CHAPTER V
HYDROMAGNETIC TRANSIENT FLOW OF VISCOELASTIC FLUID
DOWN AN INCLINED PLANE

1. INTRODUCTION

Rheology is the science of deformation and flow of matter. The aim of Rheology is to predict the deformation on flow resulting from the application of a given force system to a body or vice versa. Rheology comprises Elasticity, Fluid Dynamics and Plasticity. The sciences of Rheology and mechanics can be seen to lay the foundation to such diverse fields as hydraulics, strength of materials, structural engineering, Fluid mechanics and Plasticity. The subject of Rheology is of great technological importance as in many branches of industry the problem arises of designing apparatus to transport or to process substances which cannot be governed by the classical stress-strain velocity relations. Examples of such substances and processes are many, the extrusion of plastics, in the manufacture of rayon, nylon or other textile fibres, viscoelastic effects are encountered when the spinning solutions are transported or forced through spinnerets and in the manufacture of lubricating greases and rubber. Further viscoelastic fluids occur in the food industry, e.g. emulsions, pastes and condensed milk. Ramdas Sen [4] has studied the viscoelastic free convection boundary layer flow past an infinite plate with constant suction.

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Johri \(2\) has considered the flow of viscoelastic fluid induced by elliptic harmonic oscillations of a disc. He \(3\) also studied the unsteady channel flow of an elastico viscous liquid. Bhatia and Usha Rani \(1\) have considered the MHD unsteady Hele–Shaw flow of viscoelastic fluid. Sneddon \(5\) has studied the flow of viscous incompressible fluid down inclined plane using Fourier transform technique.

In this chapter, we study the flow of viscoelastic fluid of Maxwell type down an inclined plane under the influence of uniform transverse magnetic field. The fluid is supposed to be flowing through two heated parallel planes which are inclined. The lower plate is fixed and upper plate is moving with a transient velocity. We have evaluated velocity, liquid discharge per second, drag, temperature distribution and heat transfer. We have investigated the effects of magnetic field, relaxation time and Prandtl number on the above physical quantities.

2. FORMULATION AND SOLUTION OF THE PROBLEM

A viscoelastic liquid of the Maxwell type (that is a spring and dash-pot arranged in series) is characterised by the equations
\[ \tau_{ij} = -p \delta_{ij} + \tau'_{ij} \]  

\[ (1 + \lambda \frac{\partial}{\partial t}) \tau'_{ij} = 2 \mu \varepsilon_{ij} \]  

\[ \varepsilon_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \]

where \( \tau'_{ij} \) is the deviatoric stress-tensor, \( \varepsilon_{ij} \) is the rate of strain-tensor, \( p \) is the pressure, \( \lambda \) the relaxation time, \( \mu \) the viscous parameter and \( v_i (i = 1, 2) \) are the velocity components.

We take the origin at the mid point of the distance between the two parallel plates and \( X \) axis in the direction of motion and \( Y \) axis perpendicular to it. We introduce a uniform magnetic field of intensity \( H_0 \) in the \( Y \) direction. Now the components of velocity are \( u(y,t,0) \) in \( X \) and \( Y \) directions respectively. We assume that the fluid is of small electrical conductivity with magnetic Reynolds number much less than unity so that the induced magnetic field can be neglected in comparison with the applied magnetic field (Sparrow and Cess \( \tau' \)). The equations governing the motion and energy of viscoelastic fluid of Maxwell type under the influence of uniform transverse magnetic field (in the absence of any input electric field) are

\[ (1 + \lambda \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = \rho \sin \alpha \frac{1}{\rho} (1 + \lambda \frac{\partial}{\partial t}) \frac{\partial u}{\partial x} + \]  

\[ + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} - \frac{\sigma \mu_{e}^{2} H^{2}}{\rho} u \]  

\[ (2.2) \]
\[ 0 = g \cos \alpha + \frac{1}{\rho} \left( 1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial P}{\partial y} \]  \hspace{1cm} (2.3)

\[ \frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial y^2} + k'' \left( \frac{\partial u}{\partial y} \right)^2 \]  \hspace{1cm} (2.4)

where \( t \) is the time, \( g \) the acceleration due to gravity, \( \alpha \) the inclination of the plate to horizon, \( \lambda \) the coefficient of kinematic viscosity, \( \sigma \) the electrical conductivity of the fluid, \( \mu_e \) the magnetic permeability, \( \rho \) the density, \( k \) the thermal conductivity of the fluid, \( c \) the specific heat, \( k' = k/\sigma c \) and \( k'' = \mu/\rho c \).

From equation (2.2) it follows that

\[ g \sin \alpha = \frac{1}{\rho} \left( 1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial P}{\partial x} \]  is a function of \( t \) alone and we can write

\[ P = g \rho (x \sin \alpha - y \cos \alpha) + x \rho d \]  \hspace{1cm} (2.5)

where \( d \) is in general a function of time.

Now equation (2.2) reduces to

\[ (1 + \lambda \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = - \left( 1 + \lambda \frac{\partial}{\partial t} \right) d + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma^2 u}{\rho} \]  \hspace{1cm} (2.6)

The boundary conditions are

\[ u = 0 \hspace{0.5cm} \text{at} \hspace{0.5cm} y = -h \]  \hspace{1cm} (2.7a)

\[ u = U = f_1(y) e^{-\omega t} \hspace{0.5cm} \text{at} \hspace{0.5cm} y = h \]  \hspace{1cm} (2.7b)

\[ T = T_0 = f_2(y) e^{-2 \omega t} \hspace{0.5cm} \text{at} \hspace{0.5cm} y = \pm h \]  \hspace{1cm} (2.7c)
We introduce the following non dimensional quantities:

\[ u^* = \frac{u}{u}, \quad y^* = \frac{y}{h}, \quad t^* = \frac{t}{h}, \quad \lambda^* = \frac{\lambda}{h} \]

\[ d^* = \frac{d}{u^2}, \quad \omega^* = \frac{\omega}{u}, \quad T^* = \frac{T}{T_0} \]  \hspace{1cm} (2.8)

In view of (2.8) the equations (2.6) and (2.4) become (dropping superscripts *):

\[ (1 + \lambda \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = - (1 + \lambda \frac{\partial}{\partial t}) d + \frac{1}{R} \frac{\partial^2 y}{\partial y^2} - \mu u \]  \hspace{1cm} (2.9)

\[ \frac{\partial^2 T}{\partial y^2} = P_r \frac{R}{\partial t} \frac{\partial T}{\partial t} - P_r \frac{\partial (\frac{\partial u}{\partial y})^2}{\partial y} \]  \hspace{1cm} (2.10)

where

\[ M = \frac{c \mu^2 \rho^2 \eta^2 h}{\rho u} \] \hspace{1cm} (magnetic parameter)

\[ R = \frac{\rho u h}{y} \] \hspace{1cm} (Reynolds number)

\[ P_r = \frac{\mu c}{k} \] \hspace{1cm} (Prandtl number)

\[ E = \frac{\rho u^2}{c T_0} \] \hspace{1cm} (Eckert number)

The boundary conditions in non dimensional form are

\[ u = 0 \text{ at } y = -1 \]  \hspace{1cm} (2.11a)

\[ u = 1 = f_1(yh) \ e^{-\omega t} \text{ at } y = 1 \]  \hspace{1cm} (2.11b)

\[ T = 1 = f_0(yh) \ e^{2\omega t} \text{ at } y = \pm 1 \]  \hspace{1cm} (2.11c)
Now we consider a particular case for which \( d = d_0 \, e^{-\omega t} \), where \( d_0 \) and \( \omega \) are constants.

The equation (2.9) reduces to

\[
(1 + \lambda \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = -H e^{-\omega t} + \frac{1}{R} \frac{\partial^2 y}{\partial y^2} - Hu
\]  

(2.12)

where \( H = d_0 (1 - \lambda \omega) \)

Substituting \( u = f(y) e^{-\omega t} \), the equation (2.12) becomes

\[
\frac{\partial^2 f}{\partial y^2} + a^2 f = b
\]  

(2.13)

where \( a^2 = R (\omega (1 - \lambda \omega) - H) \)

\[
b = HR
\]  

(2.11a) and (2.11b) can be written as

\[
f = 0 \quad \text{at} \quad y = -1
\]  

(2.14)

\[
f = f_1 \quad \text{at} \quad y = 0
\]

Solving equation (2.13), using boundary conditions (2.14) we obtain

\[
f = \frac{f_1}{2} \left( \frac{\sin y}{\sin a} + \frac{\cos y}{\cos a} \right) + \frac{b}{a^2} (1 - \frac{\cos y}{\cos a})
\]  

(2.15)

Now the velocity distribution is

\[
u = \left\{ \frac{f_1}{2} \left( \frac{\sin y}{\sin a} + \frac{\cos y}{\cos a} \right) + \frac{b}{a^2} (1 - \frac{\cos y}{\cos a}) \right\} e^{-\omega t}
\]  

(2.16)

The discharge or flux of the fluid per second is

\[
Q = \int_{-1}^{1} u \, dy
\]

\[
= \left\{ \left( \frac{f_1}{a} - \frac{2b}{a^3} \right) \tan a + \frac{2b}{a^2} \right\} e^{-\omega t}
\]  

(2.17)
The drag per unit width and along the unit length, of the upper plane is

\[ \tau = \mu \left( \frac{\partial u}{\partial y} \right)_y = 1 \]

\[ = \left\{ \frac{\mu f_1 a}{2} (\cot a - \tan a) + \frac{\mu b}{a} \tan a \right\} e^{-\omega t} \quad (2.18) \]

The work expanded is

\[ w = \frac{\mu f_1 a^2}{a} \left\{ \frac{f_1 a^2}{2} \cot a + \frac{(2b - f_1 a^2)}{2} \tan a \right\} e^{-2\omega t} \quad (2.19) \]

**Temperature distribution:**

The equation (2.10) becomes

\[ \frac{\partial^2}{\partial y^2} - \frac{\partial T}{\partial t} = \frac{P_R}{\partial R} - \frac{P_E}{\partial t} \left[ \frac{f_1 a \cos a y}{2} - \frac{\sin a y}{2} \right] + \frac{b \sin a y}{a \cos a} \quad (2.20) \]

Substituting \( T = f(y) e^{-2\omega t} \) in equation (2.20), we obtain

\[ \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} = P_1 (1 + \cos 2ay) + P_2 (1 - \cos 2ay) + P_3 \sin 2ay \quad (2.21) \]

where

\[ a_2^2 = 2 \frac{P_E}{\partial x} + \omega \]

\[ P_1 = \frac{P_E \cdot \frac{a^2 f_1^2}{2}}{\sin a} \]
\[ P_2 = - \frac{P_1 E (2b - a^2 f_1)^2}{8a^2 \cos^2 a} \]

\[ P_3 = - \frac{P_1 E f_1 (2b - a^2 f_1)}{2 \sin 2a} \]

Now the boundary conditions are

\[ f = f_0 \text{ at } y = \pm 1 \]  \hspace{1cm} (2.22)

Solving equation (2.21) and using (2.22) we obtain

\[ f = (f_0 - \ell - m \cos 2a) \frac{\cos a_1 y}{\cos a_1} - \frac{n \sin 2a \sin a_1 y}{\sin a_1} + \ell + \]

\[ + m \cos 2a y + n \sin 2a y \]  \hspace{1cm} (2.23)

where

\[ \ell = \frac{P_1 + P_2}{a_1^2}, \quad m = \frac{P_1 - P_2}{(a_1^2 - 4a^2)} \]

\[ n = \frac{P_3}{(a_1^2 - 4a^2)} \]

Now the temperature distribution is

\[ T = \left[ (f_0 - \ell - m \cos 2a) \frac{\cos a_1 y}{\cos a_1} - \frac{n \sin 2a \sin a_1 y}{\sin a_1} + \ell + \right. \]

\[ + m \cos 2a y + n \sin 2a y \]

\[ \left. \right] e^{-2 \omega t} \]  \hspace{1cm} (2.24)

The rate of heat transfer:

From the point of view of applications in technology

it is of interest to know the rates of heat transfer \( q \) and \( q^* \) at lower and upper planes.
The rate of heat transfer coefficient at the lower plane is

\[ q = \left( \frac{\partial T}{\partial y} \right)_{y = -1} \]

\[ = \left( f_0 - l - n \cos 2a \right) a_1 \tan \theta_1 - n a_1 \sin 2a \cot \theta_1 + \]

\[ + 2 a_1 \sin 2a + 2 n \cos 2a \right] e^{-2 \omega t} \]  \hspace{1cm} (2.25)

The rate of heat transfer coefficient at the upper plane is

\[ q^* = \left( \frac{\partial T}{\partial y} \right)_{y = 1} \]

\[ = - \left[ \left( f_0 - l - n \cos 2a \right) a_1 \tan \theta_1 + n a_1 \sin 2a \cot \theta_1 + \right] \]

\[ + 2 a_1 \sin 2a - 2 n \cos 2a \right] e^{-2 \omega t} \]  \hspace{1cm} (2.26)

3. CONCLUSIONS

**Velocity distribution:** (Figure 1 and Table 1)

In figure 1, we have drawn velocity \( u \) against \( y \) for different values of magnetic parameter \( M \). We have observed that the velocity decreases with the increase in \( M \) and this decrement decreases as \( M \) increases. It is also seen that the velocity at every point in the upper half of the flow field is more than the velocity at the corresponding points in the lower half of the flow field. From table 1, we observe that the velocity decreases with the increase in the relaxation time \( \lambda \). We have also shown the velocities at
various points of the flow field for oldroyd fluid 
( $\lambda = 0.065$). Further we see that the velocity at every 
point in the upper half of the flow field is more than the 
velocity at the corresponding points in the lower half of 
the flow field.

**Flux of the fluid per second**: (Figures 2 and 3)

In figure 2, flux per second $Q$ is drawn against $M$ 
and in figure 3, $Q$ is drawn against $\lambda$. Closely following 
the figures 2 and 3, we observe that $Q$ decreases as $M$ or $\lambda$ 
increases.

**Drag per unit width**: (Figures 4 and 5)

In figure 4, drag $C$ is plotted against $M$ whereas $C$ is 
plotted against $\lambda$ in figure 5. We have seen that $C$ increase 
with the increase in $M$ or $\lambda$.

**Temperature distribution**: (Figures 6 to 8)

In figure 6, the temperature $T$ is plotted against $y$ 
for different values of $M$. Closely observing the figure 
we see that the temperature decreases with the increase 
in $M$ in the lower half of the flow field whereas this trend 
gets reversed at points very near the upper plate. In 
figure 7, $T$ is plotted against $y$ for different values of $\lambda$. 
It is observed that the temperature increases with the 
increase in $\lambda$. We have also shown the temperature 
distribution for the oldroyd fluid. Further we notice that
the temperature at every point in the upper half of the flow field is more than the temperature at the corresponding points in the lower half of the flow field. In figure 8, T is plotted against y for different values of $P_x E$ (Product of Prandtl and Eckert numbers). We have seen that the temperature decreases with the increase in $P_x E$ in the lower half of the flow field whereas this trend gets reversed at points very near to the upper plate.

The rate of heat transfer: (Figures 9 and 10)

In figure 9, the rate of heat transfer coefficients $q$ and $q^*$ are drawn against $M$. We have noticed that both $q$ and $q^*$ decrease with the increase in $M$. In figure 10, $q$ and $q^*$ are plotted against $\lambda$. It is found that $q$ increases with the increase in $\lambda$ whereas $q^*$ decreases.
FIG. 1 Velocity profiles for different values of $M$.
FIG. 2 Q plotted against M

FIG. 3 Q plotted against λ

1.5
1.4
1.3
0.05 0.1 0.15

1.6
1.2
0.8
0.4
0.1

0
2 3 4

M

10^{-2}

10^{-1}

10^0
FIG. 7 Temperature distribution for different values of $\lambda$
FIG. 8. Temperature profiles for different values of $P_e$: $P_e = 1, 2, 3, 4$. $T \times 10^4$ is shown on the y-axis.
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REFERENCES


