjacent vertex in the $i^{th}$ direction. This comparison is not affected if we add a constant term to $E$.

Let,

$$E_i = E + \frac{1}{2} J_{ii}$$

Then, obviously the changes in energy function $E$ is same as the changes in the energy function $E_i$ for a Hopfield network i.e., $\Delta E = \Delta E_i$.

#### 3.10.1 Change in Diagonal Elements of Synaptic Matrix

Thus it is observed that if the energy function is changed from $E$ to $E_i$, the difference in energy at two consecutive network states is not affected. For the energy function $E_i$, even if some real number is added to the diagonal elements, the energy value does not change at any state. Thus it can be concluded that, in asynchronous mode of operation, the energy function $E$ of any Hopfield network $(J, \theta)$ converges to a constant value, if and
only if, the energy function converges for any other Hopfield network \( N(r) = (J(r), \theta) \).

In other words, by adding some values to the diagonal elements of the synaptic matrix, the set of locally minimizing states of the energy function is not changed. For a synaptic matrix \( J \) if the set of locally minimizing states is denoted by \( L \). The following phenomena can be observed.

1. If \( J \) has zero diagonal elements then the set of local minima \( L \) is also the set of stable states.

2. If \( J \) has strictly positive diagonal elements then each element of \( L \) is a stable state and may be some elements which are not in \( L \) are also stable states. Moreover, the stability is ensured for asynchronous mode of operation of the Hopfield network [Bruck88].

3. If there is no restriction on the diagonal elements of \( J \) and if \( J' \) is obtained by adding some elements to the diagonal elements so that these become zero then the set of locally minimizing states \( L \) is same for \( J \) and \( J' \). But the stable states of \( J' \) are also the stable states of \( J \) and there are some additional stable states in \( J \) which do not correspond to locally minimizing points. The state transition paths are different for both the matrices as the updating rule is affected by change in diagonal elements. However, any Hopfield network, having synaptic matrix with non-zero diagonal elements, can be transformed to a Hopfield network having zero diagonal elements so that the stable states of the transformed Hopfield network corresponds to the locally minimizing states of original Hopfield network.

4. As the updating is affected by the change in the diagonal elements, the state transition for \( J \) and \( J' \) are different. Hence for two matrices \( J \) and \( J' \) differing only in diagonal entries, the associating functions of input state vector to output stable
vector are different, even though the stable states are common to both matrices. Hence, getting new stable states for any arbitrary matrix (even with unrestricted diagonal elements) is possible by adding large positive number to the diagonal elements. But however, such a scheme is useful only when the study is restricted to the set of stable states and not for associating input state to stable states.

Discussion in this section about the changes in diagonal elements of the Hopfield network is applicable when the collection of stable states of the network are considered. These observations cannot be directly extended to other areas like associative memory. The collection of stable states is of interest in the context of capacity of Hopfield network.

3.11 DIAGONAL CHANGES IN NETWORK WITH BINARY NEURONS

This section deals with aspect of diagonal element changes for Hopfield network with binary neurons are reported in this section. For the neuron state +1 the analysis is same as given in the previous sections. But for the neuron state 0, its product with the corresponding diagonal element becomes 0. In this case the local field of the neuron does not receive contribution from the diagonal element. Thus the diagonal tuning mechanism used for bipolar neurons have a limited role in case of binary neurons. This requires a different study of diagonal element changes for binary neurons.

Experiments have been conducted by starting from a high negative value for diagonal elements in the synaptic matrix for binary and bipolar neurons. The diagonal elements were gradually increased and its effect on the performance of network with bipolar and binary neurons has been observed. Asynchronous mode of operation with maximum absolute value of local field as the basis of selection has been used in these experiments. It has been observed that a Hopfield network with binary neurons attain stability earlier than bipolar neurons. This is because the values of diagonals do not have an impact
when neuron state is 0 for binary neurons.

3.12 CONCLUSION

In this chapter some experimental observations and theoretical conclusions of the study of Hopfield network is reported. It is concluded that the changes in diagonal elements and the threshold elements can be used for tuning the Hopfield network to obtain required performance. Critical values of diagonal elements (no-change-critical value and sure-change-critical value) and threshold elements ($\theta^P_i$ and $\theta^M_i$) for bipolar neurons have been estimated. It has been observed that pairwise conditional no-change-critical-value is smaller than the independent no-change-critical-value. It is also observed that pairwise conditional sure-change-critical-value is greater than the independent sure-change-critical-value. The effect of diagonal element changes in a network with binary neurons have been observed to be different from the effect of these changes on a network with bipolar neurons. The effect of diagonal changes in Hopfield network on the energy function have been studied.