CHAPTER 3

FUNCTIONAL RELATIONS OF I-FUNCTION AND DIGAMMA FUNCTIONS USING ERDÉLYI-KOBER FRACTIONAL INTEGRAL OPERATORS

3.1 Introduction

The concept of functional relations has been visualized in research papers of many authors namely Nishimoto and Saxena [91], Saxena [118], Kalla and Ross [49], Kalla and Al-saquabi [48]. Extending this concept, we obtain the functional relations of I-function and digamma function using Erdélyi-Kober fractional integral operators in this chapter.

The fields of fractional integral operators have several applications as is evident from the papers written by Nishimoto [90], Ross [107], McBride and Roach [83] and Samko, Kilbas and Marichev [116]. There are many application areas like finance, stochastic processes and many branches of applied sciences and engineering. Signal processing, modeling and control theory are other areas which have been the object of more incentive research in the last few decades. It represents a rapidly growing field both in theory and in application to real word problems.

In view of the above we have also included an application of the functional relation obtained in this chapter in obtaining a closed form solution of a fractional integral equation.
3.2 Mathematical pre-requisites

In this section besides defining Erdélyi–Kober operators, we present two lemmas whose proofs are similar to those presented in preceding chapter. The definition and computable representation of I-function has already been defined in chapter 2.

(i) Erdélyi-Kober Operators

The Erdélyi–Kober operators are fractional integral operators introduced by Erdélyi (1940) and Hermann (1940). These operators are defined as follows:

\[ E^{\alpha,\eta}_{0,x} f(x) = I^{\alpha,0,\eta}_{0, x} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} t^{-\eta} f(t) dt \]  

\[ K^{\alpha,\eta}_{x,\infty} f(x) = I^{\alpha,0,\eta}_{x, \infty} f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} t^{-\eta} f(t) dt \]  

where \( Re(\alpha) > 0 \)

(ii) Lemma 3.1

If \( Re(\rho + \min b_j/B_i) > 0, j = 1,2, \ldots, m; Re(\sigma) > 0, |\text{arg} ax| < (1/2)\pi \lambda_i, \lambda_i = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_i} B_{ji} > 0, \) where \( i = 1,2, \ldots, r \)

and \( \mu_i = \sum_{j=1}^{n} A_j + \sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{m} B_j + \sum_{j=m+1}^{q_i} B_{ji} < 0 \) or \( \mu_i = 0; i = 1,2, \ldots, r \)

and \( 0 < |at| < \beta_i^{-1}; \beta_i = \prod_{j=1}^{p_i} (A_{ji})^{\alpha_{ji}} \prod_{j=1}^{q_i} (B_{ji})^{-\beta_{ji}}, i = 1,2, \ldots, r, \) then
\[ \frac{x^{-\sigma - \eta}}{\Gamma(\sigma)} \int_0^x t^{\rho - 1}(x - t)^{\sigma - 1} \left[ a_i A_j \right]_{1,n} \cdots \left[ a_{ij} A_{ji} \right]_{n+1,p_i} \left[ b_j B_i \right]_{1,m} \cdots \left[ b_{ij} B_{ji} \right]_{m+1,q_i} \, dt \]

\[ = x^{\rho - 1} I^{m,n+1}_{p_i+1,q_i+1,r} \left[ a \right] \left[ 1 - \rho - \eta, 1 \right] \left[ a_i A_j \right]_{1,n} \cdots \left[ a_{ij} A_{ji} \right]_{n+1,p_i} \left[ b_j B_i \right]_{1,m} \cdots \left[ b_{ij} B_{ji} \right]_{m+1,q_i} \left( 1 - \rho - \eta - \sigma, 1 \right) \] (3.2.3)

(iii) Lemma 3.2

If \( \text{Re}(\rho + \min b_j B_j) > 0, j = 1, 2, \ldots, m; \text{Re}(\sigma) > 0, \text{arg} \, ax < (1/2) \pi \lambda_i, \lambda_i = \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_{ji} + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_{ji} > 0, \) where \( i = 1, 2, \ldots, r \)

and \( \mu_i = \sum_{j=1}^n A_j + \sum_{j=n+1}^p A_{ji} - \sum_{j=1}^m B_j + \sum_{j=m+1}^q B_{ji} < 0 \) or \( \mu_i = 0 \); \( i = 1, 2, \ldots, r \)

then

\[ \frac{x^\eta}{\Gamma(\sigma)} \int_x^\infty t^{\rho - 1}(t - x)^{\sigma - 1} t^{-\sigma - \eta} I^{m,n}_{p_i,q_i} \left[ a \right] \left[ 1 - \rho - \eta, 1 \right] \left[ a_i A_j \right]_{1,n} \cdots \left[ a_{ij} A_{ji} \right]_{n+1,p_i} \left[ b_j B_i \right]_{1,m} \cdots \left[ b_{ij} B_{ji} \right]_{m+1,q_i} \, dt \]

\[ = x^{\rho - 1} I^{m+1,n}_{p_i+1,q_i+1,r} \left[ a \right] \left[ 1 - \rho + \eta + \sigma, 1 \right] \left[ a_i A_j \right]_{1,n} \cdots \left[ a_{ij} A_{ji} \right]_{n+1,p_i} \left( 1 - \rho + \eta, 1 \right) \left[ b_j B_i \right]_{1,m} \cdots \left[ b_{ij} B_{ji} \right]_{m+1,q_i} \] (3.2.4)

3.3 Main Results

In the present section we shall establish some functional relations of I-function and logarithmic derivative of gamma function using Erdélyi-Kober fractional integral operators.

3.3.1 Theorem
If $\xi = (b_h + v)/B_h$, $\text{Re}(\rho + \min j/B_j) > 0$, $j = 1, 2, \ldots, m$; $\text{Re}(\sigma) \geq 0$,

$|\text{arg } ax| < (1/2)\pi\lambda_i$, $\lambda_i > 0$; $\mu_i \leq 0$,

$\lambda_i = \sum_{j=1}^n A_j - \sum_{j=n+1}^m A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_{ji} > 0$, and

$\mu_i = \sum_{j=1}^n A_j + \sum_{j=n+1}^m A_j - \sum_{j=1}^m B_j + \sum_{j=m+1}^q B_{ji} < 0$ or $\mu_i = 0$;

where $i = 1, 2, \ldots, r$, then

$$
\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{v! B_h} \chi(\xi) \times \left\{ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \right\}
$$

Proof:

Differentiating both sides of (3.2.3) with respect to $\rho$ and using Leibnitz rule, it is observed that

$$
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^{v}(ax)^{v} \chi(\xi)}{v! B_h} \{\psi(\rho + \eta + \xi) - \psi(\rho + \eta + \sigma + \xi)\}
$$
where \( \xi = \frac{b_h + v}{B_h} \)

Now L.H.S. of equation (3.3.1.1) can be written in terms of Erdélyi-Kober operator and R.H.S. can be designated by \( J(a_{p_i}, A_{p_i}; b_{q_i}, B_{q_i}; \rho, \eta, \sigma, a) \), so that it takes the form

\[
E^{\sigma, \eta}_{\lambda, x} \left( x^{\rho - 1} \ln x I^{m,n}_{p_{i_qi}} ; r(ax) \right) = J(a_{p_i}, A_{p_i}; b_{q_i}, B_{q_i}; \rho, \eta, \sigma, a) \quad (3.3.1.2)
\]

On account of analyticity and continuity Erdélyi-Kober operator at \( \xi = 0 \) and \( \eta = 0 \), we can replace \( \eta \) by \( (\eta - 1) \) and then \( \sigma \) by \( 1 - \sigma \). Hence for the differentiation of

\[
x^{\rho - 1} \ln x I^{m,n}_{p_{i_qi}} \left[ \alpha x \left( \begin{array}{c}
(a_j, A_j)_{1,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\
(b_j, B_j)_{1,m}, (b_{ji}, B_{ji})_{m+1,q_i}
\end{array} \right) \right]
\]
to an arbitrary order, we find that

\[
E^{-\sigma, \eta}_{0, x} \left( x^{\rho - 1} \ln x I^{m,n}_{p_{i_qi}} ; r(ax) \right) = J(a_{p_i}, A_{p_i}; b_{q_i}, B_{q_i}; \rho, \eta - \sigma, -\sigma, a)
\]

or

\[
E^{-\sigma, \eta}_{0, x} \left( x^{\rho - 1} \ln x I^{m,n}_{p_{i_qi}} ; r(ax) \right)
\]

\[
= x^{\rho - 1} \left[ \ln x I^{m,n+1}_{p_{i+1,q_i+1}} \left[ \alpha x \left( \begin{array}{c}
(1 - \rho - \eta - \sigma, 1), (a_j, A_j)_{1,n}, (a_{ji}, A_{ji})_{n+1,p_i} \\
(b_j, B_j)_{1,m}, (b_{ji}, B_{ji})_{m+1,q_i}
\end{array} \right) \right) \right]
\]

\[
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (ax)^h x^h (\Gamma(\tau + \eta + \sigma + \xi))}{v! B_h \Gamma(\tau + \eta + \xi)} \left[ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \right] \quad (3.3.1.3)
\]

Now we consider the following integral equation of Volterra type:

\[
\frac{x^{\rho - \eta} - \eta}{\Gamma(\rho - \eta)} \int_0^x (x - t)^{\rho - 1} t^\eta f(t) dt = x^{\rho - 1} \ln x I^{m,n}_{p_{i_qi}} ; r(ax) \quad (3.3.1.4)
\]
where, \( \text{Re}(\rho + \min b_j / B_j) > 0, \text{Re}(\sigma) \geq 0, |\arg a_i| < (1/2)\pi \lambda_i, \lambda_i > 0; \mu_i \leq 0. \)

On writing Volterra type integral (3.3.1.4) in the operator form, we have

\[
E_{0,x}^{\sigma,\eta} f(x) = x^{\rho-1} \ln x \int_{p_1, q_1}^{m, n} (ax) \tag{3.3.1.5}
\]

Operating on both sides of (3.3.1.5) with \( E_{0,x}^{-\sigma,\eta-\sigma} \), we get

\[
f(x) = E_{0,x}^{-\sigma,\eta-\sigma} \left( x^{\rho-1} \ln x \int_{p_1, q_1}^{m, n} (ax) \right)
\]

Using equation (3.3.1.3) in the above, we have

\[
f(x) = x^{\rho-1} \left[ \ln x \int_{p_1+1, q_1+1}^{m, n+1} (ax) \left[ (1 - \rho - \eta - \sigma, 1), (a_j, A_j)_{1, m}; (a_{ji}, A_{ji})_{n+1, p_i} \right] + \right.
\]

\[
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (ax)^{\xi} x(\xi)^{\Gamma(\rho+\eta+\sigma+\xi)} v! B_h \Gamma(\rho+\eta+\xi)}{B_h \Gamma(\rho+\eta+\xi)} \left[ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \right] \tag{3.3.1.6}
\]

To find the solution, we substitute the value of \( f(x) \) from equation (3.3.1.6) into (3.3.1.4). This gives

\[
\frac{x^{-\sigma-\eta}}{\Gamma(\sigma)} \int_{0}^{x} (x - t)^{\rho-1} t^\eta \times
\]

\[
\left\{ t^{\rho-1} \left[ \ln t \int_{p_1+1, q_1+1}^{m, n+1} (at) \left[ (1 - \rho - \eta - \sigma, 1), (a_j, A_j)_{1, m}; (a_{ji}, A_{ji})_{n+1, p_i} \right] + \right.\right.
\]

\[
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (at)^{\xi} x(\xi)^{\Gamma(\rho+\eta+\sigma+\xi)} v! B_h \Gamma(\rho+\eta+\xi)}{B_h \Gamma(\rho+\eta+\xi)} \left[ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \right] \right\} \right\} dt
\]
\[ = x^{\rho-1} \ln x \int_{p_i, q_i}^{m, n} r(ax) \]  

(3.3.1.7)

Now we substitute

\[ \ln t = \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{kx^k} \]  

(3.3.1.8)

where the interval of the convergence is \( 0 \leq t \leq 2x \).

Putting the value of \( \ln t \) from equation (3.3.1.8) into equation (3.3.1.7), we have

\[
\frac{x^{\sigma-\eta}}{\Gamma(\sigma)} \int_0^x (x - t)^{\sigma-1} t^{\rho + \eta - 1} \left[ \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{kx^k} \right] \times \\
\int_{p_i+1, q_i+1}^{m+1, n+1} \left[ \left(1 - \rho - \eta - \sigma, 1, (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \right) + \\
\sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (at)^\xi x^\nu \Gamma(\rho + \eta + \sigma + \xi)}{\nu! B_h \Gamma(\rho + \eta + \xi)} \{ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \} \right] \]  

dt

\[ = x^{\rho-1} \ln x \int_{p_i, q_i}^{m, n} r(ax) \]

or

\[
\frac{x^{\sigma-\eta}}{\Gamma(\sigma)} \int_0^x (x - t)^{\sigma-1} t^{\rho + \eta - 1} \ln x \times \\
\int_{p_i+1, q_i+1}^{m+1, n+1} \left[ \left(1 - \rho - \eta - \sigma, 1, (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \right) + \\
\sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (at)^\xi x^\nu \Gamma(\rho + \eta + \sigma + \xi)}{\nu! B_h \Gamma(\rho + \eta + \xi)} \{ \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \} \right] \]  

dt

\[ - \frac{x^{\sigma-\eta}}{\Gamma(\sigma)} \int_0^x (x - t)^{\sigma-1} t^{\rho + \eta - 1} \sum_{k=1}^{\infty} \frac{(x-t)^k}{kx^k} \]
For simplifying equation (3.3.10), we evaluate the three integrals on left hand side.

Thus we have

$$\frac{x^{\sigma-\eta}}{1 \sigma} \int_0^x (x - t)^{\sigma-1} t^{\rho+\eta-1} \ln x \times$$

\[\int_{p_i+1, q_i+1}^{m, n+1} \left[ \frac{(1 - \rho - \eta - \sigma, 1, (a_j, A_j)_{1, n}; (a_{ji}, A_{ji})_{n+1, p_i})}{at} \right] \left( b_j, B_j \right)_{1, m}; \left( b_{ji}, B_{ji} \right)_{m+1, q_i}, (1 - \rho - \eta, 1) \right] dt \]

$$= x^{\rho-1} \ln x \int_{p_i+1, q_i}^{m, n+1} \left[ \frac{(a_j, A_j)_{1, n}; (a_{ji}, A_{ji})_{n+1, p_i}}{ax} \right] \left( b_j, B_j \right)_{1, m}; \left( b_{ji}, B_{ji} \right)_{m+1, q_i}, (1 - \rho - \eta, 1) \right]$$

$$= x^{\rho-1} \ln x \int_{p_i+1, q_i}^{m, n+1} \left[ \frac{(a_j, A_j)_{1, n}; (a_{ji}, A_{ji})_{n+1, p_i}}{ax} \right] \left( b_j, B_j \right)_{1, m}; \left( b_{ji}, B_{ji} \right)_{m+1, q_i}, (1 - \rho - \eta, 1) \right]$$

(3.3.11)
\[
\frac{x^{-\alpha-\eta}}{1-\alpha} \int_0^x (x-t)^{\alpha-1} t^{\rho+\eta-1} \sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(at)\xi X(x(t))}{v! B_h} t^{\rho+\eta+\xi}) \times \psi(\rho + \eta + \sigma + 
\xi) - \psi(\rho + \eta + \xi) \} dt
\]

\[
= x^{\rho-1} \sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(ax)\xi X(x(\xi))}{v! B_h} \times \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \} \quad (3.3.1.12)
\]

Now putting the values of these three integrals, (3.3.1.10), (3.3.1.11) and (3.3.1.12) in the equation (3.1.2.9), we get

\[
x^{\rho-1} \ln x \int_{\nu+1, q_i}^{m, n, p_i} \left[ ax \begin{bmatrix} (a, A)_{1, n}; & (a, A)_{n+1, p_i} \\
(b, B)_{1, m}; & (b, B)_{m+1, q_i} \end{bmatrix} \right] - 
\]

\[
x^{\rho-1} \sum_{k=1}^\infty \frac{(\sigma)_k}{k!} \int_{\nu+1, q_i+1, r}^{m, n+1, p_i+1, r} \left[ ax \begin{bmatrix} (1 - \rho - \eta - \sigma, 1); & (a, A)_{1, n}; & (a, A)_{n+1, p_i} \\
(b, B)_{1, m}; & (b, B)_{m+1, q_i} \end{bmatrix} \right] + 
\]

\[
x^{\rho-1} \sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(ax)\xi X(x(\xi))}{v! B_h} \times \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \} 
\]

\[
= x^{\rho-1} \ln x \int_{\nu+1, q_i}^{m, n, p_i} \left[ ax \begin{bmatrix} (a, A)_{1, n}; & (a, A)_{n+1, p_i} \\
(b, B)_{1, m}; & (b, B)_{m+1, q_i} \end{bmatrix} \right]
\]

or

\[
\sum_{k=1}^\infty \frac{(\sigma)_k}{k!} \int_{\nu+1, q_i+1, r}^{m, n+1, p_i+1, r} \left[ ax \begin{bmatrix} (1 - \rho - \eta - \sigma, 1); & (a, A)_{1, n}; & (a, A)_{n+1, p_i} \\
(b, B)_{1, m}; & (b, B)_{m+1, q_i} \end{bmatrix} \right] + 
\]

\[
= \sum_{h=1}^m \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(ax)\xi X(x(\xi))}{v! B_h} \times \psi(\rho + \eta + \sigma + \xi) - \psi(\rho + \eta + \xi) \}
\]

This completes the proof.

### 3.3.2 Special cases
(i) If we put $\eta = 0$ and $\rho = \rho - \sigma$, then functional relation (3.3.1) reduces to well known functional relation of I-function and logarithmic derivative of gamma function obtained by applying Riemann-Liouville operator, which was discussed in chapter 2.

$$
\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k!} x^{m,n+1} \left[ ax \right] (1 - \rho, 1, (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} 
\left( b_j, B_j \right)_{1,m}; \left( b_{ji}, B_{ji} \right)_{m+1,q_i}, (1 - \rho - k, 1) 
\right] = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (ax)^\nu \chi(\xi)}{\nu! B_h} \{ \psi(\rho + s) - \psi(\rho - \sigma + s) \} \right] (3.3.2.1)
$$

where $\xi = (b_h + v)/B_h, \Re(\rho + \min b_j/B_j) > 0, j = 1, 2, ..., m; \Re(\sigma) \geq 0, \arg ax < (1/2)\pi \lambda_i, \lambda_i = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_i} B_{ji} > 0; \mu_i = \sum_{j=1}^{n} A_j + \sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{m} B_j + \sum_{j=m+1}^{q_i} B_{ji} \leq 0.$

(ii) If we put $p_i = p, q_i = q$ for all values of $i$ and $r = 1$, then the functional relation (3.3.2.1) reduces to well known result [91] for H-function:

$$
\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k!} H_{p+1,q+1}^{m,n+1} \left[ ax \right] (1 - \rho, 1, (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} 
\left( b_j, B_j \right)_{1,m}; \left( b_{ji}, B_{ji} \right)_{m+1,q_i}, (1 - \rho - k, 1) 
\right] = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (ax)^\nu \chi(\xi)}{\nu! B_h} \{ \psi(\rho + \xi) - \psi(\rho - \sigma + \xi) \} \right] (3.3.2.2)
$$

where $\xi = (b_h + v)/B_h, \Re(\rho + \min b_j/B_j) > 0, j = 1, 2, ..., m; \Re(\sigma) \geq 0, \arg ax < (1/2)\pi \lambda, \lambda > 0; \mu \leq 0. \lambda and \mu are defined by
\[ \lambda = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0 \quad \text{and} \quad \mu = \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j. \]

3.3.3 Theorem

If \( \xi = (b_h + v)/B_h \), \( Re(\rho + \min b_j/B_j) > 0 \), \( j = 1,2,\ldots,m; \ Re(\sigma) \geq 0, \)
\[ |\arg ax| < (1/2)\pi \lambda_i, \quad \lambda_i > 0; \quad \mu_i \leq 0. \]
\[ \lambda_i = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_i} B_{ji} > 0, \text{and} \]
\[ \mu_i = \sum_{j=1}^{n} A_j + \sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{m} B_j + \sum_{j=m+1}^{q_i} B_{ji} < 0 \]

or \( \mu_i = 0 \); \( i = 1,2,\ldots,r \), then
\[ \sum_{k=1}^{r} (-1)^{k(\sigma_k)} \frac{t^{m+n}}{p_i+1,q_i+1,r} \left[ \begin{array}{c} (a_j,A_j)_{1,n} ; (a_{ji},A_{ji})_{n+1,p_i} ; (1 - \rho + \eta, 1) \\ (1 - \rho + \eta - k, 1), (b_j,B_j)_{1,m} ; (b_{ji},B_{ji})_{m+1,q_i} \end{array} \right] \]
\[ = \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^{v} (ax)^{v} X(s)}{v! B_h} \{ \psi(1 - \rho + \eta + \sigma - s) - \psi(1 - \rho + \eta - s) \} \]

Proof: Differentiating both sides of (3.2.7) with respect to \( \rho \) and using Leibnitz rule,

we observe that
\[ \frac{x^n}{\Gamma(\sigma)} \int_{x}^{\infty} t^{\rho - \sigma - \eta - 1} \ln t (t - x)\sigma - 1 \ p_{m,n}^{r} p_{i,q_i} \left[ \begin{array}{c} (a_j,A_j)_{1,n} ; (a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m} ; (b_{ji},B_{ji})_{m+1,q_i} \end{array} \right] dt \]
\[
\sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (ax)^\xi x(\zeta)\Gamma(1-\rho+\eta-\xi)}{\nu!B_h \Gamma(1-\rho+\eta+\sigma-\xi)} \{\psi(1-\rho+\eta-\xi) - \psi(1-\rho+\eta+\sigma-\xi)\}
\]

where \(\psi\) represent the logarithmic derivative of gamma function and

\[
\xi = (b_h + v)/B_h.
\]

Kober fractional integral operator is given by

\[
K_{x,\infty}^{\sigma,\eta} f(x) = J_{x,\infty}^{\alpha,0,\eta} f(x) = \frac{x^\eta}{\Gamma} \int_{x}^{\infty} (t - x)^{\sigma - 1} t^{-\sigma - \eta} f(t) dt
\]

writing L.H.S. of equation (3.3.3.1) in terms of Kober fractional integral operator and designating R.H.S as \(J(a_{p_l}, A_{p_l}; b_{q_l}, B_{q_l}; \rho, \sigma, \eta, a)\), it takes the form

\[
K_{x,\infty}^{\sigma,\eta} \left( x^{\rho - 1} \ln x \right)^n_{p_l, q_l} ; r (ax) = J(a_{p_l}, A_{p_l}; b_{q_l}, B_{q_l}; \rho, \sigma, \eta, a)
\]

On account of the property of analyticity and continuity of Kober operator at \(\sigma = 0\) and \(\eta = 0\), we can replace \(\eta\) by \(\eta - \sigma\) and then \(\sigma\) by \(-\sigma\). Hence for the differentiation of \(x^{\rho - 1} \ln x \right)^n_{p_l, q_l} ; r (ax)\) of an arbitrary order, we find that

\[
K_{x,\infty}^{\sigma,\eta - \sigma} \left( x^{\rho - 1} \ln x \right)^n_{p_l, q_l} ; r (ax)
\]
\[
\sum_{n=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r (ax)^r x(\xi) \Gamma(1 - \rho + \eta + \sigma - \xi)}{v! B_h \Gamma(1 - \rho + \eta - \xi)} \{\psi(1 - \rho + \eta + \sigma - \xi) - \psi(1 - \rho + \eta - \xi)\}
\]

where \( \Re(\rho + \min b_j/B_j) > 0, \Re \sigma \geq 0, |\arg ax| < (1/2)\pi \lambda_i, \lambda_i > 0; \mu_i \leq 0.\)

On writing above integral equation in the operator form, we have

\[
K_{x,\infty}^{\sigma,\eta} f(x) = x^{\rho-1} \ln x I_{p_i, q_i}^{m,n} : r(ax)
\]

Operating on both sides by \( K_{x,\infty}^{-\sigma,\eta - \sigma} \), we get

\[
f(x) = K_{x,\infty}^{-\sigma,\eta - \sigma} \left( x^{\rho-1} \ln x I_{p_i, q_i}^{m,n} : r(ax) \right)
\]

Using equation (3.3.3.2), we get

\[
f(x) = x^{\rho-1} \ln x I_{p_i, q_i}^{m,n+1,n} : r(ax)
\]

\[
\sum_{n=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r (ax)^r x(\xi) \Gamma(1 - \rho + \eta + \sigma - \xi)}{v! B_h \Gamma(1 - \rho + \eta - \xi)} \{\psi(1 - \rho + \eta + \sigma - \xi) - \psi(1 - \rho + \eta - \xi)\}
\]
In order to find the solution, we substitute the value of \( f(x) \) from equation (3.3.6) in to (3.3.3.3) in terms of argument \( t \). Thus we have

\[
\frac{x^\eta}{\Gamma\sigma} \int_x^\infty (t - x)^{\sigma - 1} t^{-\eta} t^{\rho - 1} \times
\left[ \ln t \null_{p_i+1,q_i+1;r}^{m+1,n} \left( \begin{array}{c} (a_j, A_j)_{n+1,p_i'}^1; (a_{ji}, A_{ji})_{n+1,p_i'}^1 \left( 1 - \rho + \eta, 1 \right) \\ (1 - \rho + \eta + \sigma, 1), (b_j, B_j)_{1,m}^1; (b_{ji}, B_{ji})_{m+1,q_i}^1 \end{array} \right) \right] \\
\sum_{h=1}^{\infty} \sum_{\nu=0}^{\infty} \left( -1 \right)^\nu \frac{(x)\ln(x)\Gamma(1-\rho+\eta+\sigma-\xi)}{\nu! B_h \Gamma(1-\rho+\eta-\xi)} \{ \psi(1 - \rho + \eta + \sigma - \xi) - \psi(1 - \rho + \eta - \xi) \} \right] \, dt = x^{\rho - 1} \ln x \null_{p_i+1,q_i+1;r}^{m+1,n}(ax)
\]  

(3.3.7)

Taking

\[
\ln t = \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{k x^k}
\]

(3.3.8)

with the interval of the convergence \( 0 \leq t \leq 2x \).

Using value of \( \ln t \) from equation (3.3.8) into equation (3.3.7), we get

\[
\frac{x^\eta}{\Gamma\sigma} \int_x^\infty (t - x)^{\sigma - 1} t^{-\eta} t^{\rho - 1} \left[ \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{k x^k} \right] \times
\left[ \ln t \null_{p_i+1,q_i+1;r}^{m+1,n} \left( \begin{array}{c} (a_j, A_j)_{n+1,p_i'}^1; (a_{ji}, A_{ji})_{n+1,p_i'}^1 \left( 1 - \rho + \eta, 1 \right) \\ (1 - \rho + \eta + \sigma, 1), (b_j, B_j)_{1,m}^1; (b_{ji}, B_{ji})_{m+1,q_i}^1 \end{array} \right) \right] +
\]
\[
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (at)^v X(\xi) \Gamma(1-\rho+\eta+\sigma-\xi)}{v! B_h \Gamma(1-\rho+\eta-\xi)} \{\psi(1-\rho+\eta+\sigma-\xi) - \psi(1-\rho+\eta-
\xi)\} \] \ dt = x^{\rho-1} \ln x \ I_{p_i, q_i : r}(ax)

or

\[
x^{\eta} \int_{\sigma}^{\infty} (t-x)^{\rho-1} \int x \times \frac{(a_i, A_j)}{1, n} \cdot (a_i, A_j)_{n+1, p} \cdot (1-\rho+\eta, 1) \\
(1-\rho+\eta+\sigma, 1) \cdot (b_j, B_j)_{1, m} \cdot (b_j, B_j)_{m+1, q_i} \] \ dt

\[
\frac{x^{\eta} \int_{\sigma}^{\infty} (t-x)^{\rho-1} \int x \times} \sum_{k=1}^{\infty} \frac{(x-t)^k}{k/x^k} \] \\
\sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (at)^v X(\xi) \Gamma(1-\rho+\eta+\sigma-\xi)}{v! B_h \Gamma(1-\rho+\eta-\xi)} \{\psi(1-\rho+\eta+\sigma-\xi) - \psi(1-\rho+\eta-\xi)\} \ dt

= x^{\rho-1} \ln x \ I_{p_i, q_i : r}(ax) \tag{3.3.3.9}

For simplifying equation (3.3.3.9), we compute the value of the following integrals.

\[
x^{\eta} \int_{\sigma}^{\infty} (t-x)^{\rho-1} \int x \times \frac{(a_i, A_j)}{1, n} \cdot (a_i, A_j)_{n+1, p} \cdot (1-\rho+\eta, 1) \\
(1-\rho+\eta+\sigma, 1) \cdot (b_j, B_j)_{1, m} \cdot (b_j, B_j)_{m+1, q_i} \] \ dt
\[
= x^{\rho-1} \ln x \int_{p_i, q_i : r}^{m,n} \left[ a x \left( \begin{array}{c}
a_j, A_j \\ b_j, B_j \\
\end{array} \right)_{1,n} ; \left( \begin{array}{c}
a_{ji}, A_{ji} \\ b_{ji}, B_{ji} \\
\end{array} \right)_{n+1,p_i} \\
\right] \]

\[
\frac{x^n}{\Gamma(\sigma)} \int_x^{\infty} (t-x)^{ \sigma-1} t^{-\sigma-\eta} t^{\rho-1} \sum_{k=1}^{\infty} \frac{(x-t)^k}{k^{x^k}} \times \int_{p_i+1,q_i+1_1}^{m+1,n} \left[ a t \left( \begin{array}{c}
a_j, A_j \\ b_j, B_j \\
\end{array} \right)_{1,m} ; \left( \begin{array}{c}
a_{ji}, A_{ji} \\ b_{ji}, B_{ji} \\
\end{array} \right)_{n+1,p_i} \\
\right] dt \\
\]

\[
= x^{\rho-1} \sum_{k=1}^{\infty} \frac{(-1)^k (1)^k}{k} \int_{p_i+1,q_i+1_1}^{m+1,n} \left[ a x \left( \begin{array}{c}
a_j, A_j \\ b_j, B_j \\
\end{array} \right)_{1,m} ; \left( \begin{array}{c}
a_{ji}, A_{ji} \\ b_{ji}, B_{ji} \\
\end{array} \right)_{n+1,p_i} \\
\right] \]

\[
\frac{x^n}{\Gamma(\sigma)} \int_x^{\infty} (t-x)^{ \sigma-1} t^{-\sigma-\eta} t^{\rho-1} \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (\alpha x)^v \chi(\xi)}{v! B_h^v \Gamma(1-\rho+\eta+\xi)} \{ \psi(1-\rho+\eta+\sigma-\xi) - \psi(1-\rho+\eta-\xi) \}
\]

Now we put the values of these three integrals (3.3.3.10), (3.3.3.11) and (3.3.3.12) in the equation (3.3.3.9). Thus we get
\[ x^{\rho - 1} \ln x \int_{p_i, q_i}^{m,n} \left[ ax \left( \begin{array}{c} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{array} \right) \right] - x^{\rho - 1} \sum_{k=1}^{\infty} \frac{(-1)^k (\sigma)_k}{k} \times \]

\[ \int_{p_i+1, q_i+1}^{m,n+1} \left[ ax \left( \begin{array}{c} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i}, (1 - \rho + \eta, 1) \\ (1 - \rho + \eta - k, 1), (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{array} \right) \right] + \]

\[ x^{\rho - 1} \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (ax)^v x(\xi)}{v! B_h} \{ \psi(1 - \rho + \eta + \sigma - \xi) - \psi(1 - \rho + \eta - \xi) \} \]

\[ = x^{\rho - 1} \ln x \int_{p_i, q_i}^{m,n} \left[ ax \left( \begin{array}{c} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{array} \right) \right] \]

or

\[ \sum_{k=1}^{\infty} \frac{(-1)^k (\sigma)_k}{k} \int_{p_i+1, q_i+1}^{m,n+1} \left[ ax \left( \begin{array}{c} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i}, (1 - \rho + \eta, 1) \\ (1 - \rho + \eta - k, 1), (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i} \end{array} \right) \right] \]

\[ = \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v (ax)^v x(\xi)}{v! B_h} \{ \psi(1 - \rho + \eta + \sigma - \xi) - \psi(1 - \rho + \eta - \xi) \} \]

This completes the proof.

### 3.3.4 Special cases

(i) If we put \( \eta = 0 \), then functional relations (3.3.3) reduces to well known functional relations of I-function and logarithmic derivative of gamma function obtained by using Weyl fractional integral operator, which was discussed in chapter 2.
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k} \sigma_{k}}{k} p_{i+1,q_{i}+1}^{m,n+1} \left[ a x \right] \left[ (a_{j}, A_{j})_{1,n} ; (a_{ji}, A_{ji})_{n+1,p_{i}} ; (1 - \rho, 1) ; (1 - \rho - k, 1) ; (b_{j}, B_{j})_{1,m} ; (b_{ji}, B_{ji})_{m+1,q_{i}} \right] \\
= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} (ax)^{\nu} \chi(\xi)}{\nu! B_{h}} \left[ \psi(1 - \rho + \sigma - \xi) - \psi(1 - \rho - \xi) \right] 
\]

where \( \xi = (b_{h} + v)/B_{h}, \text{Re}(\rho + \min b_{j}/B_{j}) > 0, j = 1,2, \ldots, m; \text{Re}(\sigma) \geq 0, \)

\[|\text{arg} ax| < (1/2)\pi \lambda, \; \lambda > 0; \; \mu \leq 0. \; \lambda \text{ and } \mu \text{ are defined by}\]
\[ \lambda = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0 \text{ and } \mu = \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j. \]

### 3.4 An Application of the Functional Relation

Computable representation of I-function can also provide an alternative method of obtaining the functional relations (3.3.1) and (3.3.3) as mentioned by Saxena [118] and Chen and Shrivastava [17]. But the method chalked out above has an additional advantage that it can be used to obtain solution of the integral equations (3.3.1.5) and (3.3.3.4).

To illustrate the method more explicitly, we obtain the solution of following integral equation of Volterra type, involving Fox Wright function.

\[
_0D_x^{-\sigma} f(x) = 
\]

\[
x^{\rho-1} \ln_x \psi_q \left[ \frac{(a_1, A_1),..., (a_p, A_p)}{(b_1, B_1),..., (b_q, B_q)} \right] ax + 
\]

\[
x^{\gamma-1} \psi_q \left[ \frac{(a_1, A_1),..., (a_p, A_p)}{(b_1, B_1),..., (b_q, B_q)} \right] ax^{-\beta} \]  
(3.4.1)

Applying \(_0D_x^{-\sigma}\) to both sides of equation (3.4.1), we get

\[
f(x) = _0D_x^{-\sigma} \left\{ t^{\rho-1} \ln t \psi_q \left[ \frac{(a_1, A_1),..., (a_p, A_p)}{(b_1, B_1),..., (b_q, B_q)} \right] at \right\} (x) + 
\]

\[
_0D_x^{-\sigma} \left\{ t^{\gamma-1} \psi_q \left[ \frac{(a_1, A_1),..., (a_p, A_p)}{(b_1, B_1),..., (b_q, B_q)} \right] at^{-\beta} \right\} (x) \]  
(3.4.2)

Following relations are well-known in the literature on Fractional Calculus
\[(i) \quad 0D_x^\sigma \left\{ t^{y-1} \psi_q \left[ (a_1, A_1), \ldots, (a_p, A_p) \at \right] \right\}(x) = x^{y-\sigma} \psi_q \left[ (a_1, A_1), \ldots, (a_p, A_p) \at \right] (3.4.3)\]

and

\[(ii) \quad \frac{1}{\rho} \int_0^x (x-t)^{\sigma-1} \psi_q \left[ (a_1, A_1), \ldots, (a_p, A_p) \at \right] dt = x^{\rho+\sigma} \psi_q \left[ (1 - \rho, 1)(a_1, A_1), \ldots, (a_p, A_p) \at \right] (3.4.4)\]

Differentiating both sides of (3.4.4) with respect to $\rho$ and employing Leibnitz rule, we arrive at

\[
\frac{1}{\rho} \int_0^x (x-t)^{\sigma-1} \ln t (x-t)^{\sigma-1} \psi_q \left[ (a_1, A_1), \ldots, (a_p, A_p) \at \right] dt = x^{\rho+\sigma} \psi_q \left[ (1 - \rho, 1)(a_1, A_1), \ldots, (a_p, A_p) \at \right] +
\]

\[
\sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu(\nu)!}{\nu!B_h} \frac{\Gamma(\rho+s)}{\Gamma(\rho+\sigma+s)} \left\{ \psi(\rho + s) - \psi(\rho + \sigma + s) \right\} (3.4.5)
\]

where $s = \frac{b_h + \nu}{B_h}$

We know that Riemann - Liouville operator is given by
\[ R_{\sigma} f(x) = 0D_x^{-\sigma} f(x) = \frac{1}{\Gamma\sigma} \int_0^x (x - t)^{\sigma - 1} f(t) dt \]

Equation (3.4.5) can be written in terms of Riemann-Liouville operator as

\[ _0D_x^{-\sigma} \left( x^{\rho - 1} \ln x \right)_{\psi_q} \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right] ax \]

\[ = x^{\rho + \sigma - 1} \ln x^{-p+1} \psi_{q+1} \left[ \begin{array}{c} (1 - \rho, 1)(a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q)(1 - \rho - \sigma, 1) \end{array} \right] ax \]

\[ + \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v v!}{\Gamma(\rho + s)} \Gamma(\rho + \sigma + s) \left\{ \psi(\rho + s) - \psi(\rho + \sigma + s) \right\} \]

\[ \sum_{v=0}^{\infty} \frac{(-1)^v v!}{\Gamma(\rho + s)} \Gamma(\rho + \sigma + s) \left\{ \psi(\rho + s) - \psi(\rho + \sigma + s) \right\} \]

where \( f(t) = t^{\rho - 1} \ln t \)_{\psi_q} \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right] at \]

On account of analyticity and continuity at \( \sigma = 0 \), we can interchange the roles of

\(- \sigma \) and \( \sigma \). Hence differentiating

\[ x^{\rho - 1} \ln x \psi_q \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right] ax \]

to an arbitrary order, we find that

\[ _0D_x^{\sigma} \left( x^{\rho - 1} \ln x \right)_{\psi_q} \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right] ax \]
\[ x^{\rho-\sigma-1} \ln x_{p+1} \psi_{q+1} \left[ (1 - \rho, 1)(a_1, A_1), ..., (a_p, A_p) \right] \left[ (b_1, B_1), ..., (b_q, B_q)(1 - \rho + \sigma, 1) \right] ax + \]

\[ \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v(ax)^v X(s)}{v! B_h} \frac{\Gamma(\rho + s)}{\Gamma(\rho - \sigma + s)} \{ \psi(\rho + s) - \psi(\rho - \sigma + s) \} \]

(3.4.6)

Using (3.4.6) and (3.4.3) in (3.4.2), we get

\[ f(x) = x^{\rho-\sigma-1} \ln x_{p+1} \psi_{q+1} \left[ (1 - \rho, 1)(a_1, A_1), ..., (a_p, A_p) \right] \left[ (b_1, B_1), ..., (b_q, B_q)(1 - \rho + \sigma, 1) \right] ax + \]

\[ \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{(-1)^v(ax)^v X(s)}{v! B_h} \frac{\Gamma(\rho + s)}{\Gamma(\rho - \sigma + s)} \{ \psi(\rho + s) - \psi(\rho - \sigma + s) \} + \]

\[ x^{\gamma-\sigma-1} p_{q+1} \psi_{q+1} \left[ (a_1, A_1), ..., (a_p, A_p)(\gamma, \beta) \right] \left[ (b_1, B_1), ..., (b_q, B_q)(\sigma - \gamma, \beta) \right] ax^{-\beta} \]

which is the solution of the integral equation, under consideration.

### 3.5 Conclusion

The results derived in this chapter are further extensions of those derived in previous chapter. These results are application oriented and can play a significant role in the field of fractional calculus, engineering and technology. The method used in this chapter has an edge over other methods as it can also provide solution of the corresponding integral equations. Also we can find other functional relations taking this chapter as reference and changing fractional integral operators. In the next chapter we shall find other functional relations using Saigo fractional integral operator.