CHAPTER - 2
PROPERTIES OF IDEAL SEMIGROUP
COMPACTIFICATION

Introduction

Associated with each topological semigroup S, there is a compact semigroup called the Bohr compactification of S which is universal over the compact semigroup S containing dense continuous homomorphic images of S. The existence and uniqueness of Bohr compactification can be proved [C-H-K₁].

Wallace A.D has shown that if B is a compact semigroup and R is a closed congruence on B, then the quotient space B/R is a compact semigroup. It is proved that semigroup compactifications of a topological semigroup S are precisely the quotients of the Bohr compactifications (ß,B) of S under closed congruence on B [K].

Result: 2.1

Let S be a topological semigroup with Bohr compactification (ß,B). If (α,A) is any semigroup compactification of S, then
(a) There exists a continuous surmorphism (i.e., surjective homomorphism) \( \theta : B \rightarrow A \) such that \( \theta \beta = \alpha \).

(b) And the equivalence defined by \( \theta \) on \( B \) is a closed congruence.

(c) \((\alpha, A)\) is the quotient of \((\beta, B)\) with respect to the congruence in (b).

Thus in [K] proved that if \((\beta, B)\) is a Bohr compactification of \( S \) and \( R \) is any closed congruence on \( B \), then the quotient space \( B/R \) determines a semigroup compactification of \( S \) and conversely any semigroup compactification \((\alpha, A)\) of \( S \) is topologically isomorphic to \( B/R \) for some closed congruence \( R \) on \( B \).

We can observe [K1], if \( \omega \) is a closed ideal of a topological semigroup \( S \), then \( \omega \times \omega \cup \Delta \) is a closed congruence on \( S \).

In [K] we can observe that a topological semigroup \( S \) with Bohr Compactification has a semigroup compactification defined by \( \omega \) if and only if \( \omega \) is a closed non-singleton weak ideal of \( S \) [B].
In this situation we can observe the characterization of atoms and dual atoms in the lattice of semigroup compactification is in the stage of infancy. But in the case of existence of non-singleton subsets of topological semigroups are closed ideals, we characterize the atoms and dual atoms in the lattice of semigroup compactification of a topological semigroup.

In this chapter, we defined ideal semigroup compactification of a topological semigroup \( S \) and studied properties of ideal semigroup compactification determined by non-singleton closed \([\text{left}, \text{right}]\) ideal of Bohr compactification \((\beta, B)\) of \( S \).

**Notation: 2.2**

\((\alpha, \{0\})\) denotes the smallest semigroup compactification corresponding to closed congruence \( B \times B \) and \((\alpha, \{1\})\) denotes the greatest semigroup compactification corresponding to the closed congruence \( \Delta \). i.e., \((\alpha, \{1\}) = (\beta, B)\)
2.1 Ideal semigroup compactification

Definition: 2.1.1

Semigroup compactification of a topological semigroup $S$ is said to be an ideal semigroup compactification if and only if it is determined by closed [left, right] ideals of $(\beta, B)$.

Note: 2.1.2

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$, then the semigroup compactification $(\alpha, A)$ of $S$ is said to be an ideal semigroup compactification if and only if $(\alpha, A)$ is determined by closed [left, right] ideals of $(\beta, B)$.

Notation: 2.1.3

Let $I(S)$ denotes the family of all ideal semigroup compactification of $S$ determined by finite numbers $\{\omega_i\}$ $i=1,2,3,\ldots,n$, of closed [left,right] ideals in $(\beta, B)$. Clearly $I(S)$ be a complete lattice. In $I(S) (\alpha, \{0\})$ denotes the least
element of \( I(S) \) where \( \{0\} = B/B \times B \) and the element \( (\alpha, \{1\}) \) is the greatest element of \( I(S) \) where \( \{1\} = B/\Delta \).

Next we describe some examples for the study of ideal semigroup compactification of a topological semigroup.

**Example: 2.1.4**

Let \( S \) be a topological semigroup with Bohr compactification \( (\beta, B) \) where \( B = \{x_1, x_2, x_3, x_4\} \) with discrete topology and multiplication defined by \( ab = a \forall a \in B \).

Here we can observe all subsets are right closed ideals of \( B \) and clearly it determines ideal semigroup compactifications of \( S \). Let \( I(S) = (\alpha_i, I_i) \) be the finite collection of ideal semigroup compactifications determined by finite collection
of closed right ideals \( \{I_i\} \) of \( B \), where \( i = 1, 2, 3, ..., n \).

Let \((\alpha_1, A_1)\) be the ideal semigroup compactification determined by closed congruence 
\( R_1 = \Delta \subset B \times B \). Let \((\alpha_2, A_2)\) be the ideal semigroup compactification determined by the closed congruence 
\( R_2 = \{x_1, x_2\} \times \{x_1, x_2\} \cup \Delta \subset B \times B \). Let \((\alpha_3, A_3)\) be the ideal semigroup compactification determined by closed congruence 
\( R_3 = \{x_1, x_3\} \times \{x_1, x_3\} \cup \Delta \subset B \times B \). Let \((\alpha_4, A_4)\) be the ideal semigroup compactification determined by closed congruence 
\( R_4 = \{x_1, x_4\} \times \{x_1, x_4\} \cup \Delta \subset B \times B \). Let \((\alpha_5, A_5)\) be the ideal semigroup compactification determined by closed congruence 
\( R_5 = \{x_2, x_3\} \times \{x_2, x_3\} \cup \Delta \subset B \times B \). Let \((\alpha_6, A_6)\) be the ideal semigroup compactification determined by the closed congruence 
\( R_6 = \{x_2, x_4\} \times \{x_2, x_4\} \cup \Delta \subset B \times B \). Let \((\alpha_7, A_7)\) be the ideal semigroup compactification determined by the closed congruence 
\( R_7 = \{x_3, x_4\} \times \{x_3, x_4\} \cup \Delta \subset B \times B \). Let \((\alpha_8, A_8)\) be the ideal semigroup compactification determined by the closed congruence 
\( R_8 = \{x_1, x_2\} \times \{x_1, x_2\} \cup \{x_3, x_4\} \times \{x_3, x_4\} \). Let \((\alpha_9, A_9)\) be the ideal semigroup compactification determined by the closed congruence 
\( R_9 = \{x_1, x_3\} \times \{x_1, x_3\} \cup \{x_2, x_4\} \times \{x_2, x_4\} \).
{x_2, x_4}. Let (\alpha_{10}, A_{10}) be the ideal semigroup compactification determined by the closed congruence R_{10} = \{x_1, x_4\} \times \{x_1, x_4\} \cup \{x_2, x_3\} \times \{x_2, x_3\}. Let (\alpha_{11}, A_{11}) be the ideal semigroup compactification determined by the closed congruence R_{11} = \{x_1, x_2, x_3\} \times \{x_1, x_2, x_3\} \cup \Delta \subseteq B \times B. Let (\alpha_{12}, A_{12}) be the ideal semigroup compactification determined by the closed congruence R_{12} = \{x_1, x_2, x_4\} \times \{x_1, x_2, x_4\} \cup \Delta \subseteq B \times B. Let (\alpha_{13}, A_{13}) be the ideal semigroup compactification determined by the closed congruence R_{13} = \{x_1, x_3, x_4\} \times \{x_1, x_3, x_4\} \cup \Delta \subseteq B \times B. Let (\alpha_{14}, A_{14}) be the ideal semigroup compactification determined by the closed congruence R_{14} = \{x_2, x_3, x_4\} \times \{x_2, x_3, x_4\} \cup \Delta \subseteq B \times B. Let (\alpha_{15}, A_{15}) be the ideal semigroup compactification determined by R_{15} = B \times B. Then the lattice of ideal semigroup compactification of S are shown in figure [2.1.1].

Here \{\alpha_2, A_2\}, \{\alpha_3, A_3\}, \{\alpha_4, A_4\}, \{\alpha_5, A_5\}, \{\alpha_6, A_6\}, \{\alpha_7, A_7\}\) are determined by proper non-singleton closed (right) ideals of \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\} of B. By lattice property, we can easily observe \{R_2, R_3, R_4, R_5, R_6, R_7\} collection of closed congruences determined by proper non-singleton closed (right)
ideals of $S$ are dual atoms in the lattice of closed congruence on $B$. Dually we can observe ideal semigroup compactifications, $\{(\alpha_2,A_2), (\alpha_3,A_3), (\alpha_4,A_4), (\alpha_5,A_5), (\alpha_6,A_6), (\alpha_7,A_7)\}$ are atoms in the lattice of ideal semigroup compactification $I(S)$ of $S$.

**Example: 2.1.5**

Let $S$ be a topological semigroup with Bohr compactification $(\beta,B)$ where $B=\{x_1,x_2,x_3\}$ with left zero multiplication and discrete topology.
Then we can easily observe $(\alpha_1,A_1)$ is the ideal semigroup compactification, determined by $R_1=\Delta$, $(\alpha_2,A_2)$ is the ideal semigroup compactification determined by $R_2= \{x_1,x_2\} \times \{x_1,x_2\} \cup \Delta$, $(\alpha_3,A_3)$ is the ideal semigroup compactification determined by $R_3= \{x_1,x_3\} \times \{x_1,x_3\} \cup \Delta$, $(\alpha_4,A_4)$ is the ideal semigroup compactification determined by $R_4= \{x_2,x_3\} \times \{x_2,x_3\} \cup \Delta$, $(\alpha_5,A_5)$ is the ideal semigroup compactification determined by $R_5= \{x_1,x_2,x_3\} \times \{x_1,x_2,x_3\} = B \times B$ Then the lattice of ideal semigroup compactification of $S$ can be shown in figure [2.1.2].
Here \( \{ (\alpha_1, A_1), (\alpha_2, A_2), (\alpha_3, A_3), (\alpha_4, A_4), (\alpha_5, A_5) \} \) are family of ideal semigroup compactification of \( S \) corresponding to closed congruence \( \{ R_1, R_2, R_3, R_4, R_5 \} \) determined by closed (right) ideals of \( B \). Also we observed that \( \{ (\alpha_2, A_2), (\alpha_3, A_3), (\alpha_4, A_4) \} \) are atoms in lattice of ideal semigroup compactification of \( S \).

From the above example we can observe the following:

In Example [2.1.4], Example [2.1.5] we can observe,

a) All subsets are closed (right) ideals and determines ideal semigroup compactification of \( S \).

b) Each non-singleton distinct closed subsets \( \{ x, y \} \) determines dual atoms of \( I(S) \).

c) If \( |B| = n \), then the closed subset containing \( (n-1) \) elements determines an atom of \( I(S) \).

So we conclude the following:

1) If all non-singleton closed subsets of the Bohr compactification \( (\beta, B) \) of a topological semigroup \( S \) determines an ideal semigroup compactification, then there exist a dual atom in \( I(S) \).
2) If all subsets of Bohr compactification \((\beta, B)\) of a topological semigroup \(S\) determines an ideal semigroup compactification then there exist an atom in \(I(S)\).

As a special case, if all subsets of a semigroup are (left, right) ideals, we introduce a new definition for semigroup.

**Definition: 2.1.6 Ideal Semigroup**

A semigroup \(S\) is said to be an *ideal semigroup*, if all subsets are (left, right) ideals.

In this situation we modify our results ideal semigroup compactification in the following way.

**Result: 2.1.7**

Let \(S\) be topological semigroup with Bohr compactification \((\beta, B)\) where \(B\) is an ideal semigroup then there exist an ideal semigroup compactification. Also we describe dual atoms and atoms of \(I(S)\) in the following section.
2.2. Dual Atoms on the Lattice of Ideal Semigroup Compactification

Theorem: 2.2.1

Let \( S \) be a topological semigroup with Bohr compactification \((\beta, B)\) where \( B \) is an ideal semigroup and \( |B| > 2 \), then \((\alpha, A)\) is a dual atom of \( I(S) \) if and only if there exist non-singleton closed subsets \( \{x, y\} \) of \( B \) which determines \((\alpha, A)\).

Proof:

Let \( S \) be a topological semigroup with Bohr Compactification \((\beta, B)\) and let \((\alpha, A)\) be a dual atom of \( I(S) \). Then \((\alpha, A) \neq (\beta, B)\) and there does not exist \((\alpha_1, A_1)\) \( \in I(S) \), such that \((\alpha, A) < (\alpha_1, A_1) < (\beta, B)\). i.e., there exist no closed congruence \( R_1 \) such that \( \Delta \not< R_1 \not< R \) where \( R \) and \( \Delta \) are universal closed congruence on \( B \). i.e., there exist no closed subset of \( B \) which determines a closed congruence \( R_1 \) properly contained in \( R \). Since \( B \) is an ideal semigroup by definition all subsets are ideals and there exist no closed subset of \( B \) which determines closed congruence \( R_1 \) properly contained in \( R \) and \( \Delta \not< R \).
implies R is determined by non-singleton closed subset \( \{x,y\} \) of B such that \( \Delta \neq \{x,y\} \times \{x,y\} \cup \Delta \).

Suppose R is determined by \( R = \{x,y,z\} \times \{x,y,z\} \cup \Delta \) then, since B is an ideal semigroup, \( \{x,y\}, \{y,z\}, \{x,z\} \) are ideals of B and determines ideal semigroup compactification corresponding to closed congruences.

\( R_1 = \{x,y\} \times \{x,y\} \cup \Delta, \ R_2 = \{y,z\} \times \{y,z\} \cup \Delta \) and \( R_3 = \{x,z\} \times \{x,z\} \cup \Delta \) respectively.

Also we have \( R_1 \subset R, \ R_2 \subset R, \ R_3 \subset R \)

\[
\Delta \neq \begin{cases} R_1 \\ R_2 \\ R_3 \end{cases} \neq \ R
\]

i.e., there exist ideal semigroup compactification \((\alpha_1, A_1), (\alpha_2, A_2)\) and \((\alpha_3, A_3)\) corresponding to \( R_1, R_2 \) and \( R_3 \) respectively and

\[ (\alpha, A) < (\alpha_1, A_1) < (\beta, B) \]

\[ (\alpha, A) < (\alpha_2, A_2) < (\beta, B) \]

\[ (\alpha, A) < (\alpha_3, A_3) < (\beta, B). \]

Since \((\alpha, A)\) is a dual atom of I(S), by definition of dual atom this is not possible, so their exist no closed
subset having three elements which determines an ideal semigroup compactification, which is a dual atom of $I(S)$. So there exist non-singleton closed subset \{x,y\} which determines $(\alpha,A)$.

Conversely, let \{x,y\} be a non-singleton closed subset of $B$, since $B$ is an ideal semigroup \{x,y\} is an ideal. Then clearly \{x,y\} x \{x,y\} u $\Delta$ is a closed congruence on $B$, which determines an ideal semigroup compactification, say $(\alpha,A) \in I(S)$.

Since $B$ is an ideal semigroup and since $|B|>2$, all non-singleton closed subsets determines closed congruence on $B$. There exist semigroup compactification $(\alpha_1,A_1) \in I(S)$ such that $(\beta,B) \not\approx (\alpha,A) \not\approx (\alpha_1,A_1)$. Then $(\alpha,A)$ is a dual atom of $I(S)$, for if there exist any ideal semigroup compactification $(\alpha_1,A_1) \in I(S)$ such that $(\beta,B) > (\alpha_1,A_1) > (\alpha,A)$ then there exist closed congruence $R_1$ on $(\beta,B)$ corresponding to $(\alpha_1,A_1)$ such that $\Delta \subset R_1 \neq \{x,y\} x \{x,y\} u \Delta$, i.e., $R_1 \subset \{x,y\} x \{x,y\} u \Delta, x \neq y$. The only possibilities are $R_1 = \Delta$ or $R_1$ is determined by $\{x,y\}$, then clearly there exist no $R_1$ such that $\Delta \subset R_1 \subset \{x,y\} x \{x,y\} u \Delta$.
\{x,y\} \times \{x,y\} \cup \Delta$, that is dual atoms of I(S) are determined by non-singleton closed subset \{x,y\} of B.

In above example [2.1.4] we can observe B = \{x_1, x_2, x_3, x_4\} with discrete topology and multiplication defined by \(ab = a, \forall a \in B\), is an ideal semigroup and \(|B| = 4\). Then \((\alpha_2, A_2), (\alpha_3, A_3), (\alpha_4, A_4), (\alpha_5, A_5), (\alpha_6, A_6), (\alpha_7, A_7)\) are dual atoms determined by closed non-singleton subsets \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\} and \{x_3, x_4\} respectively of \((\beta, B)\).

In above example [2.1.4] if \(|B| = 2\) say \{x_1, x_2\} we can observe \(\Delta\) and \(B \times B\) are only closed congruence on B and there exist no closed congruence in between \(\Delta\) and \(B \times B\). Let \((\alpha, \{0\})\) and \((\beta, B)\) are ideal semigroup compactifications corresponding to \(B \times B\) and \(\Delta\) respectively. In this case there exist no dual atom in the lattice of ideal semigroup compactification. So we conclude the following result.

**Result: 2.2.2**

Let S be a topological semigroup with Bohr compactification \((\beta, B)\) where B is an ideal semigroup and there exist dual atoms in I(S). Also we can easily observe from the example [2.1.6] the
with $|B| \leq 2$, then there exist no dual atoms in the lattice of ideal semigroup compactification.

**Example: 2.2.3**

Let $S$ be topological semigroup with Bohr compactification $(\beta, B)$ where $B=\{x, y\}$ with left zero multiplication.

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<tr>
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The lattice of ideal semigroup compactification is shown in the figure

\[ \Delta \]

\[ \text{B} \times \text{B} \]

**Result: 2.2.4**

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$ where $B$ is an ideal semigroup and $|B| > 2$, then there exist dual atoms in $I(S)$. Also we can easily observe from the example [2.1.5] the
cardinality of dual atoms in the lattice of ideal semigroup compactification is 3 i.e. \( \frac{3!}{2!(3-2)!} \).

In the example [2.1.4] and by the result [2.2.4] we can observe the dual atoms in the lattice of ideal semigroup compactification of \((\beta, B)\) are, determined by non-singleton closed subsets, \(\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\) i.e., there are 6 dual atoms. i.e., \(\frac{4!}{(4-2)!2!} = 6\)

Next we generalize the result in the following way.

**Result: 2.2.5**

Let \(S\) be a topological semigroup with Bohr Compactification \((\beta, B)\) where \(B\) is an ideal semigroup and if \(|B| = n\), then there exist \(\frac{n!}{(n-2)!2!}\) dual atoms in the lattice of ideal semigroup compactification \(I(S)\) of \(S\).

**Result: 2.2.6**

Let \(S\) be a topological semigroup with Bohr compactification \((\beta, B)\) with \(|B| = 3\) then dual atoms and atoms of \(I(S)\) are same.
Proof:

From the above example [2.1.5] we can observe
\[ \{x_1, x_2\} \times \{x_1, x_2\} \cup \Delta, \{x_1, x_3\} \times \{x_1, x_3\} \cup \Delta \] and \[ \{x_2, x_3\} \times \{x_2, x_3\} \cup \Delta \]
determines ideal semigroup compactification \((\alpha_2, A_2), (\alpha_3, A_3),(\alpha_4, A_4)\) respectively such that

\[ \Delta \neq \{x_1, x_2\} \times \{x_1, x_2\} \cup \Delta \neq B \times B \]

\[ \Delta \neq \{x_1, x_3\} \times \{x_1, x_3\} \cup \Delta \neq B \times B \]

\[ \Delta \neq \{x_2, x_3\} \times \{x_2, x_3\} \cup \Delta \neq B \times B \]

We can clearly observe \((\alpha_2, A_2), (\alpha_3, A_3)\) and \((\alpha_4, A_4)\) are both dual atoms and atoms in the lattice of ideal semigroup compactification.

2.3. Atoms in the Lattice of Ideal Semigroup Compactification

Example:-

In the above example [2.1.4] in section 2.1 we can observe the dual atoms determined by the closed congruences are

\[ \{x_1, x_2\} \times \{x_1, x_2\} \cup \{x_3, x_4\} \times \{x_3, x_4\} \]
\{x_1, x_3\} \times \{x_1, x_3\} \cup \{x_2, x_4\} \times \{x_2, x_4\},
\{x_2, x_3\} \times \{x_2, x_3\} \cup \{x_1, x_4\} \times \{x_1, x_4\},
\{x_1, x_2, x_3\} \times \{x_1, x_2, x_3\} \cup \{x_1, x_2, x_4\} \times \{x_1, x_2, x_4\} \cup \Delta,
\{x_2, x_3, x_4\} \times \{x_2, x_3, x_4\} \cup \Delta, \{x_1, x_3, x_4\} \times \{x_1, x_3, x_4\} \cup \Delta,

respectively. Then the corresponding ideal semigroup compactifications \((\alpha_8, A_8), (\alpha_9, A_9), (\alpha_{10}, A_{10}), (\alpha_{11}, A_{11}), (\alpha_{12}, A_{12}), (\alpha_{13}, A_{13}), (\alpha_{14}, A_{14})\) are the atoms in the lattice of ideal semigroup compactification.

**Theorem: 2.3.1**

Let \(S\) be a topological semigroup with Bohr compactification \((\beta, B)\) where \(B\) is an ideal semigroup and \(|B| > 2\), then \((\alpha, A)\) is an atom of \(I(S)\) if and only if there exist non-singleton closed maximal subset \(C\) of \(B\) such that \(|C| < |B|\) which determines \((\alpha, A)\).

**Proof:**

Let \(S\) be a topological semigroup with Bohr compactification \((\beta, B)\) where \(B\) is an ideal semigroup and let \((\alpha_0, A_0)\) be an atom of \(I(S)\), Then \((\alpha_0, A_0)\) is the smallest semigroup compactification corresponding to \(B \times B\) on \(B\). Let \(R_0\) be a closed
congruence corresponding to \((\alpha_0,A_0)\). Since \((\alpha_0,A_0)\) is an atom, by definition there does not exist \((\alpha_1,A_1)\) such that \((\alpha_0,A_0) > (\alpha_1,A_1) > (\alpha,\{0\})\). There exist no closed congruence \(R_1\) such that \(R_0 \subset R_1 \subset B \times B\) i.e., there exist no closed subset of \(B\) which determines a closed congruence \(R_1\) properly contains \(R_0\). Since \(B\) is an ideal semigroup, all subsets are ideals and there exist no closed subset of \(B\) which determines closed congruence \(R_1\) properly contains \(R_0\) and \(R_0 \neq B \times B\) implies \(R_0\) is determined by non-singleton closed subset of \(B\) say \(C\) such that \(|C| < |B|\).

Also \(C\) is maximal closed subset of \(B\). For, suppose, \(C\) is not maximal, i.e., there exist a proper closed subset \(C^1\) of \(B\) such that \(C \neq C^1\). Since \(B\) is an ideal semigroup, clearly \(C^1\) determines an ideal semigroup compactification say \((\alpha^1,C^1)\) such that

\[
(\alpha^1,C^1) \neq (\alpha,\{0\}) \text{ and } (\alpha_0,A_0) > (\alpha^1,C^1) > (\alpha,\{0\});
\]

Since \(C^1\) is proper, \((\alpha^1,C^1) \neq (\alpha,\{0\})\) then the possibility is \((\alpha_0,A_0) > (\alpha^1,C^1) > (\alpha,\{0\})\). This contradicts assumption that \((\alpha_0,A_0)\) is an atom of \(I(S)\). Hence this is not possible
Therefore \((\alpha_0, A_0)\) corresponds a maximal proper closed subset 'C' of B such that \(|C| < |B|\).

Conversely suppose that there exist a maximal closed subset \(C\) of B. Since B is an ideal semigroup, \(C\) is an ideal, then clearly \(C \times C \cup \Delta\) is a closed congruence on B, which determines an ideal semigroup compactification say \((\alpha_0, A_0) \in I(S)\). Since B is an ideal semigroup and \(C\) is maximal subset, by the definition of maximal subset there exist no closed subset in between \(C\) and B i.e., there exist no closed congruence in between \((\alpha_0, A_0)\) and \((\alpha, \{0\})\) i.e., \((\alpha_0, A_0)\) is an atom.