CHAPTER - 3

LATTICE PROPERTIES ON WEAK* IDEAL SEMIGROUP COMPACTIFICATIONS

Introduction

In topological spaces, the theory of compactifications is well developed starting with the work of A. Tychonoff (1930) [TY]. Contribution of R.E Chandler (1906) [CH] and R.C Walker (1974) are mainly on the theory of Hausdorff compactification. If $X$ is a locally compact space, all Hausdorff compactifications of $X$ are obtained by considering Hausdorff quotients of $\beta X - X$. In [MA] Magill K.D.Jr. observed the properties of the lattice of compactifications of locally compact space $X$. In [K] studied lattice of semigroup compactification of a topological semigroup $S$. We observed weak ideals, joint weak ideals and complimentary joint weak ideals are developed in the theory of semigroup [K₁]

In this section, we study the family of all weak* ideal semigroup compactifications $W*I(S)$ determined by weak ideals, joint weak ideals and
complementary joint ideals and investigate some results about meet and join of $W^*I(S)$.

**Definition: 3.1**

A non-empty non-singleton subset $\omega$ of a semigroup $S$ is said to be *weak ideal* if and only if $ax, bx, \in \omega$ or $ax = bx$ and $xa, xb, \in \omega$ or $xa = xb$ for all $a, b \in \omega$ and for every $x \in S$.

**Definition: 3.2**

Two non-empty subsets $\omega_1, \omega_2$ of a semigroup $S$ are said to be *joint weak ideals* if either $ax, bx, \in \omega_1$, or $ax, bx \in \omega_2$, or $ax = bx$ and $xa, xb, \in \omega_1$ or $xa, xb, \in \omega_2$ or $xa = xb$ for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$.

**Definition: 3.3**

Two joint ideals $\omega_1, \omega_2$ of a semigroup $S$ are said to be *complementary* if they are disjoint and $\omega_1 \cup \omega_2 = S$. 
Weak* ideal semigroup compactification (W*I(S))

In this section we consider W*I(s) the family of all semigroup compactifications determined by weak ideals, joint weak ideals and complementary joint weak ideals of Bohr compactifications and we define this collection as a weak* ideal semigroup compactification.

Definition: 3.4

Weak* Ideal Semigroup Compactification means semigroup compactification determined by weak ideals, joint weak ideals, complementary joint weak ideals of a topological semigroup.

3.1 Meet and Join of weak* ideal semigroup compactification

Here we describe some results about meet and join of weak* ideal semigroup compactifications W*I(S) from the following example.
**Example: 3.1.1**

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$ where $B = \{1/2, 1/3, 1/4, 1\}$ with discrete topology and multiplication defined by $xy = \max \{1/2, xy\}$.

Then the closed congruences are $R_1 = \Delta$, 

- $R_2 = (\{1/2, 1/3\} \times \{1/2, 1/3\}) \cup \Delta$, 
- $R_3 = (\{1/2, 1/4\} \times \{1/2, 1/4\}) \cup \Delta$, 
- $R_4 = (\{1/2, 1\} \times \{1/2, 1\}) \cup \Delta$, 
- $R_5 = (\{1/3, 1/4\} \times \{1/3, 1/4\}) \cup \Delta$, 
- $R_6 = (\{1/2, 1/3, 1\} \times \{1/2, 1/3, 1\}) \cup \Delta$, 
- $R_7 = (\{1/2, 1/4, 1\} \times \{1/2, 1/4, 1\}) \cup \Delta$, 
- $R_8 = (\{1/2, 1/3, 1/4\} \times \{1/2, 1/3, 1/4\}) \cup \Delta$, 
- $R_9 = (\{1/2, 1\} \times \{1/2, 1\}) \cup (\{1/3, 1/4\} \times \{1/3, 1/4\})$, 
- $R_{10} = B \times B$.
\begin{array}{c|cccc}
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 \\
\hline
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
\end{array}

Let \( W^* I(S) \) denotes the family of all semigroup compactification determined by weak ideals, joint weak ideals and complementary joint weak ideals determined by, say \( K_2 = \{1/2, 1/3\} \), 
\[ K_3 = \{1/2, 1/4\}, \quad K_4 = \{1/2, 1\}, \quad K_5 = \{1/3, 1/4\}, \]
\[ K_6 = \{1/2, 1/3, 1\}, \quad K_7 = \{1/2, 1/4, 1\}, \]
\[ K_8 = \{1/2, 1/3, 1/4\}, \quad K_9 = \{1/2, 1\} \cup \{1/3, 1/4\}. \]

Then \( (\alpha, K_2) \), \( (\alpha, K_3) \), \( (\alpha, K_4) \), \( (\alpha, K_5) \) etc be the semigroup compactifications determined by the above closed congruences. We observed
\[ (\alpha, K_6) = (\alpha, K_2) \land (\alpha, K_4) = (\alpha, K_2 \cup K_4); \quad \text{since}, \quad K_2 \cap K_4 \neq \emptyset \]
\[ (\alpha, K_7) = (\alpha, K_3) \land (\alpha, K_4) = (\alpha, K_3 \cup K_4); \quad \text{since} \quad K_3 \cap K_4 \neq \emptyset. \]
\[ (\alpha, K_8) = (\alpha, K_2) \land (\alpha, K_3) \land (\alpha, K_5) = (\alpha, K_2 \cup K_3 \cup K_5) \; \text{since} \]
\[ K_2 \cap K_3 \neq \emptyset, \quad K_2 \cap K_5 \neq \emptyset, \quad K_3 \cap K_5 \neq \emptyset, \]
\[ (\alpha, K_9) = (\alpha, K_4) \land (\alpha, K_5) = (\alpha, K_4, K_5); \quad \text{since} \quad K_4 \cap K_5 = \emptyset. \]
\[ (\alpha, K_3) \lor (\alpha, K_4) = (\alpha, K_3 \cap K_4); \]
Example: 3.1.2

Let $S$ be a topological semigroup with Bohr compactifications $(\beta, B)$ where $B = \{a, b, c, d, e\}$ with discrete topology and usual multiplication where,

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1/3 \\ 0 & 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

The closed congruences are

$$R_1 = \Delta, \quad R_2 = (\{b, c\} \times \{b, c\}) \cup \Delta, \quad R_3 = (\{b, d\} \times \{b, d\}) \cup \Delta,$$

$$R_4 = (\{b, e\} \times \{b, e\}) \cup \Delta,$$
\[ R_5 = (\{c,d\} \times \{c,d\}) \cup \Delta, \quad R_6 = (\{c,e\} \times \{c, e\}) \cup \Delta, \]
\[ R_7 = (\{d, e\} \times \{d, e\}) \cup \Delta, \]
\[ R_8 = (\{b,c\} \times \{b,c\}) \cup (\{d,e\} \times \{d,e\}) \cup \Delta, \]
\[ R_9 = (\{b,d\} \times \{b,d\}) \cup (\{c,e\} \times \{c,e\}) \cup \Delta, \]
\[ R_{10} = (\{b,e\} \times \{b,e\}) \cup (\{c,d\} \times \{c,d\}) \cup \Delta, \]
\[ R_{11} = (\{b,c,d\} \times \{b,c,d\}) \cup \Delta, \]
\[ R_{12} = (\{b,c,e\} \times \{b,c,e\}) \cup \Delta, \]
\[ R_{13} = (\{b,d,e\} \times \{b,d,e\}) \cup \Delta, \]
\[ R_{14} = (\{c,d,e\} \times \{c,d,e\}) \cup \Delta, \]
\[ R_{15} = (\{b,c,d,e\} \times \{b,c,d,e\}), \quad R_{16} = B \times B \]
Let $K_2 = \{b,c\}$, $K_3 = \{b,d\}$, $K_4 = \{b,e\}$, $K_5 = \{c,d\}$, $K_6 = \{c,e\}$, $K_7 = \{d,e\}$, $k_8 = \{b,c\} \cup \{d,e\}$ etc.

Let $W^*(S)$ denotes the family of all semigroup compactifications determined by the above closed congruence denoted as $(\alpha,K_2)$, $(\alpha,K_3)$, $(\alpha,K_4)$, etc. We can observe the following from the lattice of weak* ideal semigroup compactifications.

\[(\alpha,K_8) = (\alpha,K_2) \wedge (\alpha,K_7) = (\alpha,K_2,K_7), \text{ since } K_2 \cap K_7 = \emptyset\]
\[(\alpha,K_9) = (\alpha,K_3) \wedge (\alpha,K_6) = (\alpha,K_3,K_6) \text{ since } K_3 \cap K_6 = \emptyset\]
\[(\alpha,K_{10}) = (\alpha,K_4) \wedge (\alpha,K_5) = (\alpha,K_4,K_5) \text{ since } K_4 \cap K_5 = \emptyset\]
\[(\alpha,K_{11}) = (\alpha,K_2) \wedge (\alpha,K_3) \wedge (\alpha,K_5) = (\alpha,K_2 \cup K_3 \cup K_5), \text{ since } K_2 \cap K_3 \neq \emptyset, K_2 \cap K_5 \neq \emptyset, K_3 \cap K_5 \neq \emptyset.
\[(\alpha,K_{12}) = (\alpha,K_2) \wedge (\alpha,K_4) \wedge (\alpha,K_6) = (\alpha,K_2 \cup K_4 \cup K_6) \text{ since } K_2 \cap K_4 \neq \emptyset, K_2 \cap K_6 \neq \emptyset, K_4 \cap K_6 \neq \emptyset.
\[(\alpha,K_{13}) = (\alpha,K_3) \wedge (\alpha,K_4) \wedge (\alpha,K_7) = (\alpha,K_2 \cup K_4 \cup K_7) \text{ since } K_3 \cap K_4 \neq \emptyset, K_4 \cap K_7 \neq \emptyset, K_5 \cap K_7 \neq \emptyset.
\[(\alpha,K_{14}) = (\alpha,K_5) \wedge (\alpha,K_6) \wedge (\alpha,K_7) = (\alpha,K_5 \cup K_6 \cup K_7) \text{ etc.}
\[(\alpha,K_3) \vee (\alpha,K_6) = (\alpha,K_3 \cap K_6)
\[(\alpha,K_2) \vee (\alpha,K_3) \vee (\alpha,K_5) = (\alpha,K_2 \cap K_3 \cap K_5) \text{ etc.}

From the above examples we can conclude the following result:-
Result: 3.1.3

Let $S$ be a topological semigroup with Bohr compactifications $(\beta, B)$, where $B$ is finite. Let $K_1, K_2, \ldots, K_n$ are closed non-singleton subsets of $B$ and $W^*I(S)$ is the family of all weak*ideal semigroup compactification of $S$, then:-

Theorem-1

$$(\alpha, K_1) \land (\alpha, K_2) \land \ldots \land (\alpha, K_n) = (\alpha, K_1, K_2, \ldots, K_n);$$
if $K_1, K_2, \ldots, K_n$ are complementary joint weak ideals
(i.e., $K_1 \cap K_2 \cap \ldots \cap K_n = \emptyset$ and $K_1 \cup K_2 \cup \ldots \cup K_n = B$)

Theorem-2

$$(\alpha, K_1) \land (\alpha, K_2) \land \ldots \land (\alpha, K_n) = (\alpha, K_1 \cup K_2 \cup \ldots \cup K_n);$$
if $K_1 \cap K_2 \neq \emptyset, K_1 \cap K_3 \neq \emptyset, \ldots, K_{n-1} \cap K_n \neq \emptyset$;
if $K_1, K_2, \ldots, K_n$ are joint weak ideals

Theorem-3

$$(\alpha, K_1) \lor (\alpha, K_2) \lor \ldots \lor (\alpha, K_n) = (\alpha, K_1 \cap K_2 \cap \ldots \cap K_n),$$
if $K_1, K_2, \ldots, K_n$ are weak ideals.

Theorem-1

Proof: Let $K_1$ and $K_2$ be complementary ideals
i.e., $K_1 \cap K_2 = \emptyset$. Then we can easily prove that
$$(\alpha, K_1) \land (\alpha, K_2) = (\alpha, K_1, K_2).$$
Suppose that $K_1 \cap K_2 = \emptyset$, it follows immediately that $(\alpha, K_1) \geq (\alpha, K_1, K_2)$ and $(\alpha, K_2) \geq (\alpha, K_1, K_2)$.

Suppose that $(\gamma, \gamma_s)$ is any other weak* ideal semigroup compactification determined by complementary ideals $K_1$ and $K_2$ such that $(\alpha, K_1) \geq (\gamma, \gamma_s)$ and $(\alpha, K_2) \geq (\gamma, \gamma_s)$. Then $K_1$ is a subset of some set corresponding to the compactification $(\gamma, \gamma_s)$.

Then $(\alpha, K_1, K_2) \geq (\gamma, \gamma_s)$. Therefore in this particular case (complementary).

$(\alpha, K_1) \wedge (\alpha, K_2) = (\alpha, K_1, K_2)$

Since $K_1 \times K_1 \subseteq K_1 \times K_1 \cup K_2 \times K_2$ and $K_2 \times K_2 \subseteq K_1 \times K_1 \cup K_2 \times K_2$

$(\alpha, K_1) \geq (\alpha, K_1, K_2)$, $(\alpha, K_2) \geq (\alpha, K_1, K_2)$

Therefore $(\alpha, K_1) \wedge (\alpha, K_2) = (\alpha, K_1, K_2)$

This is true for all $K_1, K_2, \ldots, K_n \subseteq B$, if $K_1, K_2, \ldots, K_n = \emptyset$

Thus we proved $(\alpha, K_1) \wedge (\alpha, K_2) \wedge \ldots \wedge (\alpha, K_n) = (\alpha, K_1 \cup K_2 \cup \ldots \cup K_n)$; if $K_1, K_2, \ldots, K_n$ are complementary ideals.

**Theorem-2**

**Proof:** Let $K_1$ and $K_2$ be joint weak ideals such that $K_1 \cap K_2 \neq \emptyset$
Let \( p \in K_1 \cap K_2 \), where \( K_1 \) & \( K_2 \) are non-singleton subset of \( B \). Let \( K_1 = \{p,q\}, K_2 = \{p,r\} \), then clearly \( \{q,r\} \) determines a weak*ideal semigroup compactification.

Consider \( K_1 \cup K_2 = \{p,q,r\} \). Clearly \( K_1 \subseteq K_1 \cup K_2 \) and \( K_2 \subseteq K_1 \cup K_2 \), then the weak*ideal semigroup compactification determined by \( K_1 \cup K_2 = \{p,q,r\} \); that is \((a, K_1 \cup K_2)\).

Then \((a, K_1) \land (a, K_2) = (a, K_1 \cup K_2)\), if \( K_1 \) and \( K_2 \) are joint weak ideals. This is true for all \( K_1, K_2, \ldots, K_n \subseteq B \), such that \( K_1 \cap K_2 \cap \ldots \cap K_n \neq \phi \).

Thus we proved,

\[
(a, K_1) \land (a, K_2) \land \ldots \ldots \land (a, K_n) = (a, K_1 \cup K_2 \cup \ldots \cup K_n).
\]

**Theorem 3**

**Proof:** Let \( K_1 \) and \( K_2 \) are weak ideals \((a, K_1)\) determines a weak*ideal semigroup compactification. Similarly \((a, K_2)\) determines a weak* ideal semigroup compactification.

Clearly \((a, K_1) \lor (a, K_2) \leq (a, K_1)\)

and \((a, K_1) \lor (a, K_2) \leq (a, K_2)\)

This is possible in the case that \( K_1 \cap K_2 \subseteq K_1 \) and \( K_1 \cap K_2 \subseteq K_2 \) (from definition of weak ideals).
More over $K_1 \cap K_2$ determines weak*ideal semigroup compactification. In this case it is clear from example also that

$$(\alpha, K_1) \lor (\alpha, K_2) = (\alpha, K_1 \cap K_2)$$

This is true for all weak ideals

$K_1, K_2, \ldots, K_n \subseteq B$

Hence $(\alpha, K_1) \lor (\alpha, K_2) \lor \ldots \lor (\alpha, K_n) = (\alpha, K_1 \cap K_2 \cap \ldots \cap K_n)$.

### 3.2 Dual atoms and atoms in the weak* Ideal semigroup compactification

**Definition: 3.2.1**

A non-empty subset $\omega$ of a semigroup $S$ is said to be a weak* ideal in $S$ if $\omega$ is a weak ideal or $\omega$ is a joint weak ideal or $\omega$ is a complementary joint weak ideal of $S$ such that $1 < |\omega| < |S|$.

**Definition :3.2.2**

Let $S$ be a topological semigroup and $\Omega^*$ is the family of all weak* ideals of $S$. An element $\omega \in \Omega^*$ is called maximal in $\Omega^*$ provided, there does not exist an $\omega^1 \in \Omega^*$ such that $\omega^1$ properly
containing \( \omega \). Similarly an element \( \omega \in \Omega^* \) is called minimal in \( \Omega^* \) provided, there does not exist an \( \omega^1 \in \Omega^* \) such that \( \omega^1 \) properly contained in \( \omega \).

**Definition: 3.2.3**

Let \( S \) be a topological semigroup with Bohr compactification \((\beta, B)\), \( W^* I(S) \) denotes the family of all weak* ideal semigroup compactifications of \( S \), then an element \((a, A) \in W^* I(S)\) is a dual atom of \( W^* I(S) \) provided \((a, A) < (\beta, B)\) and there does not exist \((\gamma, C)\) in \( W^* I(S) \) such that \((a, A) < (\gamma, C) < (\beta, B)\).

**Definition: 3.2.4**

An element \((a, A) \in W^* I(S)\) is an atom of \( W^* I(S) \) provided \((a, A) > (a, \{0\})\), \((a, \{0\})\) is the smallest semigroup compactification of \( S \) in \( W^* I(S) \) and there does not exist \((\gamma, C)\) for which \((a, A) > (\gamma, C) > (a, \{0\})\).

In example [4.1.2], the collection of weak* ideals are \{b, c\}, \{b, d\}, \{c, d\}, \{b, e\}, \{d, e\}, \{c, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\} and \{b, c, d, e\}. 66
In this example dual atoms corresponding to weak* ideals are \{b,c\}, \{b,e\}, \{b,d\}, \{c,d\}, \{c,e\} and \{d,e\} which are minimal elements in $\Omega^*$ and the atom corresponding to weak* ideal is \{b,c,d,e\} which is maximal in $\Omega^*$

**Example: 3.2.5**

Let $S$ be a topological semigroup with Bohr Compactification $(\beta, B) = (\beta, Z_8)$ where $Z_8$ is a compact semigroup with multiplication modulo 8 and with discrete topology.
Fig 3.2.1
The collection of weak* ideals are \{0,4\}, \{2,6\}, \{1,5\} \cup \{3,7\}, \{0,4\} \cup \{2,6\} \cup \{1,7\} \cup \{3,5\} \cup \{2,6\}, \{1,3\} \cup \{5,7\} \cup \{2,6\}, \{0,4\} \cup \{1,5\} \cup \{3,7\}, \{1,5\} \cup \{3,7\} \cup \{2,6\}, \{0,2,4,6\}, \{0,4\} \cup \{2,6\} \cup \{1,5\} \cup \{3,7\}, \{1,7\} \cup \{3,5\} \cup \{0,2,4,6\}, \{1,3\} \cup \{5,7\} \cup \{0,2\}, \{1,5\} \cup \{3,7\} \cup \{0,2,4,6\}, \{1,3,5,7\} \cup \{2,6\}, \{1,3,5,7\} \cup \{0,2,4,6\}. Denote \((\alpha,\{0\})\) is the smallest semigroup compactification and \((\alpha,\{1\})\) is the largest semigroup compactification. Here dual atoms corresponding to weak* ideals are \{0,4\}, \{2,6\}, \{1,5\} \cup \{3,7\} which are minimal elements in \(\Omega^*\) and the atoms corresponding to weak * ideal \{1,3,5,7\} \cup \{0,2,4,6\} which is a maximal element of \(\Omega^*\).

**Note: 3.2.6**

If \(\omega\) is a set of joint weak ideals which is minimal then \((\alpha,A)\) corresponding to \(\omega\) is an atom of \(W^*I(S)\).

If \(\omega\) is a set of complementary joint weak ideals which is minimal in \(\Omega^*\) then \((\alpha,A)\) corresponding to \(\omega\) is an atom of \(W^*I(S)\).
**Theorem: 3.2.7**

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$ where $B$ is finite and $\Omega^*$ be the collection of all weak* ideals then $(\alpha, A) \in W^*I(S)$ is determined by $\omega$ is a dual atom (atom) of $W^*I(S)$ if and only if there exist a closed non-singleton proper weak* ideal $\omega$ minimal (maximal) in $\Omega^*$.

**Proof:**

Let $|B| = n$ where $n$ is finite, $\Omega^*$ be the collection of weak* ideals, and let $\omega$ be a closed non-singleton proper weak* ideal minimal in $\Omega^*$.

**Case-1**

If $\omega$ is a weak* ideal which is minimal in $\Omega^*$, then by the definition of minimality there exist no weak* ideal contained in $\omega$ and $(\omega \times \omega) \cup \Delta$ is a non-trivial closed congruence on $B$ i.e., $\Delta \neq (\omega \times \omega) \cup \Delta$ and there exist no non-trivial closed congruence properly contained in $(\omega \times \omega) \cup \Delta$. 
If not, let $R^1$ be a non-trivial closed congruence properly contained in $(\omega \times \omega) \cup \Delta$. i.e., $R^1 \not\subseteq (\omega \times \omega) \cup \Delta \subseteq B \times B$. Since $R^1$ is non-trivial $R^1$ is determined by at least one non-singleton subset $A$ (say) of $B$. If not let $|A|=1$, $R^1$ is determined by $A$ is $\Delta$. This is not possible, since $R^1 \neq \Delta$, then the possible cases of $R^1$ are the following.

Case (a):-

$R^1$ is determined by a subset $A$ of $B$ with $1 < |A| < n$. If $|A|=2$, then $R^1 = \{a, b\} \times \{a, b\} \cup \Delta \subseteq \omega \times \omega \cup \Delta$.

Since $A \neq B$ and $\{a, b\} \subseteq \omega$ and since $R^1$ is a congruence for all $a, b \in A$, $ax, bx \in A$ or $ax = bx$ and $xa, xb \in A$ or $xa = xb$ for all $x \in B$.

i.e., $A = \{a, b\}$ is a weak ideal, $\{a, b\} \subseteq \omega$ is a contradiction. Similarly we have a contradiction if $R^1$ is determined by any non-empty subset $A$ of $B$ with $1 \leq |A| < n$.

Case (b):-

If $R^1$ is determined by two non-singleton subsets $A_1 = \{a, b\}$, $A_2 = \{c, d\}$ i.e., $\{a, b\} \times \{a, b\} \cup (\omega \times \omega) \cup \Delta$. 

\{c,d\} \times \{c,d\} \cup \Delta \text{ is a closed congruence contained in } (\omega \times \omega) \cup \Delta. \text{ Since } a \neq b, \ c \neq d. \ i.e., \{a,b,c,d\} \subset \omega \text{ and since } R^1 \text{ is a congruence for all } x \in B, \ ax, bx \in \{a,b\} \text{ or } ax, bx \in \{c,d\} \text{ or } ax = bx \text{ and } xa, xb \in \{a,b\} \text{ or } xa, xb \in \{c,d\} \text{ or } xa = xb. \ i.e. \{a,b,c,d\} \text{ is a set of joint weak ideals contained in } \omega \text{ which is a contradiction for the minimality of } \omega. \text{ Similarly, we have a contradiction if } R^1 \text{ is determined by any collection of subsets of } B \text{ atleast one of which is non- singleton.}

Case (c):

If } R^1 \text{ is determined by two non- singleton subsets } A_1, A_2 \text{ of } B \text{ such that } A_1 \cup A_2 = B. \text{ Let } A_1 = \{a,b\}, A_2 = \{c,d\} \text{ then } R^1 = \{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \text{ is a closed congruence. Then } \{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \subset \omega \times \omega \cup \Delta. \text{ Since } a \neq b, c \neq d, \{a,b,c,d\} \subset \omega. \text{ Since } R^1 \text{ is a congruence for all } x \in B \text{ and } a,b \in A_1 \text{ or in } A_2 \text{ i.e. } ax, bx \in A_1 \text{ or } ax, bx \in A_2 \text{ and } xa, xb \in A_1 \text{ or } xa, xb \in A_2. \ i.e., \{A_1, A_2\} \text{ is a set of complementary joint ideals contained in } \omega, \text{ which is a contradiction. Similarly we have a contradiction if } R^1 \text{ is determined by any}
disjoint collection of subsets of $B$ whose union is $B$, at least one of which is non-singleton. Thus in all these possible cases, since $\omega$ is a minimal weak* ideal there exist no non-trivial closed congruence properly contained in $(\omega \times \omega) \cup \Delta$.

Therefore, $(\alpha, A)$ the weak* ideal semigroup determined by $\omega$ is a dual atom of $W^* I(S)$

Let $\omega$ be a closed non-singleton proper weak* ideal of $B$ which is maximal in $\Omega^*$.

Therefore, $\Delta \subset (\omega \times \omega) \cup \Delta \subset B \times B$ is a closed congruence on $B$ and there exist no proper closed congruence properly contains $(\omega \times \omega) \cup \Delta$. If not let $R^1$ can be closed congruence properly contains $(\omega \times \omega) \cup \Delta$. If not let $R^1$ be a closed congruence properly contains $(\omega \times \omega) \cup \Delta$ i.e., $\Delta \neq (\omega \times \omega) \cup \Delta \neq R^1 \neq B \times B$. Since $R^1$ is non-trivial, the possible cases of $R^1$ are same as in the case (1) and we have a contradiction. Similarly if $R^1$ is determined by any non-singleton subset $A$ of $B$ with $1 < |A| < n$ and also if $R^1$ is determined by any disjoint
collection of subset B at least one of which is non-singleton.

Therefore, there exist no proper closed congruence properly contains \((\omega \times \omega) \cup \Delta\). Then \((\alpha, A)\) is the weak* ideal semigroup compactification corresponding to \((\omega \times \omega) \cup \Delta\) and is clearly an atom of \(W*I(S)\), by similar argument.

Conversely, let \((\alpha, A)\) is an atom of \(W*I(S)\), by the definition of atom \((\alpha, A) > (\alpha, \{0\})\) and there does not exist \((\alpha_1, A_1)\) in \(W*I(S)\) for which \((\alpha, A) > (\alpha_1, A_1) > (\alpha, \{0\})\). i.e., there exist no closed congruence \(R_1\) such that \(R \subseteq R_1 \neq B \times B\)

\(\Rightarrow R\) is the maximal closed congruence

\(\Rightarrow R\) is determined by maximal weak ideal or joint weak ideal or complementary joint ideal i.e., \(R\) is determined by maximal weak* ideal.

Therefore, \((\alpha, A)\) is determined by maximal weak* ideal.
3.3. Some results about meet and join of ideal semigroup compactification.

Lemma: 3.3.1.

Let $S$ be a topological Semigroup with Bohr compactification $(\beta, B)$ where $B$ is an ideal semigroup. Let $K_1$ and $K_2$ be two non-empty non-singleton closed subset of $(\beta, B)$ then,

1. $\alpha(S:K_1) \land \alpha(S:K_2) = \alpha(S:K_1,K_2)$ if $K_1 \cap K_2 = \emptyset$

2. $\alpha(S:K_1) \land \alpha(S:K_2) = \alpha(S:K_1 \cup K_2)$ if $K_1 \cap K_2 \neq \emptyset$

Proof:

First suppose that $K_1 \cap K_2 = \emptyset$ i.e., non-empty non-singleton closed subsets of ideal Bohr compactification $(\beta, B)$ are disjoint. Since $B$ is an ideal semigroup $K_1$ and $K_2$ are ideals and it determines ideal semigroup compactification and closed congruence determined by $K_1$ contained in closed congruence determined by $K_1,K_2$, then clearly

$\alpha(S:K_1) \geq \alpha(S:K_1,K_2)$ and $\alpha(S:K_2) \geq \alpha(S:K_1,K_2)$. 

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Suppose \((\gamma, C)\) be an another ideal semigroup compactification of \(S\) such that
\[
\alpha(S:K_1) \geq (\gamma, C) \quad \text{and} \quad \alpha(S:K_2) \geq (\gamma, C)
\]

Since the closed congruence determined by \(K_1\) contained in closed congruence \(R\) corresponding to \((\gamma, C)\) and closed congruence determined by \(K_2\) contained in closed congruence \(R\) corresponding to \((\gamma, C)\)

i.e., \(K_1 \times K_1 \cup \Delta \subset R\), \(K_2 \times K_2 \cup \Delta \subset R\)

Then \((K_1 \times K_1) \cup (K_2 \times K_2) \cup \Delta \subset R\), which implies that
\[
\alpha(S:K_1, K_2) \geq (\gamma, C)
\]

Therefore in this particular case, \(\alpha(S:K_1) \wedge \alpha(S:K_2)\)

\[
= \alpha(S:K_1, K_2).
\]

Now suppose that \(K_1 \cap K_2 \neq \emptyset\), then it is immediate that \(\alpha(S:K_1) \geq \alpha(S:K_1 \cup K_2)\) and
\[
\alpha(S:K_2) \geq \alpha(S:K_1 \cup K_2)
\]

Suppose that \((\gamma, C)\) is any other ideal semigroup compactification of \(S\) such that \(\alpha(S:K_1) \geq (\gamma, C)\) and \(\alpha(S:K_2) \geq (\gamma, C)\).
Then closed congruence corresponding to the subset $K_1$ contained in closed congruence corresponds to $(\gamma, C)$, and closed congruence corresponding to the subset $K_2$ contained in closed congruence corresponding to $(\gamma, C)$. Since $K_1 \cap K_2 \neq \emptyset$ and $K_1$ and $K_2$ are closed ideals $K_1 \cup K_2$ is a closed ideal and determines ideal semigroup compactification $\alpha(S:K_1 \cup K_2)$. Hence in this case $\alpha(S:K_1) \wedge \alpha(S:K_2) = \alpha(S:K_1 \cup K_2).

**Note: 3.3.2**

We say a congruence $R_1$ refines a congruence $R_2$ and write $R_1 \succeq R_2$ if $R_1 \subseteq R_2$. We say a closed subset $K_1$ refines a closed subset $K_2$ and write $K_1 \succeq K_2$ if $K_1 \subseteq K_2$.

**Result: 3.3.3**

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$ and $B$ is an ideal semigroup. Let $K_i$ denote the set of non-singleton closed subsets of $B$ corresponding to the ideal semigroup compactifications $(\alpha_i, A_i)$ of $S$, then $(\alpha_1, A_1) \succeq (\alpha_2, A_2)$.
if and only if $K_1$ refines $K_2$, where $K_1$ and $K_2$ are non-singleton closed subset of $B$ corresponding to $(a_1, A_1)$ and $(a_2, A_2)$ respectively.

**Proof:**

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$ and $B$ is an ideal semigroup. Let $K_1$ and $K_2$ be non-singleton closed subsets of $B$ such that $K_1 \supseteq K_2$, i.e., $K_1 \subseteq K_2$. Since $B$ is an ideal semigroup $K_1$ and $K_2$ are closed ideals then $K_1 \times K_1 \cup \Delta$ and $K_2 \times K_2 \cup \Delta$ are closed congruences such that $(K_1 \times K_1) \cup \Delta \subseteq (K_2 \times K_2) \cup \Delta$.

Let $(\alpha_1, A_1)$ and $(\alpha_2, A_2)$ be the ideal semigroup compactification corresponding to $K_1$ and $K_2$ respectively. Then $B/(K_1 \times K_1) \cup \Delta \sim (\alpha_1, A_1)$ and $B/(K_2 \times K_2) \cup \Delta \sim (\alpha_2, A_2)$. Then by Induced homomorphism theorem [C-H-K], there exist a continuous surmorphism $\theta : A_1 \to A_2$ such that $\theta \alpha_1 = \alpha_2$. Hence $(\alpha_1, A_1) \geq (\alpha_2, A_2)$.

Conversely suppose that $(\alpha_1, A_1) \geq (\alpha_2, A_2)$. Then by definition[1.2.3] there exist a continuous surmorphism $\theta : A_1 \to A_2$ such that $\theta \alpha_1 = \alpha_2$. Again
let \((\beta, B)\) be the Bohr compactification of \(S\), then \((\alpha_1, A_1)\) and \((\alpha_2, A_2)\) are the quotient spaces of \(B\). It follows that \(f_1 : B \to A_1\) is a continuous dense homomorphism such that \(f_1\beta = \alpha_1\) and \(f_2\beta = \alpha_2\).

Now let \(R_1\) and \(R_2\) be relations on \(B\) defined by \(f_1\) and \(f_2\) respectively as

\[
R_1 = \{(x, y) \in B \times B : f_1(x) = f_1(y)\}
\]

\[
R_2 = \{(x, y) \in B \times B : f_2(x) = f_2(y)\}
\]

Then \(R_1 = (f_1 \times f_1)^{-1}(\Delta_{A_1})\) is closed. Since \(\Delta_{A_1}\), diagonal on \(A_1 \times A_1\) of Hausdorff space is closed.

Again \(R_1\) is a congruence, for \((a, b) \in R_1 \Rightarrow f_1(a) = f_1(b)\)

\((c, d) \in R_1 \Rightarrow f_1(c) = f_1(d)\)

then \(f_1(ac) = f_1(a)f_1(c) = f_1(b)f_1(d) = f_1(bd)\)

Hence \((a, c, b, d) \in R_1\)

Hence \(R_1\) is a closed congruence on \(B\). Similarly we observed \(R_2 = (f_2 \times f_2)^{-1}(\Delta_{A_2})\) is a closed congruence.
on B. Then clearly $R_1$ and $R_2$ are closed congruence defined by $f_1$ and $f_2$ respectively. Then $(\alpha_1, A_1)$ topologically isomorphic to $B/R_1$ and $(\alpha_2, A_2)$ topologically isomorphic to $B/R_2$ and given $\theta: A_1 \to A_2$ such that $\theta \alpha_1 = \alpha_2$.

i.e., $\theta f_1 \beta = \theta \alpha_1 = f_2 \beta$.

i.e., $\theta f_1 = f_2$

Then closed congruence determined by $f_1(R_1)$ closed congruence determined by $f_2(R_2)$ i.e., $R_1 \subseteq R_2$.

i.e. $\{(x, y) \in B \times B : f_1(x) = f_1(y)\} \subseteq \{(x, y) \in B \times B : f_2(x) = f_2(y)\}$.

Since $B$ is an ideal semigroup all subsets are ideals, then we get the following cases.

Case - 1

$R_1$ is of the form $K_1 \times K_1 \cup \Delta$ and $R_2$ is of the form $K_2 \times K_2 \cup \Delta$ with $K_1 \cap K_2 \neq \emptyset$, then clearly $K_1 \subseteq K_2$ since $K_2 = K_1 \cup K_2$.

Case - 2

$R_1$ is of the form $K_1 \times K_1 \cup \Delta$ and $R_2$ is of form $K_1 \times K_1 \cup K_2 \times K_2$, with $K_1 \cap K_2 = \emptyset$. Then clearly $K_1 \subseteq \{K_1, K_2\}$.

In this case we observed $K_1 \subseteq K_2$, then $K_1$ refines $K_2$. 