CHAPTER - I

PRELIMINARIES, BASIC CONCEPTS AND CONTRIBUTIONS

Brief description and introduction 1.1: The notion of intuitionistic fuzzy sets introduced by Atanassov [1986, 1989, 1994] is one among them. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degrees of membership and of non-membership. Both degrees belong to the interval $[0; 1]$, and their sum should not exceed 1.

BCK-algebras and BCI-algebras are two important classes of logical algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras.


Section 1.2: Intuitionistic fuzzy translations of intuitionistic fuzzy T-ideal in BCI-algebra

In the second chapter, intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy T-ideals in BCK/BCI-algebras are discussed. Relations among intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy T-ideals in BCK/BCI-algebras are also investigated.

In this section, some elementary aspects that are necessary are included.

In a BCI-algebra $X$, define a partial ordering “$\leq$” by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra $X$ satisfies $0 * x = 0$ for all $x \in X$, then $X$ is a BCK-algebra.

Any BCK-algebra $X$ satisfies the following axioms for all $x, y, z \in X$.

(1) $(x * y) * z = (x * z) * y$; (2) $((x * z) * (y * z)) * (x * y) = 0$

(3) $x * 0 = x$; (4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0$, $(z * y) * (z * x) = 0$.

Throughout this chapter, $X$ always means a BCK/BCI-algebra without any specification.

**Definition 1.2.1:** A non-empty subset $S$ of $X$ is a sub-algebra of $X$ if $x * y \in S$ for any $x, y \in S$.

**Definition 1.2.2:** A non-empty subset $I$ of $X$ is said to be a T-ideal of $X$ if it satisfies (I1) and (I3) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

**Definition 1.2.3:** A BCI-algebra is associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

**Definition 1.2.4:** A fuzzy set $A = \{< x, \mu_A(x) > : x \in X\}$ in $X$ is called a fuzzy subalgebra of $X$ if it satisfies the inequality $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

**Definition 1.2.5:** A fuzzy set $A = \{< x, \mu_A(x) > : x \in X\}$ in $X$ is a fuzzy ideal of $X$ if it satisfies
(F1) $\mu_A(0) \geq \mu_A(x)$ and (F2) $\mu_A(x) \geq \min \{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

**Definition 1.2.6:** A fuzzy set $A = \{< x, \mu_A(x), v_A(x) > : x \in X\}$ in $X$ is a fuzzy $T$-ideal of $X$ if it satisfies

(F1) and (F3) $\mu_A(x * z) \geq \min \{\mu_A((x * y)* z), \mu_A(y)\}$ for all $x, y, z \in X$.

**Definition 1.2.7:** An intuitionistic fuzzy set $A = \{< x, \mu_A(x), v_A(x) > : x \in X\}$ in $X$ is an intuitionistic fuzzy sub-algebra of $X$ if it satisfies the following two conditions:

(F4) $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ and (F5) $v_A(x * y) \leq \max \{v_A(x), v_A(y)\}$ for all $x, y \in X$.

**Definition 1.2.8:** An intuitionistic fuzzy set $A = \{< x, \mu_A(x), v_A(x) > : x \in X\}$ in $X$ is called an intuitionistic fuzzy ideal [12] of $X$ if it satisfies

(F6) $\mu_A(0) \geq \mu_A(x)$, $v_A(0) \leq v_A(x)$, (F7) $\mu_A(x) \geq \min \{\mu_A(x * y), \mu_A(y)\}$ and

(F8) $v_A(x) \leq \max \{v_A(x * y), v_A(y)\}$ for all $x, y \in X$.

**Definition 1.2.9:** An intuitionistic fuzzy set $A = \{< x, \mu_A(x), v_A(x) > : x \in X\}$ in $X$ is called an intuitionistic fuzzy T-ideal of $X$ if it satisfies (F6) and

(F9) $\mu_A(x * z) \geq \min \{\mu_A((x * y) * z), \mu_A(y)\}$

(F10) $v_A(x * z) \leq \max \{v_A((x * y) * z), v_A(y)\}$ for all $x, y, z \in X$.

**Notation 1.2.10:** The symbol $A = (\mu_A, v_A)$ is used for the intuitionistic fuzzy subset $A = \{< x, \mu_A(x), v_A(x) > : x \in X\}$. Throughout this chapter, $\Xi = \inf \{v_A(x)|x \in X\}$ for any intuitionistic fuzzy set $A = (\mu_A, v_A)$ of $X$.

**Definition 1.2.11:** Let $A = (\mu_A, v_A)$ be an intuitionistic fuzzy subset of $X$ and $\alpha \in [0, \Xi]$. An object having the form $A_{\alpha}^T = ((\mu_A)^T_{\alpha}, (v_A)^T_{\alpha})$ is called an intuitionistic fuzzy $\alpha$–translation of $A$ if

$(\mu_A)^T_{\alpha}(x) = \mu_A(x) + \alpha$ and $(v_A)^T_{\alpha}(x) = v_A(x) - \alpha$, for all $x \in X$. 

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Contribution on $\alpha$–translation of intuitionistic fuzzy T-ideal

**Theorem 1.2.12:** If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy T-ideal of $X$, then the intuitionistic fuzzy $\alpha$–translation $A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha})$ of $A$ is an intuitionistic fuzzy T-ideal of $X$ for all $\alpha \in [0, 1]$. 

**Theorem 1.2.13:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha$–translation $A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha})$ of $A$ is an intuitionistic fuzzy T-ideal of $X$ for some $\alpha \in [0, 1]$. Then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy T-ideal of $X$.

**Theorem 1.2.14:** If the intuitionistic fuzzy $\alpha$-translation $A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha})$ of $A$ is an intuitionistic fuzzy T-ideal of $X$ for some $\alpha \in [0, \mathfrak{T}]$, then it must be an intuitionistic fuzzy ideal of $X$.

**Theorem 1.2.15:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha$–translation $A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha})$ of $A$ is an intuitionistic fuzzy T-ideal of $X$ for some $\alpha \in [0, \mathfrak{T}]$. If $(x * b) * a = 0$, for all $x, a, b \in X$, then $(\mu_A)^T_{\alpha}(x) \geq \min\{((\mu_A)^T_{\alpha}(a), (\mu_A)^T_{\alpha}(b))\}$ and $(\nu_A)^T_{\alpha}(x) \leq \max\{((\nu_A)^T_{\alpha}(a), (\nu_A)^T_{\alpha}(b))\}$.

**Corollary 1.2.16:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha$–translation $A^T_{\alpha} = ((\mu_A)^T_{\alpha}, (\nu_A)^T_{\alpha})$ of $A$ is an intuitionistic fuzzy T-ideal of $X$ for some $\alpha \in [0, \mathfrak{T}]$. If $(... (x * a_1) * ... ) * a_n = 0$ for all $x, a_1, a_2, ... a_n \in X$, then

$(\mu_A)^T_{\alpha}(x) \geq \min\{((\mu_A)^T_{\alpha}(a_1), (\mu_A)^T_{\alpha}(a_2), .... (\mu_A)^T_{\alpha}(a_n))\}$ and

$(\nu_A)^T_{\alpha}(x) \leq \max\{((\nu_A)^T_{\alpha}(a_1), (\nu_A)^T_{\alpha}(a_2), .... (\nu_A)^T_{\alpha}(a_n))\}$. 

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**Result 1.2.17:** Now a condition for the statement “an intuitionistic fuzzy ideal of X to be an intuitionistic fuzzy T-ideal of X” for an intuitionistic fuzzy $\alpha$—translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is given.

**Further properties on $\alpha$—translation of intuitionistic fuzzy T-ideal**

**Theorem 1.2.18:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy $\alpha$—translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy T-ideal of X for $\alpha \in [0, \mathfrak{T}]$. If it satisfies the condition $(\mu_A)^T_\alpha(x \ast y) \geq (\mu_A)^T_\alpha(x)$ and $(\nu_A)^T_\alpha(x \ast y) \leq (\nu_A)^T_\alpha(x)$ for all $x, y \in X$, then the intuitionistic fuzzy $\alpha$—translation $A^T_\alpha$ of A is an intuitionistic fuzzy T-ideal of X.

**Theorem 1.2.19:** If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of associative BCI-algebra X such that the intuitionistic fuzzy $\alpha$—translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy T-ideal of X for $\alpha \in [0, \mathfrak{T}]$, then the intuitionistic fuzzy $\alpha$—translation $A^T_\alpha$ of A is an intuitionistic fuzzy T-ideal of X.

**Theorem 1.2.20:** If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X such that the intuitionistic fuzzy $\alpha$—translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy T-ideal of X for $\alpha \in [0, \mathfrak{T}]$, then the sets $T_{\mu_A} = \{x | x \in X and (\mu_A)^T_\alpha(x) = (\mu_A)^T_\alpha(0)\}$ and $T_{\nu_A} = \{x | x \in X and (\nu_A)^T_\alpha(x) = (\nu_A)^T_\alpha(0)\}$ are T-ideals of X.
Theorem 1.2.21: Let the intuitionistic fuzzy $\alpha -$translation $A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A be an intuitionistic fuzzy $T-$ideal of $X$ for $\alpha \in [0, \mathcal{I}]$. If $x \leq y$, then $(\mu_A)_\alpha^T(x) \geq (\mu_A)_\alpha^T(y)$ and $(\nu_A)_\alpha^T(x) \leq (\nu_A)_\alpha^T(y)$ [($(\mu_A)_\alpha^T$ is order-reversing and $(\nu_A)_\alpha^T$ is order-preserving)].

Theorem 1.2.22: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha -$translation $A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A is an intuitionistic fuzzy $T-$ideal of $X$ for $\alpha \in [0, \mathcal{I}]$. Then the following assertions are equivalent:

(i) $A^T_\alpha$ is an intuitionistic fuzzy $T-$ideal of $X$.

(ii) $(\mu_A)_\alpha^T((x \ast y) \ast z) \geq (\mu_A)_\alpha^T((x \ast z) \ast (y \ast 0))$ and

$(\nu_A)_\alpha^T((x \ast y) \ast z) \leq (\nu_A)_\alpha^T((x \ast z) \ast (y \ast 0))$, for all $x, y, z \in X$.

(iii) $(\mu_A)_\alpha^T((x \ast y) \ast z) \geq \min\{(\mu_A)_\alpha^T((x \ast z) \ast (y \ast 0)), (\mu_A)_\alpha^T(z)\} \text{ and }$

$(\nu_A)_\alpha^T((x \ast y) \ast z) \leq \max\{(\nu_A)_\alpha^T((x \ast z) \ast (y \ast 0)), (\nu_A)_\alpha^T(z)\}$ for all $x, y, z \in X$.

Theorem 1.2.23: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of $X$ such that the intuitionistic fuzzy $\alpha -$translation $A^T_\alpha = ((\mu_A)_\alpha^T, (\nu_A)_\alpha^T)$ of A is an intuitionistic fuzzy $T-$ideal of $X$ for $\alpha \in [0, \mathcal{I}]$. Then the following assertions are equivalent:

(i) $A^T_\alpha$ is an intuitionistic fuzzy $T-$ideal of $X$.

(ii) $(\mu_A)_\alpha^T((x \ast z) \ast y) \geq (\mu_A)_\alpha^T((x \ast z) \ast (y \ast 0))$ and

$(\nu_A)_\alpha^T((x \ast z) \ast y) \leq (\nu_A)_\alpha^T((x \ast z) \ast (y \ast 0))$ for all $x, y \in X$.

(iii) $(\mu_A)_\alpha^T((x \ast y) \ast z) \geq \min\{(\mu_A)_\alpha^T((x \ast z) \ast (y \ast 0)), (\mu_A)_\alpha^T(z)\}$ and

$(\nu_A)_\alpha^T((x \ast y) \ast z) \leq \max\{(\nu_A)_\alpha^T((x \ast z) \ast (y \ast 0)), (\nu_A)_\alpha^T(z)\}$ for all $x, y, z \in X$. 
Properties on intuitionistic fuzzy T-ideal extension

**Definition 1.2.24:** Let $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ be two intuitionistic fuzzy subset of $X$. If $A \leq B$, $\mu_A(x) \leq \mu_B(x)$ and $\upsilon_A(x) \geq \upsilon_B(x)$ for all $x \in X$, then $B$ is an intuitionistic fuzzy extension of $A$.

**Definition 1.2.25:** Let $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ be two intuitionistic fuzzy subset of $X$. Then $B$ is an intuitionistic fuzzy T-ideal extension of $A$ if the following assertions are valid:

(i) $B$ is an intuitionistic fuzzy extension of $A$.

(ii) If $A$ is an intuitionistic fuzzy T-ideal of $X$, then $B$ is an intuitionistic fuzzy T-ideal of $X$.

From the definition of intuitionistic fuzzy $\alpha$-translation, we get $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$ and $(\upsilon_A)^T_\alpha(x) = \upsilon_A(x) - \alpha$ for all $x \in X$.

**Theorem 1.2.26:** Let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy T-ideal of $X$ and $\alpha \in [0, \infty)$. Then the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\upsilon_A)^T_\alpha)$ of $A$ is an intuitionistic fuzzy T-ideal extension of $A$. An intuitionistic fuzzy T-ideal extension of an intuitionistic fuzzy T-ideal $A$ may not be represented as an intuitionistic fuzzy $\alpha$-translation of $A$, that is, the converse of Theorem is not true in general as seen in the following example.
Theorem 1.2.27: let $A = (\mu_A, \nu_A)$ the intuitionistic fuzzy T-ideal of X and $\alpha \in [0, \Xi]$. Then the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy T-ideal extension of A. Then intersection of intuitionistic fuzzy T-ideal extensions of an intuitionistic fuzzy T-ideal A of X is an intuitionistic fuzzy T-ideal extension of A. But the union of intuitionistic fuzzy T-ideal extensions of an intuitionistic fuzzy T-ideal A of X is not an intuitionistic fuzzy T-ideal extension of A as seen in the following example.

Definition 1.2.28: For an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of X, $\alpha \in [0, \Xi]$ and $t, s \in [0,1]$ with $t \geq \alpha$, Let $U_\alpha(\mu_A; t) = \{x | x \in X \text{ and } \mu_A(x) \geq t - \alpha\}$ and

$$L_\alpha(\nu_A; s) = \{x | x \in X \text{ and } \nu_A(x) \geq s + \alpha\}$$

Theorem 1.2.29: If A is an intuitionistic fuzzy T-ideal of X, then it is clear that $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are T-ideals of X for all $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t \geq \alpha$. But, if we do not give a condition that A is an intuitionistic fuzzy T-ideal of X, then $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are not T-ideals of X as seen in the following example.

Level sets in intuitionistic fuzzy $\alpha$-translation

Theorem 1.2.29: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and $\alpha \in [0, \Xi]$. Then the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A)^T_\alpha, (\nu_A)^T_\alpha)$ of A is an intuitionistic fuzzy T-ideal of X if and only if $U_\alpha(\mu_A; t)$ and $L_\alpha(\nu_A; s)$ are T-ideals of X for $t \in Im(\mu_A)$ and $s \in Im(\nu_A)$ with $t \geq \alpha$. 

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**Theorem 1.2.30:** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy T-ideal of $X$ and let $\beta \in [0, \Xi]$. For every intuitionistic fuzzy T-ideal extension $B = (\mu_B, \nu_B)$ of the intuitionistic fuzzy $\beta$-translation $A^T_\beta = ((\mu_A^T_\beta, (\nu_A^T_\beta)$ of A, there exists $\alpha \in [0, \Xi]$ such that $\alpha \geq \beta$ and B is an intuitionistic fuzzy T-ideal extension of the intuitionistic fuzzy $\alpha$-translation $A^T_\alpha = ((\mu_A^T_\alpha, (\nu_A^T_\alpha)$ of A.

**Section 1.3: Direct product of intuitionistic fuzzy T-ideals of BCI-algebra**

**Aim:** In the third chapter, the notion of direct product of intuitionistic fuzzy T-ideal is introduced in BCI-algebras and some related properties are investigated. Characterizations of direct product of intuitionistic fuzzy T-ideals of BCI-algebras are given. And also the notion of upper s-level cut of $\mu_{A \times B}$ and lower t-level cut of $\lambda_{A \times B}$ are introduced. Further $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$ for an intuitionistic fuzzy set of BCI-algebras $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B)$ if and only if upper and lower level sets are T-ideals of BCI-algebra $X_1 \times X_2$ for any $s, t \in [0, 1]$.

**Definition 1.3.1:** An intuitionistic fuzzy ideal $A = (\mu_A, \lambda_A)$ of $X$ is an intuitionistic fuzzy closed-ideal of $X$ if the following axiom satisfies:

$$\text{(IF4)} \quad \mu_A(0 \ast x) \geq \mu_A(x) \text{ and } \lambda_A(0 \ast x) \leq \lambda_A(x), \text{ for all } x \in X.$$ 

**Definition 1.3.2:** An intuitionistic fuzzy subset $A = (\mu_A, \lambda_A)$ of a BCI-algebra $X$ is an intuitionistic fuzzy T-ideal of $X$ if

$$\text{(IFT1)} \quad \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x),$$

$$\text{(IFT2)} \quad \mu_A(x \ast z) \geq \min \{\mu_A((x \ast y) \ast z), \mu_A(y)\},$$

$$\text{(IFT3)} \quad \lambda_A(x \ast z) \leq \max \{\lambda_A((x \ast y) \ast z), \lambda_A(y)\}, \text{ for all } x, y, z \in X.$$
**Definition 1.3.3:** An intuitionistic fuzzy subset $A = (\mu_A, \lambda_A)$ of a BCI-algebra $X$ is an intuitionistic fuzzy closed $T$–ideal of $X$, if it satisfies (IFT2), (IFT3) and the following:

(IFT4) $\mu_A(0 \ast x) \geq \mu_A(x)$ and $\lambda_A(0 \ast x) \leq \lambda_A(x)$, for all $x \in X$

**Theorem 1.3.4:** Every intuitionistic fuzzy ideal of BCI-algebra $X$ is intuitionistic fuzzy sub-algebra of $X$.

**Theorem 1.3.5:** A non-empty subset $I$ of BCI-algebra $X$ is an ideal of $X$ if and only if $\bar{I} = \langle X_p, \bar{X_p} \rangle$ is an intuitionistic fuzzy ideal.

**Direct product of intuitionistic Fuzzy T-ideal**

**Definition 1.3.6:** Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are two intuitionistic fuzzy sets in BCI-algebras $X_1$ and $X_2$ respectively. The direct product of intuitionistic fuzzy sets $A$ and $B$ is denoted by $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$, and defined as $\mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$ and $\lambda_{A \times B}(x, y) = max\{\lambda_A(x), \lambda_B(y)\}$ for all, $(x, y) \in X_1 \times X_2$.

**Definition 1.3.7:** An IFS $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ of $X_1 \times X_2$ is an intuitionistic fuzzy sub-algebra of $X_1 \times X_2$ if (DIF1) $\mu_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \geq min\{\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2)\}$

(DIF2) $\lambda_{A \times B}((x_1, y_1) \ast (x_2, y_2)) \leq max\{\lambda_{A \times B}(x_1, y_1), \lambda_{A \times B}(x_2, y_2)\}$,

for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. 

Definition 1.3.8: An IFS \( A \times B = (\mu_{A \times B}, \lambda_{A \times B}) \) of \( X_1 \times X_2 \) is an intuitionistic fuzzy T-ideal of \( X_1 \times X_2 \) if (DIF3) \( \mu_{A \times B}(0,0) \geq \mu_{A \times B}(x,y) \) and \( \lambda_{A \times B}(0,0) \leq \lambda_{A \times B}(x,y) \).

(DIF4) \( \mu_{A \times B}((x_1,y_1) \ast (x_3,y_3)) \geq \min\{\mu_{A \times B}((x_1,y_1) \ast (x_2,y_2)) \ast (x_3,y_3)), \mu_{A \times B}(x_2,y_2)) \}

(DIF5) \( \lambda_{A \times B}((x_1,y_1) \ast (x_3,y_3)) \leq \max\{\lambda_{A \times B}((x_1,y_1) \ast (x_2,y_2)) \ast (x_3,y_3)), \lambda_{A \times B}(x_2,y_2)) \}, \)

for all \((x_1,y_1),(x_2,y_2),(x_3,y_3) \in X_1 \times X_2 \).

Definition 1.3.9: An IFS \( A \times B = (\mu_{A \times B}, \lambda_{A \times B}) \) of \( X_1 \times X_2 \) is an intuitionistic fuzzy closed T-ideal of \( X_1 \times X_2 \) if it satisfies the (DIF3), (DIF4) and (DIF5) and the following

(DIF6) \( \mu_{A \times B}((0,0) \ast (x,y)) \geq \mu_{A \times B}(x,y) \) and \( \lambda_{A \times B}((0,0) \ast (x,y)) \leq \lambda_{A \times B}(x,y) \)

Theorem 1.3.10: Let \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) be two intuitionistic fuzzy sub-algebra of BCI-algebras \( X_1 \) and \( X_2 \) respectively. Then \( A \times B = (\mu_{A \times B}, \lambda_{A \times B}) \) is an intuitionistic fuzzy sub-algebra of BCI-algebra \( X_1 \times X_2 \).

Theorem 1.3.11: Let \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) be two intuitionistic fuzzy ideals of BCI-algebras \( X_1 \) and \( X_2 \) respectively. Then \( A \times B = (\mu_{A \times B}, \lambda_{A \times B}) \) is an intuitionistic fuzzy ideal of BCI-algebra \( X_1 \times X_2 \).

Properties on direct product of intuitionistic fuzzy T-ideals

Theorem 1.3.12: Let \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) are two intuitionistic fuzzy T-ideals of BCI-algebras \( X_1 \) and \( X_2 \) respectively. Then \( A \times B = (\mu_{A \times B}, \lambda_{A \times B}) \) is an intuitionistic fuzzy T-ideal of BCI-algebra \( X_1 \times X_2 \).
**Theorem 1.3.13**: Let $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be two intuitionistic fuzzy closed T-ideals of BCK/BCI-algebras $X_1$ and $X_2$, respectively. Then $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ is an intuitionistic fuzzy closed T-ideal of BCI-algebra $X_1 \times X_2$.

**Lemma 1.3.14**: If $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ is intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$. Then $(a, b) \leq (x, y) \Rightarrow \mu_{A \times B}(x, y) \geq \mu_{A \times B}(a, b)$ and $\lambda_{A \times B}(x, y) \leq \lambda_{A \times B}(a, b)$ for all $(a, b) \in X_1 \times X_2$.

**Theorem 1.3.15**: Let $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ and $C \times D = \langle \mu_{C \times D}, \lambda_{C \times D} \rangle$ is an intuitionistic fuzzy closed T-ideal of BCI-algebras $X_1$ and $X_2$ respectively. Then $A \times B) \cap (C \times D) = \langle \mu_{(A \times B) \cap (C \times D)}, \lambda_{(A \times B) \cap (C \times D)} \rangle$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$.

**Theorem 1.3.16**: Let $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be two intuitionistic fuzzy T-ideals of BCI-algebras $X_1$ and $X_2$ respectively. Then $\cap(A \times B) = \langle \mu_{A \times B}, \overline{\mu}_{A \times B} \rangle$ is intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$, where $\overline{\mu}_{A \times B} = 1 - \mu_{A \times B}$.

**Properties on some new products in intuitionistic fuzzy T-ideals**

**Theorem 1.3.17**: Let $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be two intuitionistic fuzzy T-ideals of BCI-algebras $X_1$ and $X_2$ respectively. Then $\cap(A \times B) = \langle \overline{\lambda}_{A \times B}, \lambda_{A \times B} \rangle$ is intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$, where $\overline{\lambda}_{A \times B} = 1 - \lambda_{A \times B}$.

**Theorem 1.3.18**: Let $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ is an intuitionistic fuzzy T-ideals of BCK-algebras $X_1$ and $X_2$ respectively. Then $(A \times B)^m = \langle \mu_{(A \times B)^m}, \lambda_{(A \times B)^m} \rangle$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$. 

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Theorem 1.3.19: Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two intuitionistic fuzzy closed T-ideals of BCI-algebras $X_1$ and $X_2$ respectively. Then $A \times B = (\lambda_{A \times B}, \lambda_{A \times B})$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$ if and only if $A \times B = (\mu_{A \times B}, \mu_{A \times B})$ and $A \times B = (\lambda_{A \times B}, \lambda_{A \times B})$ are intuitionistic fuzzy T-ideals of BCI-algebra $X_1 \times X_2$.

Theorem 1.3.20: If $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ and $C \times D = (\mu_{C \times D}, \lambda_{C \times D})$ is an intuitionistic fuzzy closed T-ideal of BCI-algebras $X_1$ and $X_2$ respectively. Then $(A \times B) \cup (C \times D) = (\mu_{(A \times B) \cup (C \times D)}, \lambda_{(A \times B) \cup (C \times D)})$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$.

Upper and lower level sets

Definition 1.3.21: Let $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in BCI-algebra $X$. The set $U(\mu_A, t) = \{x \in X: \mu_A(x) \geq t\}$ is the upper $t$-level of $\mu_A$ and $L(\lambda_A, s) = \{x \in X: \lambda_A(x) \leq s\}$ is the upper $s$-level of $\lambda_A$ for any $s, t \in [0, 1]$.

Definition 1.3.22: Let $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ is an intuitionistic fuzzy set in BCI-algebra $X_1 \times X_2$. For any $s, t \in [0, 1]$, the set $U(\mu_{A \times B}, t) = \{(x, y) \in X_1 \times X_2: \mu_{A \times B}(x) \geq t\}$ is the upper $t$-level of $\mu_{A \times B}$ and the set $L(\lambda_{A \times B}, s) = \{(x, y) \in X_1 \times X_2: \lambda_{A \times B}(x) \leq s\}$ is the upper $s$-level of $\lambda_{A \times B}$.

Theorem 1.3.23: Let $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$. Then $A \times B = (\mu_{A \times B}, \lambda_{A \times B})$ is an intuitionistic fuzzy T-ideal of BCI-algebra $X_1 \times X_2$ if and only if, upper and lower level sets are T-ideals of BCI-algebra $X_1 \times X_2$ for $s, t \in [0, 1]$. 

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**Theorem 1.3.24:** Let $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ be an intuitionistic fuzzy set of BCI-algebra $X_1 \times X_2$. Then $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ is an intuitionistic fuzzy $T$-ideal of BCI-algebra $X_1 \times X_2$ if and only if upper level and lower level sets are $T$-ideals of BCI-algebra $X_1 \times X_2$ for any $s, t \in [0, 1]$.

**Theorem 1.3.25:** Let $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ be an intuitionistic fuzzy set of BCI-algebra $X_1 \times X_2$. Then $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ is an intuitionistic fuzzy closed $H$-ideal of BCI-algebra $X_1 \times X_2$ if and only if upper level and lower level sets are $T$-ideals of BCI-algebra $X_1 \times X_2$ for any $s, t \in [0, 1]$.

**Theorem 1.3.26:** If $A \times B = \langle \mu_{A \times B}, \lambda_{A \times B} \rangle$ be an intuitionistic fuzzy $T$-ideal of BCI-algebra $X_1 \times X_2$, then the sets $I = \{ (x, y) \in X_1 \times X_2 / \mu_{A \times B}(x, y) = \mu_{A \times B}(0,0) \}$ and $K = \{ (x, y) \in X_1 \times X_2 / \lambda_{A \times B}(x, y) = \lambda_{A \times B}(0,0) \}$ are $T$-ideal of $X_1 \times X_2$.

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**Section 1.4: Structures on fuzzy ideal and fuzzy a-ideal**

**Aim:** In Tavoosi and Kordi [2008], they introduced the notions of $T$-fuzzy a-ideals in BCI-algebra, and proved few of their properties. In this paper, some algebraic properties on fuzzy ideals and fuzzy a-ideals of BCI-algebra are investigated.

**Definition 1.4.1:** A non-empty subset $I$ of $X$ is an ideal of $X$ if it satisfies:

(I1) $0 \in I$ and (I2) $x * y \in I, y \in I$ imply $x \in I$.  

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Definition 1.4.2: A non-empty subset $I$ of $X$ is an $a$-ideal of $X$ if it satisfies condition (I1) and (I3) $(x * z) * (0 * y) \in I, z \in I$ imply $y * x \in I$.

Definition 1.4.3: A fuzzy set $A$ with membership function $\mu_A$ in a BCI-algebra $X$ is a fuzzy $a$-ideal of $X$ if (Fa1) $\mu_A(0) > \mu_A(x)$; (Fa2) $\mu_A(y * x) > \min\{\mu_A((x * z) * (0 * y)), \mu_A ss(z)\}$ for all $x, y, z \in X$.

Definition 1.4.4: Let $A$ and $B$ be two fuzzy ideal of BCI algebra $X$. The fuzzy set $(A \cap B)$ with membership function $\mu_{A \cap B}$ is defined by $\mu_{A \cap B}(x) = \min \{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

Definition 1.4.5: Let $A$ and $B$ be two fuzzy ideal of BCI algebra $X$. The fuzzy set $(A \cup B)$ with membership function $\mu_{A \cup B}$ is defined by $\mu_{A \cup B}(x) = \max \{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

Definition 1.4.6: Let $A$ and $B$ be two fuzzy ideal of BCI algebra $X$ with membership functions $\mu_A$ and $\mu_B$ respectively. Then $A$ is contained in $B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

Definition 1.4.7: Let $A$ be a fuzzy ideal of BCI-algebra $X$. The fuzzy set $A^m$ with membership function $\mu_{A^m}$ is defined by $\mu_{A^m}(x) = (\mu_A(x))^m$, for all $x \in X$.

Some algebraic properties on fuzzy ideals

Theorem 1.4.8: If $A$ is a fuzzy ideal of BCI algebra $X$, then $A^m$ is a fuzzy ideal of $X$.

Theorem 1.4.9: Let $A$, $B$ be two fuzzy ideal of BCI- algebra $X$. Then $A \cap B$ is a fuzzy-ideal of $X$.

Theorem 1.4.10: Let $A$ and $B$ be two fuzzy ideal of BCI algebra $X$. If one is contained in other, then $A \cup B$ is a fuzzy-ideal of $X$. 

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Theorem 1.4.11: If $A$ is a fuzzy $a$-ideal of BCI algebra $X$, then $A^m$ is a fuzzy $a$-ideal of $X$.

Theorem 1.4.12: If $A$, $B$ be two fuzzy $a$-ideal of BCI algebra $X$, $A \cap B$ is a fuzzy $a$-ideal of $X$.

Theorem 1.4.13: If $A$ and $B$ be two fuzzy $a$-ideal of BCI algebra $X$, If one is contained in other, then $A \cup B$ is a fuzzy $a$-ideal of $X$.

Theorem 1.4.14: Let $\mu$ and $\nu$ be fuzzy ideals of a BCI-algebra $X$ such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If $\mu$ is a fuzzy a-ideal of $X$, then so is $\nu$.

Properties on level sets in fuzzy $a$-ideal

Theorem 1.4.15: A fuzzy set $\mu$ of a BCI-algebra $X$ is a fuzzy $a$-ideal of $X$ if and only if $\mu_t$ is either empty or a $-a$-ideal of $X$ for all $t \in [0, 1]$.

Theorem 1.4.16: Let $I$ be an $a$-ideal of a BCI-algebra $X$. Then there exists a fuzzy $a$-ideal $\mu$ of $X$ such that $\mu_t = I$, for some $t \in (0, 1]$.

Theorem 1.4.17: A fuzzy $a$-ideal $\mu$ of a BCI-algebra $X$ is a fuzzy sub algebra of $X$.

Theorem 1.4.18: Let $X$ be a BCI-algebra. Then the following are equivalent:

(i) every $a$-ideal of $X$ is closed; (ii) every fuzzy $a$-ideal of $X$ is a closed fuzzy ideal of $X$.

Theorem 1.4.19: Let $\mu$ be a $-a$-fuzzy ideal of a BCI-algebra $X$. If for any $x, y \in X,\mu(x \ast (x \ast (y \ast (y \ast x)))) \geq \mu((x \ast (x \ast y)) \ast (y \ast x))$ then $\mu$ is a $-a$-fuzzy BCI-positive implicative ideal.
Properties on cosets in fuzzy $a$-ideal

**Definition 1.4.20:** Let $\mu$ be a fuzzy ideal of a BCI-algebra $X$ and $x \in X$. Then the fuzzy subset $\mu x$ which is defined by $\mu x(y) = \min\{\mu(x \ast y), \mu(y \ast x)\}$ is a fuzzy coset of $\mu$. The set of all fuzzy cosets of $\mu$ is denoted by $X/\mu$.

**Theorem 1.4.21:** A BCI-algebra is positive implicative (weakly positive implicative) if and only if it satisfies $x \ast y = ((x \ast y) \ast y) \ast (0 \ast y)$.

**Theorem 1.4.22:** Let $\mu$ be a closed fuzzy ideal of a BCI-algebra $X$. Then $\mu$ is a fuzzy BCI-positive implicative ideal if and only if $(X/\mu, \ast, \mu_0)$ is a positive implicative BCI-algebra.

**Theorem 1.4.23:** Let $\mu$ be a fuzzy BCI-positive implicative ideal and $\mu$ be a closed fuzzy ideal of a BCI-algebra $X$. Then the following are equivalent:

(i) $\mu$ is fuzzy BCI-positive implicative; (ii) For all $x, y \in X, \mu((x \ast (x \ast y)) \ast (y \ast x)) = \mu(x \ast (x \ast (y \ast x))))$.

Section 1.5: Interval-valued Intuitionistic fuzzy $a$-ideal of BCI-algebra

**Definition 1.5.1:** An intuitionistic fuzzy set $A=<\mu_A, \nu_A>$ of a BCI-algebra $X$ is an intuitionistic fuzzy $a$-ideal if it satisfies (F1) and (F4) $\mu_A(y\ast x) \geq \min\{\mu_A((x\ast z)\ast (0\ast y)), \mu_A(z)\}$, (F5) $\nu_A(y\ast x) \leq \max\{\nu_A((x\ast z)\ast(0\ast y)), \nu_A(z)\}$, for all $x, y, z \in X$.

**Definition 1.5.2:** An interval-valued intuitionistic fuzzy set $A$ defined on $X$ is given by
A = \{(x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]) : \forall x \in X\}, where \(\mu_A^L\) and \(\mu_A^U\) are two membership functions and \(\nu_A^L\) and \(\nu_A^U\) are two non-membership functions of X such that \(\mu_A^L \leq \mu_A^U\) and \(\nu_A^L \geq \nu_A^U\), \forall x \in X.

Let \(\bar{\mu}_A(x) = [\mu_A^L, \mu_A^U]\) and \(\bar{\nu}_A(x) = [\nu_A^L, \nu_A^U]\), \forall x \in X and let D \([0,1]\) denote the family of all closed subintervals of \([0,1]\). If \(\mu_A^L(x) = \mu_A^U(x) = c, 0 \leq c \leq 1\) and if \(\nu_A^L(x) = \nu_A^U(x) = k, 0 \leq k \leq 1\), then we have \(\bar{\mu}_A(x) = [c, c]\) and \(\bar{\nu}_A(x) = [k, k]\).

Thus \(\bar{\mu}_A(x)\) and \(\bar{\nu}_A(x)\) \(\in [0,1]\), \forall x \in X\), and therefore the interval-valued intuitionistic fuzzy set A is given by \(A=\{(x, \bar{\mu}_A(x), \bar{\nu}_A(x)) : \forall x \in X\}\), where \(\bar{\mu}_A : X \rightarrow D[0,1]\) and \(\bar{\nu}_A : X \rightarrow D\ [0,1]\). Now let us define what is known as refined minimum, refined maximum of two elements in D \([0,1]\). we also define the symbols “\(\leq\)”, “\(\geq\)” and “\(=\)” in the case of two elements in D \([0,1]\).

Consider two elements \(D_1 = [a_1, b_1]\) and \(D_2 = [a_2, b_2] \in D[0,1].\)

Then \(\text{r min} (D_1, D_2) = [\min \{a_1, a_2\}, \min \{b_1, b_2\}]\),
And \(\text{r max} (D_1, D_2) = [\max \{a_1, a_2\}, \max \{b_1, b_2\}]\)

\(D_1 \geq D_2 \iff a_1 \geq a_2, b_1 \geq b_2\);

\(D_1 \leq D_2 \iff a_1 \leq a_2, b_1 \leq b_2\) and \(D_1 = D_2\).

**Definition 1.5.3:** An interval-valued intuitionistic fuzzy set A = \{(x, \bar{\mu}_A(x), \bar{\nu}_A(x)) : \forall x \in X\} in BCI-algebra X is an interval-valued intuitionistic fuzzy a-ideal of X if it satisfies

\((\text{FI}_1) \bar{\mu}_A(0) \geq \bar{\mu}_A(x), \bar{\nu}_A(0) \leq \bar{\nu}_A(x)\); \((\text{FI}_2) \bar{\mu}_A(y * x) \geq \text{r min} \{\bar{\mu}_A ((x * z) * (0 * y)), \bar{\mu}_A(z)\},\)
Contribution on interval valued intuitionistic fuzzy a-ideal

Theorem 1.5.4: Let $A$ be an $i$-$v$ intuitionistic fuzzy $A$-ideal of $X$. If there exists a sequence $\{x_n\}$ in $X$ such that $\lim \mu_A(x_n) = [1,1]$ and $\lim \nu_A(x_n) = [0,0]$, then $\mu_A(0) = [1,1]$ and $\nu_A(0) = [0,0]$.

Theorem 1.5.5: An $i$-$v$ intuitionistic fuzzy set $A = [\langle \mu_A^L, \mu_A^U \rangle, \langle \nu_A^L, \nu_A^U \rangle]$ in $X$ is an $i$-$v$ intuitionistic fuzzy ideal of $X$ if and only if $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \nu_A^L, \nu_A^U \rangle$ are intuitionistic fuzzy ideals of $X$.

Theorem 1.5.6: An $i$-$v$ intuitionistic fuzzy set $A = [\langle \mu_A^L, \mu_A^U \rangle, \langle \nu_A^L, \nu_A^U \rangle]$ in $X$ is an $i$-$v$ intuitionistic fuzzy $A$-ideal of $X$ if and only if $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \nu_A^L, \nu_A^U \rangle$ are intuitionistic fuzzy $A$-ideals of $X$.

Cartesian product of interval valued intuitionistic fuzzy a-ideal

Definition 1.5.7: Let $\mu_A$ and $\mu_B$ be two membership functions and $\nu_A$ and $\nu_B$ be two non-membership functions of each $x \in X$ to an $i$-$v$ subsets $A$ and $B$, respectively. Then $\mu_A \times \mu_B$ is membership function and $\nu_A \times \nu_B$ is non-membership function of each element $(x, y) \in X \times X$ to the set $A \times B$ and defined by $\langle \mu_A \times \nu_B \rangle (x, y) = \mu_A(x) \times \nu_A(y)$ and $\langle \nu_A \times \mu_B \rangle (x, y) = \nu_A(x) \times \mu_B(y)$.

Definition 1.5.8: Let $A = [\langle \mu_A^L, \mu_A^U \rangle, \langle \nu_A^L, \nu_A^U \rangle]$ and $B = [\langle \mu_B^L, \mu_B^U \rangle, \langle \nu_B^L, \nu_B^U \rangle]$ be two interval valued intuitionistic fuzzy subsets in a set $X$. Then the cartesian product of $A \times B$ is defined by

$$(\forall) \nu_A(y \times x) \leq \max \{\nu_A ((x \times z) \times (0 \times y)), \nu_A(z)\}.$$
\[ A \times B = \{ ((x, y), (\bar{\mu}_A \times \bar{\mu}_B), (\bar{\nu}_A \times \bar{\nu}_B)) : \forall (x, y) \in X \times X \} \text{, Where } A \times B : X \times X \rightarrow D [0, 1]. \]

**Theorem 1.5.9:** Let \( A = \{ (\mu^L_A, \mu^U_A), (\nu^L_A, \nu^U_A) \} \) and \( B = \{ (\mu^L_B, \mu^U_B), (\nu^L_B, \nu^U_B) \} \) be two interval-valued intuitionistic fuzzy subsets in a set \( X \). Then \( A \times B \) is an interval-valued intuitionistic fuzzy \( A \)-ideal of \( X \times X \).

**Definition 1.5.10:** Let \( \bar{\mu}_B, \text{and } \bar{\nu}_B \) respectively, be an interval-valued membership and non-membership function of each element \( x \in X \) to the set \( B \). Then strongest interval-valued intuitionistic fuzzy set relation on \( X \) that is a membership function relation \( \bar{\mu}_A \) on \( \bar{\mu}_B \) and non-membership function relation \( \bar{\nu}_A \) on \( \bar{\nu}_B \) and \( \mu_{A_x}, \nu_{A_y} \) whose interval-valued membership and non-membership function, of each element \( (x, y) \in X \times X \) and defined by

\[ \bar{\mu}_{A_x}(x, y) = r \min \{ \bar{\mu}_B(x), \bar{\mu}_B(y) \} \text{ and } \bar{\nu}_{A_y}(x, y) = r \max \{ \bar{\nu}_B(x), \bar{\nu}_B(y) \} \]

**Definition 1.5.11:** Let \( B = \{ (\mu^L_B, \mu^U_B), (\nu^L_B, \nu^U_B) \} \) be an interval-valued subset in a set \( X \). Then the strongest interval-valued intuitionistic fuzzy relation on \( X \) that is an interval-valued \( A \) on \( B \) is \( A_B \) and defined by, \( A_B = \{ (\mu^L_{A_y}, \mu^U_{A_y}), (\nu^L_{A_y}, \nu^U_{A_y}) \} \).

**Theorem 1.5.12:** Let \( B = \{ (\mu^L_B, \mu^U_B), (\nu^L_B, \nu^U_B) \} \) be an interval-valued subset in a set \( X \) and \( A_B = \{ (\mu^L_{A_y}, \mu^U_{A_y}), (\nu^L_{A_y}, \nu^U_{A_y}) \} \) be the strongest interval-valued intuitionistic fuzzy relation on \( X \). Then \( B \) is an interval-valued intuitionistic \( A \)-ideal of \( X \) if and only if \( A_B \) is an interval-valued intuitionistic fuzzy \( A \)-ideal of \( X \times X \).
Theorem 1.5.13: If $\mu_A$ is an interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$, then $\mu_{A'}$ is also interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$.

Theorem 1.5.14: If $\mu_A$ is an interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$, then $\mu_{A \cap B}$ is also an interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$.

Theorem 1.5.15: If $\mu_A$ is an interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$, then $\mu_{A \cup B}$ is also an interval-valued intuitionistic fuzzy $A$-ideal of BCI-algebra $X$.

Section 1.6: International fuzzification of $h$-ideal of BCI-algebra

Aim: In this chapter, the concept of an intuitionist fuzzy set is applied to $H$-ideals in BCI-algebras. The notion of an intuitionist fuzzy $H$-ideal of a BCI-algebra is introduced, and investigated some related properties. Relations between an intuitionist fuzzy ideal and an intuitionist fuzzy $H$-ideal are provided. Characterizations of an intuitionist fuzzy $H$-ideal are given. Using a collection of $H$-ideals, intuitionist fuzzy $H$-ideals are established.

Definition 1.6.1: An algebra $(X; *, 0)$ of type $(2, 0)$ is a BCI-algebra if it satisfies the following conditions:

(BCI-1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$;
(BCI-2) $(\forall x, y \in X)$ $(x * (x * y)) * y = 0)$;
(BCI-3) $(\forall x \in X)$ $(x * x = 0)$;
(BCI-4) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$. 
**Definition 1.6.2:** A non-empty subset $I$ of $(X, *, 0)$ is a $h$-ideal of $X$ if it satisfies the following axioms: (1) $0 \in I$; 2. $(\forall x, z \in X) \ (\forall y \in I) \ (x * (y * z)) \in I$ and $y \in I$ imply $x * y \in I$.

**Definition 1.6.3:** A fuzzy subset $A$ with membership function $\mu_A$ in a BCI-algebra $(X, *, 0)$ is a fuzzy $H$-ideal of $X$ if it satisfies the following axioms:

1. $\mu_A(0) \geq \mu_A(x)$, $\forall \ x \in X$; and 2. $\mu_A(x * z) \geq \min \{\mu_A(x * (y * z)), \mu_A(y)\}$, $\forall x, y, z \in X$.

**Definition 1.6.4:** An intuitionistic fuzzy set $A = \{<x, \mu_A(x), \lambda_A(x) >: \ x \in X\}$ in a BCI-algebra $(X, *, 0)$ is an intuitionistic fuzzy $h$-ideal of $X$ if it satisfies the following axioms:

1. $\mu_A(0) \geq \mu_A(x)$, $\forall \ x \in X$; 2. $\mu_A(x * z) \geq \min \{\mu_A(x * (y * z)), \mu_A(y)\}$,

3. $\lambda_A(x * z) \leq \max \{\lambda_A(x * (y * z)), \lambda_A(y)\}$, for all $x, y, z \in X$.

**Definition 1.6.5 (Upper and lower $\alpha$-cut):** Let $A = (X, \mu_A, \lambda_A)$ be an intuitionist fuzzy set in a BCI-algebra $(X, *, 0)$. The set $U(\mu_A; s) = \{x \in X: \mu_A(x) \geq s\}$ is the upper $s$-level of $\mu_A$ and the set $L(\lambda_A; t) = \{x \in X: \lambda_A(x) \leq t\}$ is the lower $t$-level of $\lambda_A$.

**Definition 1.6.6:** The power of fuzzy set $A$ with membership function $\mu_A$ is again a fuzzy set with membership function $\mu_A^m$ defined by $\mu_A^m(x) = (\mu_A(x))^m$ for all $x \in X$.

**Definition 1.6.7:** Let $f$ be a mapping with domain a set $X$ and $A = (X, \mu_A, \lambda_A)$ an intuitionistic fuzzy set in $X$. Then the fuzzy sets $u$ and $v$ on $f(X)$ defined by $u(y) = sup_{x \in f^{-1}(y)} \mu_A(x)$ and $v(y) = inf_{x \in f^{-1}(y)} \lambda_A(x)$ for all $y \in f(X)$ is image of $A$ under $f$. If $u, v$ are fuzzy sets in $f(X)$, then the fuzzy sets $\mu_A = (u \circ f)$ and $\lambda_A = (v \circ f)$ are the pre-image of $u$ and $v$ under $f$. 

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**Definition 1.6.8:** A map $f: X \rightarrow Y$ of BCI-algebra is homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that if $f$ is homomorphism of BCI-algebra, then $f(0) = 0^1$

**Definition 1.6.9:** Let $f: X \rightarrow Y$ be onto homomorphism of BCI-algebras be a homomorphism of BCI-algebra for any intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $Y$, A new intuitionistic fuzzy set $A^f = (\mu^f_A, \lambda^f_A)$ in $X$ is defined by $\mu^f_A(x) = \mu_A(f(x))$ and $\lambda^f_A(x) = \lambda_A(f(x))$, for all $x \in X$.

**Theorem 1.6.10:** Let $f: X \rightarrow X^1$ be an onto homomorphism of BCI-algebra. If $A^1 = (X^1, u, v)$ is an intuitionistic fuzzy $H$-ideal of $X^1$, then the pre-image of $A^1$ under $f$ is an intuitionistic fuzzy $H$-ideal of $X$.

**Theorem 1.6.11:** $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy $H$-ideal of a BCI-algebra $(X, *, 0)$ if and only if the non-empty upper $s$-level cut $U(\mu_A; s)$ and the non-empty lower $t$-level cut $L(\lambda_A; t)$ are $H$-ideals of $X$, for any $s, t \in [0, 1]$.

**Theorem 1.6.12:** $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy closed $H$-ideal of a BCI-algebra $X$ if and only if the non-empty upper $s$-level cut $U(\mu_A; s)$ and the non-empty lower $t$-level cut $L(\lambda_A; t)$ are closed $H$-ideal of $X$, for any $s, t \in [0, 1]$.

**Theorem 1.6.13:** Let $f: X \rightarrow Y$ be a homomorphism of BCI-algebra. If an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic T-fuzzy $H$-ideal of $Y$, then an intuitionistic T-fuzzy $H$-ideal $A^f = (\mu^f_A, \lambda^f_A)$ in $X$ is an intuitionistic T-fuzzy $H$-ideal of $X$.

**Theorem 1.6.14:** Let $f: X \rightarrow Y$ be an epimorphism of BCI – algebra and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in $Y$. If $A^f = (\mu^f_A, \lambda^f_A)$ is an intuitionistic fuzzy $H$-ideal of $X$, then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy $H$-ideal of $Y$.
Union, and intersection properties on intuitionistic fuzzy h-ideal

**Theorem 1.6.15:** Let an intuitionistic fuzzy set \( A = (X, \mu_A, \lambda_A) \) in \( X \) be an intuitionistic fuzzy H-ideal of \( X \). If the inequality \( z \leq y \) holds in \( X \), then \( \mu_A(x) \geq \mu_A(y) \), \( \lambda_A(x) \leq \lambda_A(y) \).

**Theorem 1.6.16:** If \( A \) be an intuitionistic fuzzy H-ideal of BCI-algebra \( X \), then \( A^m \) is an intuitionistic fuzzy H-ideal of \( X \).

**Theorem 1.6.17:** If \( A \) and \( B \) are two intuitionistic fuzzy H-ideals of \( X \), then \( A \cap B \) an intuitionistic fuzzy H-ideal of \( X \).

**Theorem 1.6.18:** If \( A \) and \( B \) are two intuitionistic fuzzy H-ideals of BCI-algebra \( X \), and one is contained in other, then \((A \cup B)\) is an intuitionistic fuzzy H-ideal of \( X \).

**Theorem 1.6.19:** Every intuitionistic fuzzy H-ideal an intuitionistic fuzzy ideal.

**Theorem 1.6.20:** Every intuitionistic T-fuzzy H-ideal is an intuitionistic T-fuzzy sub-algebra.

**Other properties on intuitionistic fuzzy H-ideal**

**Theorem 1.6.21:** An intuitionistic fuzzy set \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy H-ideal of \( X \) if and only if the fuzzy sets \( \mu_A \) and \( \overline{\lambda_A} \) are fuzzy H-ideals of \( X \).

**Theorem 1.6.22:** Let \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy H-ideal of a BCI-algebra \( X \), Then so is \( \Box A = (X, \mu_A, \overline{\mu_A}) \).

**Theorem 1.6.23:** A \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy H-ideal of a BCI-algebra \( X \) if and only if \( \Box A = (X, \mu_A, \overline{\mu_A}), \) \( \Diamond A = (X, \overline{\lambda_A}, \lambda_A) \) are intuitionistic fuzzy H-ideals of a BCI-algebra \( X \).

**Theorem 1.6.23:** If \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy closed H-ideal of a BCI-algebra \( X \), then so is \( \Box A = (X, \mu_A, \overline{\mu_A}) \).
**Theorem 1.6.24:** A = (X, μ_A, λ_A) be an intuitionistic fuzzy closed H-ideal of a BCI-algebra X if and only if □A = (X, μ_A, \( \mu_{\text{A}} \)) and ◊A = (X, \( \lambda_{\text{A}} \)) are intuitionistic fuzzy closed H-ideals of BCI-algebra X.

**Section 1.7: Some examples on intuitionistic fuzzy h-ideal**

**Definition 1.7.1:** Let IFSs(X) denotes the family of all IFS on the universe X, and A, B ∈ IFS(X) be given as A={< x, μ_A(x), λ_A(x) : x ∈ X}, B = {< x, μ_B(x), λ_B(x) : x ∈ X}. Then the union

\[ A \cup B \] is defined as \[ A \cup B = \{ < x, \mu_{A \cup B}(x), \lambda_{A \cup B}(x) > : x \in X \} \]

where \( \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)) \) : x ∈ X} and \( \lambda_{A \cup B}(x) = \min(\lambda_A(x), \lambda_B(x)) \) : x ∈ X}.

**Example 1.7.2:** If A = (X, μ_A, λ_A) and B = (X, μ_B, λ_B) are intuitionistic fuzzy H-ideals of X, then the union A ∪ B need not be an intuitionistic fuzzy H-ideal of X.

**Definition 1.7.3:** Let IFSs(X) denotes the family of all IFS on the universe X, and let A,B ∈ IFS(X) be given as A= {< x, μ_A(x), λ_A(x) : x ∈ X}, B = { < x, μ_B(x), λ_B(x) : x ∈ X}. Then the intersection A ∩ B is defined as \[ A \cap B = \{ < x, \mu_{A \cap B}(x), \lambda_{A \cap B}(x) > : x \in X \} \]

where \( \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)) \) : x ∈ X} and \( \lambda_{A \cap B}(x) = \max(\lambda_A(x), \lambda_B(x)) \) : x ∈ X}.

**Example 1.7.4:** If A = (X, μ_A, λ_A) and B = (X, μ_B, λ_B) are intuitionistic fuzzy H-ideals of X, the intersection (A ∩ B) need not be an intuitionistic fuzzy H-ideal of X.
Definition 1.7.5: Let IFSs(X) denotes the family of all IFS on the universe X, and let A,B ∈ IFS(X) be given as \( A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \} \), \( B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \} \). The algebraic sum \( A + B \) is defined as \( A + B = \{ < x, \mu_{A+B}(x), \lambda_{A+B}(x) > : x \in X \} \) where \( \mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \) and \( \lambda_{A+B}(x) = \lambda_A(x) \cdot \lambda_B(x) \).

Example 1.7.6: If \( A = (X, \mu_A, \lambda_A) \) and \( B = (X, \mu_B, \lambda_B) \) are intuitionistic fuzzy H-ideals of X, then \( A \oplus B \) is need not be an intuitionistic fuzzy P-ideal of X.

Definition 1.7.7: Let IFSs(X) denotes the family of all IFS on the universe X, and let A,B ∈ IFS(X) be given as \( A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \} \), \( B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \} \). Then algebraic product \( A \cdot B \) is defined as \( A \cdot B = \{ < x, \mu_{A \cdot B}(x), \lambda_{A \cdot B}(x) > : x \in X \} \) where \( \mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x) \) and \( \lambda_{A \cdot B}(x) = \lambda_A(x) + \lambda_B(x) - \lambda_A(x) \cdot \lambda_B(x) \).

Example 1.7.8: If A and B are intuitionistic fuzzy H-ideal of X, then \( A \otimes B \) need not be an intuitionistic fuzzy H-ideal of X.

Hamacher intersection, and Hamacher union in intuitionistic h-ideal

Definition 1.7.9: Let IFSs(X) denotes the family of all IFS on the universe X, and let A,B ∈ IFS(X) be given as \( A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \} \), \( B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \} \). Then the hamacher intersection \( A \cap B \) as defined as \( A \cap B = \{ < x, \mu_{A \cap B}(x), \lambda_{A \cap B}(x) > : x \in X \} \),
where \( \mu_{A \cap B}(x) = \left\{ \frac{\mu_A(x) \cdot \mu_B(x)}{\gamma + (1-\gamma)(\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x))} \right\}, \gamma \geq 0 : x \in X \)

\( \lambda_{A \cap B}(x) = \left\{ \frac{\lambda_A(x) \cdot \lambda_B(x)}{\gamma + (1-\gamma)(\lambda_A(x) + \lambda_B(x) - \lambda_A(x) \cdot \lambda_B(x))} \right\}, \gamma \geq 0 : x \in X \)

**Example 1.7.10:** If \( A = \langle X, \mu_A, \lambda_A \rangle \) and \( B = \langle X, \mu_B, \lambda_B \rangle \) are intuitionistic fuzzy H-ideals of \( X \), then the hamacher intersection need not be an intuitionistic fuzzy H-ideal of \( X \).

**Definition 1.7.11:** Let IFSs \( (X) \) denotes the family of all IFS on the universe \( X \), and let \( A, B \in \text{IFS}(X) \) be given as \( A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \} \), \( B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \} \). Then the hamacher union \( A \cup B \) is defined as \( A \cup B = \{ < x, \mu_{A \cup B}, \lambda_{A \cup B} > : x \in X \} \) where

\[
\mu_{A \cup B} = \left\{ \frac{\mu_A(x) + \mu_B(x) - (2-\gamma)\mu_A(x) \cdot \mu_B(x)}{1 - (1-\gamma)\mu_A(x) \cdot \mu_B(x)} \right\}, \gamma \geq 0 : x \in X \text{ and}
\]

\[
\lambda_{A \cup B} = \left\{ \frac{\lambda_A(x) + \lambda_B(x) - (2-\gamma)\lambda_A(x) \cdot \lambda_B(x)}{1 - (1-\gamma)\lambda_A(x) \cdot \lambda_B(x)} \right\}, \gamma \geq 0 : x \in X \}

**Example 1.7.12:** If \( A = \langle X, \mu_A, \lambda_A \rangle \) and \( B = \langle X, \mu_B, \lambda_B \rangle \) are intuitionistic fuzzy H-ideals of \( X \), then the union \( A \cup B \) need not be an intuitionistic fuzzy H-ideal of \( X \).

**Definition 1.7.13:** Let IFSs \( (X) \) denotes the family of all IFS on the universe \( X \), and let \( A, B \in \text{IFS}(X) \) be given as \( A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \} \), \( B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \} \). Then \( A \# B \) is defined as a set \( \{ < x, \mu_{A \# B}(x), \lambda_{A \# B}(x) > : x \in X \} \) where

\[
\mu_{A \# B}(x) = \left\{ \frac{2\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x)} : x \in X \right\} \text{ and}
\]

\[
\lambda_{A \# B}(x) = \left\{ \frac{2\lambda_A(x) \cdot \lambda_B(x)}{\lambda_A(x) + \lambda_B(x)} : x \in X \right\} \}
Example 1.7.14: If $A = \langle X, \mu_A, \lambda_A \rangle$ and $B = \langle X, \mu_B, \lambda_B \rangle$ are intuitionistic fuzzy H-ideals of $X$, then $A \# B$ is not an intuitionistic fuzzy H-ideal of $X$.

Definition 1.7.15: Let $\text{IFSs}(X)$ denotes the family of all IFSs on the universe $X$, and $A, B \in \text{IFS}(X)$ be $A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \}$, $B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \}$. Then $A \$ B$ is defined as $\{ < x, \mu_{A\$ B}(x), \lambda_{A\$ B}(x) > : x \in X \}$ where

$$\mu_{A\$ B}(x) = \sqrt{\mu_A(x) \mu_B(x)} : x \in X$$

And

$$\lambda_{A\$ B}(x) = \sqrt{\lambda_A(x) \lambda_B(x)} : x \in X$$

Example 1.7.16: If $A = \langle X, \mu_A, \lambda_A \rangle$ and $B = \langle X, \mu_B, \lambda_B \rangle$ are intuitionistic fuzzy H-ideals of $X$, then $A \$ B$ is not an intuitionistic fuzzy H-ideal of $X$.

Bounded sum, bounded difference, and simple disjunctive sum

Definition (Bounded Sum) 1.7.17: Let $\text{IFSs}(X)$ denotes the family of all IFSs on the universe $X$, and let $A, B \in \text{IFS}(X)$ be given as $A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \}$, $B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \}$. Then the bounded sum $A \oplus B$ is defined as

$$A \oplus B = \{ < x, \mu_{A \oplus B}(x), \lambda_{A \oplus B}(x) : x \in X \}$$

where $\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x) : x \in X\}$ and $\lambda_{A \oplus B}(x) = \min\{\lambda_A(x) + \lambda_B(x) : x \in X\}$
**Example 1.7.18:** If $A = \langle X, \mu_A, \lambda_A \rangle$ and $B = \langle X, \mu_B, \lambda_B \rangle$ are intuitionistic fuzzy H-ideals of $X$, then the bounded sum need not be an intuitionistic fuzzy H-ideal of $X$.

**Definition (Bounded difference) 1.7.19:** Let $\text{IFSs}(X)$ denotes the family of all IFSs on the universe $X$, and let $A, B \in \text{IFS}(X)$ be given as $A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \}$, $B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \}$. Then the bounded difference $A \subset B$ is defined as $A \subset B = \{ < x, \mu_{A \subset B}(x), \lambda_{A \subset B}(x) > : x \in X \}$, where

$$
\mu_{A \subset B}(x) = \min\{\mu_A(x), 1 - \mu_B(x) : x \in X\}
$$

$$
\lambda_{A \subset B}(x) = \min\{\lambda_A(x), 1 - \lambda_B(x) : x \in X\}
$$

**Example 1.7.20:** If $A = \langle X, \mu_A, \lambda_A \rangle$ and $B = \langle X, \mu_B, \lambda_B \rangle$ are intuitionistic fuzzy H-ideals of $X$, the bounded difference need not be an intuitionistic fuzzy H-ideal of $X$.

**Definition (Simple disjunctive Sum) 1.7.21:** Let $\text{IFSs}(X)$ denotes the family of all IFSs on the universe $X$, and let $A, B \in \text{IFS}(X)$ be given as $A = \{ < x, \mu_A(x), \lambda_A(x) > : x \in X \}$, $B = \{ < x, \mu_B(x), \lambda_B(x) > : x \in X \}$. Then the bounded sum $A \hat{\oplus} B$ is defined as $A \hat{\oplus} B = \{ < x, \mu_{A \hat{\oplus} B}(x), \lambda_{A \hat{\oplus} B}(x) > : x \in X \}$, where

$$
\mu_{A \hat{\oplus} B}(x) = \max\{\min\{\mu_A(x), 1 - \mu_B(x)\}, \min\{1 - \mu_A(x), \mu_B(x)\} : x \in X\}
$$

$$
\lambda_{A \hat{\oplus} B}(x) = \max\{\min\{\lambda_A(x), 1 - \lambda_B(x)\}, \min\{1 - \lambda_A(x), \lambda_B(x)\} : x \in X\}
$$

**Example 1.7.22:** If $A = \langle X, \mu_A, \lambda_A \rangle$ and $B = \langle X, \mu_B, \lambda_B \rangle$ are intuitionistic fuzzy H-ideal of $X$, then the simple disjunctive sum is an intuitionistic fuzzy H-ideal of $X$. 

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Three different direct products in intuitionistic fuzzy h-ideal

**Definition 1.7.23:** Let IFSs(X) denotes the family of all IFSs on the universe X, and A, B ∈ IFS(X) be A= {< x, , μA(x), λA(x)> : x ∈X}, B = {< x, , μB(x), λB(x)> : x ∈ X}. The A ∗ B is defined as A ∗ B = {< x, , μA+B(x), λA+B > : x ∈ X}, where

\[ μ_{A+B}(x) = \left\{ \frac{μ_A(x)+μ_B(x)}{2[μ_A(x)+μ_B(x)+1]} : x \in X \right\} \]

and \[ λ_{A+B} (x) = \left\{ \frac{2λ_A(x)λ_B(x)}{λ_A(x)+λ_B(x)} : x \in X \right\} \]

**Example 1.7.24:** If A = ⟨ X, μA, λA⟩ and B = ⟨ X, μB, λB⟩ are intuitionistic fuzzy H-ideals of X, then A ∗ B need not be an intuitionistic fuzzy H-ideal of X.

**Definition 1.7.25:** Let IFS (X) denotes the family of all IFS (X) on the universe X, and A, B ∈ IFS (X) be given as A= {< x, , μA(x), λA(x)> : x ∈X}, B = {< x, , μB(x), λB(x)> : x ∈ X}. The A ⊙ B is defined as A ⊙ B = {< x, , μA⊙B(x), λA⊙B > : x ∈ X}, where

\[ μ_{A⊙B}(x) = \left\{ \frac{1}{2} [μ_A(x) + μ_B(x)] : x \in X \right\} \]

and \[ λ_{A⊙B} (x) = \left\{ \frac{1}{2} [λ_A(x) + λ_B(x)] : x \in X \right\} \]

**Example 1.7.26:** If A = ⟨ X, μA, λA⟩ and B = ⟨ X, μB, λB⟩ are intuitionistic fuzzy H-ideal of X, then A ⊙ B need not be an intuitionistic fuzzy H-ideal of X.

**Definition 1.7.27:** Let IFS (X) denotes the family of all IFS (X) on the universe X, and let A, B ∈ IFS(X) be given as A= {< x, , μA(x), λA(x)> : x ∈X}, B = {< x, , μB(x), λB(x), λB(x)> : x ∈ X}. The A ⊕ B is defined as A ⊕ B = {< x, , μA⊕B(x), λA⊕B > : x ∈ X}, where

\[ μ_{A⊕B}(x) = \left\{ \max [0, μA(x) + μB(x)-1] : x \in X \right\} \]

and \[ λ_{A⊕B} (x) = \left\{ \max [0, λA(x) + λB(x)-1] : x \in X \right\} \]

**Example 1.7.28:** If A = ⟨ X, μA, λA⟩ and B = ⟨ X, μB, λB⟩ are intuitionistic fuzzy H-ideals of X, then the bounded sum is not an intuitionistic fuzzy H-ideal of X.