Chapter 1

Preliminaries

In this chapter we present some definitions and elementary results of lattice theory, regular semigroups, regular rings and category theory. Moreover a quick review of the concepts that are necessary to introduce regular semigroups through normal category is made. The notations and terminology are generally as in [10] or [14].

1.1 Lattices

Here we introduce the definition and some basic ideas on lattices. The treatment and concept on lattice theory are as in Birkhoff[1].

Definition 1.1.1. A lattice is a partially ordered set $L$ any two of whose elements have a greatest lower bound (denoted by $\land$ and called 'meet') and a least upper bound (denoted by $\lor$ and called 'join'). A sublattice of $L$ is a subset which contains with any two elements their join and their meet. A lattice is said to contain a greatest element $g$ if $x \leq g$ for all $x \in L$. A lattice is said to contain a least element $l$ if $l \leq x$ for all $x \in L$. Usually greatest
element is denoted by 1 and least element by 0.

An isomorphism between two partially ordered sets $P_1$ and $P_2$ is a one one correspondence $\theta$ which preserves the order. That is, $x \leq y$ if and only if $\theta(x) \leq \theta(y)$. If $L_1$ and $L_2$ are lattices then $\theta : L_1 \to L_2$ is a homomorphism if $\theta$ preserves the order, joins and meets. That is, for all $x, y \in L_1$

(a) $x \leq y$ if and only if $\theta(x) \leq \theta(y)$

(b) $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$

(c) $\theta(x \vee y) = \theta(x) \vee \theta(y)$

Further if $\theta$ is a one one correspondence then it is called an isomorphism.

**Definition 1.1.2.** A complement of an element $x$ of a lattice $L$ with 0 and 1 is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. $L$ is called complemented if all its elements have complements. A lattice $L$ is called relatively complemented if all its closed intervals are complemented. That is, $L$ is relatively complemented if and only if, given $a \preceq x \preceq b$, an element $y$ exists such that $x \wedge y = a$ and $x \vee y = b$. This element is a relative complement of $x$ in the closed interval $[a, b]$.

### 1.2 Regular Semigroups

This section contains some basic concepts and results on regular semigroups. We assume familiarity with basic definitions and results as in [7], [2] and [14].

**Definition 1.2.1.** A nonempty set $S$ with a binary operation $(x, y) \to xy$ from $S \times S \to S$ is said to be a semigroup if $x(yz) = (xy)z$ for all $x, y, z \in S$. A nonempty subset $T$ of $S$ is called a subsemigroup of $S$ if $T$ is a semigroup under the binary operation induced by that on $S$. 
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An element $x$ in a semigroup $S$ is called regular if there is an element $x'$ in $S$ such that $xx'x = x$. A semigroup $S$ is called regular if all its elements are regular.

An element $e \in S$ is called an idempotent if $e^2 = e$. If $T$ is a subset of a regular semigroup $S$ then the set of idempotents in $T$ is denoted by $E(T)$. If $S$ is a regular semigroup, then it is well known that $E(S)$ is nonempty (see [2]).

If $A$ is a nonempty subset of a semigroup $S$ then $SA$ denote the set \{sa : s \in S \text{ and } a \in A\}. If A is the singleton set \{a\}, the $SA$ is denoted by $Sa$ and $AS$ by $aS$. A nonempty subset $A$ of a semigroup $S$ is called a left ideal if $SA \subseteq A$ and a right ideal if $AS \subseteq A$.

If there exists $1 \in S$ such that $1x = x = x1$ for all $x \in S$ then this element $1$ is called the identity of $S$. A semigroup may or may not contain $1$. An element $z \in S$ is said to be a left [right] zero element of $S$ if $zx = z$ [$xz = x$] for all $x \in S$. An element which is both left and right zero is called the zero element and is denoted by $0$.

The principal left (right) ideal generated by $a$ is denoted by $S^1a[aS^1]$. Here $S^1a = Sa \cup \{a\}$. In case $S$ is regular, $S^1a = Sa$ and $aS^1 = aS$. The Green’s equivalences are used to describe the structure of a semigroup (see [7]) and are defined as follows. For $a, b \in S$,

\[ a \mathcal{L} b \iff S^1a = S^1b. \]
\[ a \mathcal{R} b \iff aS^1 = bS^1. \]
\[ a \mathcal{J} b \iff S^1aS^1 = S^1bS^1. \]
\[ \mathcal{H} = \mathcal{L} \cap \mathcal{R}. \]
\[ \mathcal{D} = \mathcal{L} \lor \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}. \]

The different equivalence classes containing $a$ is denoted respectively by
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$L_a, R_a, J_a, H_a$ and $D_a$. Partial orders are defined on $S/L, S/R, S/J$ as follows.

$$L_a \leq L_b \iff S^1 a \subseteq S^1 b$$
$$R_a \leq R_b \iff aS^1 \subseteq bS^1$$
$$J_a \leq J_b \iff S^1 aS^1 \subseteq S^1 bS^1.$$ 

The following result on Green’s relations is used often.

**Theorem 1.2.2.** (cf.[7]) Let $e \in E(S), x \in S$. Then

(i) if $e \mathcal{L} x$ then $xe = x$

(ii) if $e \mathcal{R} x$ then $ex = x$ and

(iii) if $e \mathcal{H}$ then $ex = x = xe$

If $e, f \in E(S)$ then $e \mathcal{L} f$ if and only if $ef = e$ and $fe = f$. Also $e \mathcal{R} f$ if and only if $ef = f$ and $fe = e$ and $e \mathcal{H} f$ if and only if $e = f$.

**Definition 1.2.3.** Let $S$ be a semigroup.

On $E(S)$ we define the relations $\omega^l$ and $\omega^r$ by $e \omega^l f \iff e f = e \text{ and } e \omega^r f \iff f e = e$, for $e, f \in E(S)$. Also we define $\omega = \omega^r \cap \omega^l$ and write $e \leq f$ if $e \omega f$.

If $S$ is a regular semigroup, then $E(S)$ has the following properties (see [13]). By a quasiorder we mean a reflexive and transitive relation.

**Theorem 1.2.4.** (i) $\omega^r$ and $\omega^l$ are quasiorders on $E(S)$.

For $e, f, g \in E(S)$

(ii) $f \in \omega^r (e) \implies fe \in E(S)$ and $f \mathcal{R} fe \omega e$.

(iii) $g \omega^l f, f, g \in \omega^r (e) \implies ge \omega f e$. 
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(iv) \( g \circ f \in \omega \) if and only if \( g \circ (ge) = (gf)e \).

(v) \( g \circ f, \ f, \ g \in \omega \) if and only if \( (fg)e = (fe)(ge) \).

The following result is useful in considering the partial orders in \( S \rightarrow L \) and \( S \rightarrow \mathcal{R} \).

For \( e, f \in E(S) \)

(i) \( R_e \leq R_f \) if and only if \( e \in \omega \) if and only if \( e \in \omega \).

(ii) \( L_e \leq L_f \) if and only if \( e \in \omega \).

(iii) If \( R_e \leq R_f \) then there exists \( e' \in R_e \) such that \( e' \leq f \).

(iv) If \( L_e \leq L_f \) then there exists \( e' \in L_e \) such that \( e' \leq f \).

Proof. (i) \( R_e \leq R_f \) if and only if \( eS \leq fS \) if and only if \( e \in \omega \).

(ii) \( L_e \leq L_f \) if and only if \( Se \leq Sf \) if and only if \( e \in \omega \).

(iii) Set \( e' = ef \). Then \( e'^2 = (ef)(ef) = e(fe)f = ef = e' \) so that \( e' \in E(S) \).

Also \( e'f = e' \) and \( fe' = f(fe) = (fe)f = ef = e' \) so that \( e' \leq f \). Again \( e'e = (ef)e = e(fe) = ee = e \) and \( ee' = e(fe) = ef = e' \). Therefore \( e' \in \mathcal{R} \) showing that \( e' \in R_e \).

(iv) Here the choice is \( e' = fe \). Then as before we get \( e'^2 = e' \), \( e'e = (fe)e = fe = e' \), \( ee' = e(fe) = (ef)e = ee = e \) and \( e'f = (fe)f = f(fe) = fe = e' \), \( fe' = f(fe) = fe = e' \). Hence \( e' \in E(S), e' \in L_e, e' \leq f \). □

1.3 Categories

In this section we review some definitions and results from category theory.

The definitions which we present here are as in [10]. A category \( \mathcal{C} \) consists of

(a) A class of objects denoted by \( \mathcal{V} \)
(b) For every ordered pair of objects \((X, Y)\), a set \(\mathcal{C}(X, Y)\) of morphisms with domain \(X\) and codomain \(Y\). If \(f \in \mathcal{C}(X, Y)\) we write \(f : X \to Y\).

(c) For every triple of objects \(X, Y\) and \(Z\) and a pair of morphisms, \(f : X \to Y\) and \(g : Y \to Z\), a composite \(fg\) is defined where \(fg : X \to Z\). This composition satisfies the following properties.

\[
\begin{align*}
\text{(i) Associativity:} & \quad \text{If } f : X \to Y, \ g : Y \to Z \text{ and } h : Z \to W, \text{ then } f(g h) = (f g) h. \\
\text{(ii) Identity:} & \quad \text{For every object } Y \text{ there is a morphism } 1_Y : Y \to Y \text{ such that if } f : X \to Y \text{ then } f 1_Y = f \text{ and if } h : Y \to Z, \text{ then } 1_Y h = h.
\end{align*}
\]

Note 1: \(1_Y\) is called the identity morphism on \(Y\) and it is unique.

Note 2: Corresponding to any class \(X\), there is an associated category \(\mathcal{C}\) with \(v\mathcal{C} = X\) and for any \(x, x' \in X\), \(\mathcal{C}(x, x') = \phi\) if \(x \neq x'\) and \(\mathcal{C}(x, x) = \{1_x\}\) where \(1_x\) is the identity morphism on \(x\).

If the class of objects is a set, the category is referred to as a small category. Categories that involve in this discussion are small so that we are using the terminology that are applicable to small categories. So \(v\mathcal{C}\) is always a set. If \(f : X \to Y\) and \(g : Y \to X\) are such that \(fg = 1_X\), then \(g\) is called the right inverse of \(f\) and \(f\) is called the left inverse of \(g\).

An inverse of \(f \in \mathcal{C}(X, Y)\) is a morphism which is both a left inverse and a right inverse. A morphism \(f : X \to Y\) is called an isomorphism if there is a morphism \(g : Y \to X\) which is an inverse of \(f\) and in this case \(X\) and \(Y\) are said to be isomorphic or equivalent.

**Definition 1.3.1.** A subcategory \(\mathcal{C}' \subseteq \mathcal{C}\) is a category such that

\[
\begin{align*}
(a) & \quad \text{Each object of } \mathcal{C}' \text{ is an object of } \mathcal{C}.
\end{align*}
\]
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(b) If \( X', Y' \in v\mathcal{C}' \), then \( \mathcal{C}'(X', Y') \subseteq \mathcal{C}(X', Y') \)

(c) If \( f' \in \mathcal{C}'(X', Y'), g' \in \mathcal{C}'(Y', Z') \), then \( f'g' \) in \( \mathcal{C}' \) and \( \mathcal{C} \) are the same.

\( \mathcal{C}' \subseteq \mathcal{C} \) is called a full subcategory of \( \mathcal{C} \) if \( \mathcal{C}' \) is a subcategory of \( \mathcal{C} \) and \( \mathcal{C}'(X', Y') = \mathcal{C}(X', Y') \) for all \( X', Y' \in v\mathcal{C}' \). Functors are maps between categories which preserve identities and composites. This is explained further in the following steps.

Definition 1.3.2. Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. A covariant functor \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \) consists of an object function which assigns to each object \( X \) of \( \mathcal{C} \), an object \( F(X) \) in \( \mathcal{D} \) and a morphism function which assigns to each morphism \( f : X \rightarrow Y \) of \( \mathcal{C} \) a morphism \( F(f) : F(X) \rightarrow F(Y) \) of \( \mathcal{D} \) such that

(i) \( F(1_X) = 1_{F(X)} \)

(ii) \( F(f \circ g) = F(f) \circ F(g) \)

A contravariant functor \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \) consists of an object function which assigns to each object \( X \) of \( \mathcal{C} \), an object \( F(X) \) in \( \mathcal{D} \) and a morphism function which assigns to each morphism \( f : X \rightarrow Y \) of \( \mathcal{C} \) a morphism \( F(f) : F(Y) \rightarrow F(X) \) of \( \mathcal{D} \) such that

(i) \( F(1_X) = 1_{F(X)} \)

(ii) \( F(f \circ g) = F(g) \circ F(f) \)

For any category \( \mathcal{C} \), \( \mathcal{C}^{op} \) denotes the category with the same object set and morphisms reversed. So if \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a contravariant functor then \( F : \mathcal{C}^{op} \rightarrow \mathcal{D} \) is a covariant functor.

In our discussion functor always means covariant functor (unless otherwise specified). If \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a functor, sometimes we denote the object
function by $vF$. Thus $vF = F|v\mathcal{C}$. If $\mathcal{D}$ is subcategory of $\mathcal{C}$, then the inclusions of objects and morphisms of $\mathcal{D}$ into those of $\mathcal{C}$ is a functor called the inclusion functor. For any category $\mathcal{C}$, there always exists a functor, denoted by, $1_\mathcal{C}$ where the object function is the identity map on the object set and the morphism function is the identity map on the morphism set of $\mathcal{C}$.

**Definition 1.3.3.** Let $\mathcal{C}$ be a category and $\textbf{Set}$ denotes the category of sets whose objects are sets and morphisms are mappings between sets. For a fixed $c \in \mathcal{C}$, and $f \in \mathcal{C}(c', c'')$, let $\mathcal{C}(c, f)$ denote the function from $\mathcal{C}(c, c')$ to $\mathcal{C}(c, c'')$ defined by

$$\mathcal{C}(c, f)(g) = gf \quad \text{for every } g \in \mathcal{C}(c, c').$$

Then the assignments $c' \rightarrow \mathcal{C}(c, c')$ and $f \rightarrow \mathcal{C}(c, f)$ defines a functor denoted by $\mathcal{C}(c, -)$ from $\mathcal{C}$ to $\textbf{Set}$. This functor $\mathcal{C}(c, -)$ is called the covariant hom-functor determined by the object $c$.

$$\begin{array}{ccc}
C & \xrightarrow{g} & C' \\
\downarrow{1_c} & & \downarrow{f} \\
C & \xrightarrow{sf} & C''
\end{array} \quad (1.1)$$

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called $v$-injective if $vF$ is injective. That is one one on objects. Also it is called $v$-surjective if $vF$ is surjective. $F$ is called faithful if for each $c, c' \in v\mathcal{C}$ the restriction of $F$ to $\mathcal{C}(c, c')$ is injective. $F$ is called full if for each $c, c' \in v\mathcal{C}$, $F$ maps $\mathcal{C}(c, c')$ onto $\mathcal{D}(F(c), F(c'))$. An isomorphism of categories is a full and faithful functor $F$ in which $vF$ is a bijection.
Comparison of functors is done by means of suitable maps between functors. These are called natural transformations.

Definition 1.3.4. Let $F$ and $G$ be covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A natural transformation $\eta : F \rightarrow G$ is a function from $v\mathcal{C}$ to the morphism class of $\mathcal{D}$ such that for every $X \in v\mathcal{C}$, $\eta_X : F(X) \rightarrow G(X)$ is a morphism in $\mathcal{D}$ and for every morphism $f : X \rightarrow Y$ of $\mathcal{C}$, the following diagram commutes.

![Diagram](1.2)

That is $F(f)\eta_Y = \eta_X G(f)$.

If $F$ and $G$ are contravariant functors then $\eta : F \rightarrow G$ is a natural transformation if the following diagram commutes.

![Diagram](1.3)

For each $X \in v\mathcal{C}$ the morphism $\eta_X : F(X) \rightarrow G(X)$ in $\mathcal{D}$ is called the component of $\eta$ at $X$. If every component of $\eta$ is an isomorphism, then it is called a natural isomorphism. In this case the functors $F$ and $G$ from $\mathcal{C}$ to $\mathcal{D}$ are called naturally equivalent and we write $F \approx G$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if there exists functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $1_{\mathcal{C}} \approx FG$ and $GF \approx 1_{\mathcal{D}}$. In this case $F$ is called an equivalence from $\mathcal{C}$ to $\mathcal{D}$.
1.3.1 Functor Categories

If $\mathcal{C}$ and $\mathcal{D}$ are small categories, there is an associated category in which every functor from $\mathcal{C}$ to $\mathcal{D}$ is an object and every natural transformation between two such functors is a morphism. This category is denoted by $[\mathcal{C}, \mathcal{D}]$. Any subcategory of $[\mathcal{C}, \mathcal{D}]$ is called a functor category from $\mathcal{C}$ to $\mathcal{D}$. The functor category $[\mathcal{C}, \textbf{Set}]$ is the category of all set valued functors. Also it has the notation $\mathcal{C}^*$. If $F$ and $G$ are two functors from $\mathcal{C}$ to $\mathcal{D}$ then the set of all morphisms from $F$ to $G$ in the functor category $[\mathcal{C}, \mathcal{D}]$ is sometimes denoted by $\text{Nat}(F, G)$. In this category, the composition is defined componentwise. That is, if $T_1 \in \text{Nat}(F, G), T_2 \in \text{Nat}(G, H)$ then $T_1 \circ T_2 \in \text{Nat}(F, H)$ is a natural transformation defined by $(T_1 \circ T_2)(c) = T_1(c)T_2(c)$, for all $c \in \mathcal{C}$.

1.3.2 Monomorphisms and Epimorphisms

A morphism $f \in \mathcal{C}(c, c')$ is called a monomorphism if it is right cancellable. That is, for $g, h \in \mathcal{C}(d, c)$, $gf = hf$ implies $g = h$. A morphism $f \in \mathcal{C}(c, c')$ is called a epimorphism if it is left cancellable. That is, for $g, h \in \mathcal{C}(c', d)$, $fg = fh$ implies $g = h$.

A morphism $f \in \mathcal{C}(c, c')$ is called a split monomorphism if there exists a morphism $g \in \mathcal{C}(c', c)$, such that $fg = 1_c$. That is $f$ has a right inverse. A morphism $f \in \mathcal{C}(c, c')$ is called a split epimorphism if there exists a morphism $g \in \mathcal{C}(c', c)$, such that $gf = 1_c$. That is $f$ has a left inverse.

The following theorem is useful in dealing with monomorphisms and epimorphisms.
Theorem 1.3.5. Let $f, g$ be morphisms in $\mathcal{C}$ such that $fg$ exists. Then

(i) if $fg$ is a monomorphism then $f$ is a monomorphism

(ii) if $fg$ is an epimorphism then $g$ is an epimorphism.

Proof. (i) Suppose that for morphisms $h, k$ in $\mathcal{C}$, $hf = kf$. Then $h(fg) = (hf)g = (kf)g = k(fg)$. Since $fg$ is a monomorphism, this gives $h = k$. Hence $f$ is a monomorphism.

(ii) Suppose $fg$ is an epimorphism. For morphisms $h, k$ in $\mathcal{C}$ suppose that $gh = gk$. Then $(fg)h = f(gh) = f(gk) = (fg)k$. Since $fg$ is an epimorphism this gives $h = k$. Therefore $g$ is an epimorphism.

1.3.3 Kernel of a Morphism

Definition 1.3.6. A zero object in a category $\mathcal{C}$ is an object $Z$ with the property that for each object $A$ of $\mathcal{C}$ there is precisely one morphism from $Z$ to $A$ and precisely one morphism from $A$ to $Z$. This morphism is usually denoted by $0$.

Theorem 1.3.7. In a category $\mathcal{C}$ any two zero objects are isomorphic.

Proof. Let $Z$ and $Z'$ be zero objects in the category $\mathcal{C}$. Then $\mathcal{C}(Z, Z')$ has precisely one morphism, say, $0$ and $\mathcal{C}(Z', Z)$ has precisely one morphism, say $0'$. Therefore $00' = 1_Z = 0 : Z \to Z$ and $0'0 = 1_{Z'}$. Thus any two zero objects are isomorphic.

Note that if $\mathcal{C}$ is category with a zero object $Z$ then for any $A, B \in \mathcal{V}C$ there is a unique morphism from $A$ to $B$ which factors through $Z$. This
morphism is called the zero morphism from $A$ to $B$. This is also denoted by $0$. We may denote the zero object as well as the zero morphism by $0$. The situation will make clear what the $0$ stands for.

Definition 1.3.8. Let $\mathcal{C}$ be a category with a zero object and $\alpha : A \to B$. We will call a morphism $u : K \to A$, a kernel of $\alpha$ if $u\alpha = 0$ and if for every morphism $u' : K' \to A$ such that $u'\alpha = 0$ there is a unique morphism $\gamma : K' \to K$ such that $\gamma u = u'$.

![Diagram](image)

Sometimes we refer to the object $K$ as the kernel object and $u$ as the kernel morphism or the kernel inclusion.

Definition 1.3.9. Let $\mathcal{C}$ be a category with a zero object and $\alpha : A \to B$. We will call a morphism $u : B \to K$, the cokernel of $\alpha$ if $\alpha u = 0$ and if for every morphism $u' : B \to K'$ such that $\alpha u' = 0$ there is a unique morphism $\gamma : K \to K'$ such that $u\gamma = u'$.

![Diagram](image)

Here we refer to $K$ as the cokernel object.
1.3.4 Image of a Morphism

Definition 1.3.10. The image of a morphism \( f : A \to B \) in \( \mathcal{C} \) is denoted by \( \text{im} f \) and is defined as a triple \((f', I, u)\) where \( I \) is an object in \( \mathcal{C} \) and \( f' : A \to I \), and \( u : I \to B \) are morphisms in \( \mathcal{C} \) with \( u \) a monomorphism such that \( f = f'u \) and if \( u' : J \to B \) is any other monomorphism with \( f = f''u' \) for some \( f'' : A \to J \), then there exists a unique morphism \( \gamma : I \to J \) such that \( \gamma u' = u \). We write \( I \) or \((I, u)\) for \( \text{im} f \) sometimes.

\[
\begin{array}{c}
A & \xrightarrow{f} & B \\
\downarrow{f'} & & \downarrow{u} \\
I & & J
\end{array}
\quad
\begin{array}{c}
A & \xrightarrow{f'} & I \\
\downarrow & & \downarrow \\
B & & J
\end{array}
\quad
\begin{array}{c}
A & \xrightarrow{f''} & B \\
\downarrow{u'} & & \downarrow{u} \\
J & & I
\end{array}
\]

Note that in this case \( f'' = f'\gamma \).

Definition 1.3.11. The coimage of a morphism \( f : A \to B \) is denoted by \( \text{coim} f \) and is defined as a triple \((v, T, \theta)\) where \( v : A \to T \) is an epimorphism, and \( \theta : T \to B \) is such that \( f = v\theta \) and if \( v' : A \to S \) is an epimorphism with \( f = v'\phi \) for some \( \phi : S \to B \), then there exists a unique morphism \( \gamma : S \to T \) such that \( v = v'\gamma \). We write \( T \) or \((v, T)\) for \( \text{coim} f \).

\[
\begin{array}{c}
A & \xrightarrow{f} & B \\
\downarrow{v} & & \downarrow{\theta} \\
T & & I
\end{array}
\quad
\begin{array}{c}
A & \xrightarrow{v'} & S \\
\downarrow & & \downarrow \\
B & & T
\end{array}
\quad
\begin{array}{c}
A & \xrightarrow{v} & S \\
\downarrow{\phi} & & \downarrow{v'} \\
T & & I
\end{array}
\]

It may be noted that an epimorphism \( A \to T \) is a coimage of a morphism \( f \) if it is an image of \( f \) in the dual category.
Lemma 1.3.12. Let \( f : A \to B \). Suppose \((f_1, I, u)\) and \((f_2, J, u')\) be images of \( f \). Let \( \alpha : I \to J \) and \( \beta : J \to I \) be morphisms such that \( u = \alpha u' \) and \( u' = \beta u \). Then \( \alpha \) and \( \beta \) are isomorphisms and \( \alpha = \beta^{-1} \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{u} & & \downarrow{u'} \\
I & \xleftarrow{\alpha} & J
\end{array}
\]

Proof. By definition of image of \( f \), it is clear that \( u \) and \( u' \) are monomorphisms. Now, \( u = \alpha u' = \alpha(\beta u) = (\alpha \beta)u \) and \( u' = \beta u = (\beta \alpha)u' \) and hence \( \alpha \beta = 1_I \) and \( \beta \alpha = 1_J \) from which it follows that \( \alpha \) and \( \beta \) are isomorphisms and \( \alpha = \beta^{-1} \). \( \square \)

1.4 Subobjects in a Category

Here we familiarise the concept of subobject relation in a category (see \[14\]). We consider a subobject as an object in the category rather than an equivalence class of monomorphisms, as is usually taken.

1.4.1 Preorders

A preorder is a category \( P \) such that for any \( p, p' \in P \), the morphism class \( P(p, p') \) contains at most one morphism. So the relation \( \leq \) defined on \( vP \) by \( p \leq p' \iff P(p, p') \neq \emptyset \) is a quasiorder. Also in a preorder \( P \), \( p \) and \( p' \) are isomorphic if and only if \( P(p, p') \neq \emptyset \neq P(p', p) \). So the relation \( \leq \) is a partial order if and only if \( P \) does not contain any non-trivial isomorphisms.
1.4. Subobjects in a Category

That is, the only isomorphisms of \( P \) are identity morphisms. In this case \( P \) is called a strict preorder.

**Definition 1.4.1.** Let \( \mathcal{C} \) be a category. A choice of subobjects in \( \mathcal{C} \) is a subcategory \( \mathcal{P} \subseteq \mathcal{C} \) satisfying the following:

(a) \( \mathcal{P} \) is a strict preorder with \( v_P = v_C \)

(b) Every \( f \in \mathcal{P} \) is a monomorphism in \( \mathcal{C} \).

(c) If \( f, g \in \mathcal{P} \) and if \( f = hg \) for some \( h \in \mathcal{C} \) then \( h \in \mathcal{P} \).

When \( \mathcal{P} \) satisfies these conditions, the pair \( (\mathcal{C}, \mathcal{P}) \) is called a category with subobjects.

1.4.2 Induced Partial order on \( v_C \)

Let \( \mathcal{C} \) be a category with subobjects. Then by definition 1.4.1, there is a choice of a \( v \)-full subcategory \( \mathcal{P} \) of monomorphisms which is a strict preorder. This induces a partial order on the vertex set \( v_C \) by defining \( A \leq B \) if there exists a morphism in \( \mathcal{P} \) from \( A \) to \( B \). In this case we say that \( A \) is a subobject of \( B \). We also write \( A \subset B \).

If \( A \leq B \) then the unique morphism from \( A \) to \( B \) in the subcategory \( \mathcal{P} \) is denoted by \( j(A, B) \) and is called the inclusion from \( A \) to \( B \). In a category with subobjects, a morphism \( e : B \to A \) is said to be a retraction if \( A \subseteq B \) and \( j(A, B)e = 1_A \). We will denote this retraction \( e : B \to A \) by \( e(b, A) \). It is not necessary that retractions exists for all subobject inclusions \( A \subseteq B \). In case \( A \subseteq B \) and a retraction \( e(b, A) : B \to A \) exists then \( A \) is said to be retract of \( B \).
1.4.3 Normal Factorisations

Factorisation of morphisms in a category are often found useful in studying the category. We consider here a specialised factorisation called normal factorisation in which the factors have some particular properties.

**Definition 1.4.2.** A normal factorisation of a morphism \( f \in \mathcal{C} \) is a factorisation of the form \( f = euj \) where \( e \) is a retraction, \( u \) is an isomorphism and \( j \) is an inclusion.

We note that every morphism in a category \( \mathcal{C} \) may not have a normal factorisation. Moreover when a morphism \( f \) has a normal factorisation it need not be unique. But there is some element of uniqueness in a normal factorisation as is given in the next Proposition.

**Proposition 1.4.3.** (cf. Proposition 5, Ch II, [14]) Let \( f = euj = e'u'j' \) be two normal factorisations of \( f \) in \( \mathcal{C} \). Then \( j = j' \) and \( eu = e'u' \).

**Proof.** Let \( f : X \to Y \) and \( X \xrightarrow{e} A \xrightarrow{u} B \xrightarrow{j} Y \) and \( X \xrightarrow{e'} A' \xrightarrow{u'} B' \xrightarrow{j'} Y \) be the factorisations. Since \( e \) is a retraction there exists an inclusion \( i = j(A, x) \) such that \( ie = 1_A \). Since \( u \) is an isomorphism, \( q = u^{-1}i \) is such that \( qe u = u^{-1}ie u = u^{-1}1_A u = u^{-1}u = 1_B \). Therefore \( j = 1_B j = qe u j = qe' u' j' \). So by axiom (c) of the category with subobjects (see Definition 1.4.1), \( qe' u' : B \to B' \) is an inclusion. In a similar way we can get that there is an inclusion from \( B' \) to \( B \). So by condition (a) of Definition 1.4.1, we see that \( B = B' \). This implies that \( j = j' \). Hence \( euj = e'u'j \) and since \( j \) is a monomorphism we get \( eu = e'u' \). \( \square \)

**Definition 1.4.4.** Let \( f = euj \) be a normal factorisation of \( f \). Then \( f^\circ = eu \) is called the epi part of \( f \) and \( \text{cod} f^\circ \) is often called the image of \( f \).
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In view of proposition 1.4.3, \( f^0 \) is independent of the factorisation. It may also be noted that \((f^0, \text{cod } f^0, j)\) is an image of \( f \) in the sense of Definition 1.3.10.

Note: In a normal factorisation \( f = euj \) any or all of the factors can be identity and often the identity factor is left out in the presentation. For example, if \( j \) is the identity, the factorisation can be written as \( f = eu \) or if \( e \) is the identity, \( f = uj \) etc. It may be noted that \( f^0 = eu \) is an epimorphism.

**Proposition 1.4.5.** If \( f = g^0 \) for some morphism \( g \) then the normal factorisation of \( f \) is \( f = eu \) where \( e \) is a retraction and \( u \) is an isomorphism.

**Proof.** If \( f = euj \) be a normal factorisation and \( g = e'u'j' \) be a normal factorisation of \( g \). Since \( f = g^0 \), \( f = e'u' \). Now \( f = e'u' = e'u'i \) where \( i: \text{cod } f \rightarrow \text{cod } f \) is the identity which is also an inclusion. So \( e'u'i \) is also a normal factorisation of \( f \). Therefore by Proposition 1.4.3 \( f = e'u' = eu \).

This completes the proof. 

**Proposition 1.4.6.** If \( f: a \rightarrow b \) is an isomorphism then it has a unique normal factorisation as \( f = e fj \) where \( e = e(a, a) = 1_a \) and \( j = j(b, b) = 1_b \).

**Proof.** Suppose that \( f = euj \) be a normal factorisation of \( f \). Already \( f = e(a, a)fj(b, b) \) is a normal factorisation of \( f \). So by Proposition 1.4.5 \( j = j(b, b) = 1_b \) and \( eu = e(a, a)f = f \). So \( e = fu^{-1} \) is an isomorphism. It follows that \( e = 1 \). Hence the result.

If \( f \) is a morphism we write \( \text{im } f = \text{cod } f^0 \). We have the following result on images.

**Proposition 1.4.7.** If \( f, g \) are morphisms having normal factorisations then image of \( f^0g = \text{image of } g \)
Proof. Let \( f^0 g = euj \) be a normal factorisation of \( f^0 g \). Then image of \( f^0 g \) is \( \text{cod} eu \). Let \( f = e_1 u_1 j_1 \) and \( g = e_2 u_2 j_2 \) be normal factorisations of \( f \) and \( g \) respectively. Then \( f^0 g = e_1 u_1 e_2 u_2 j_2 \). It follows that \( (f^0 g)^0 = e_1 u_1 e_2 u_2 = f^0 g^0 \). So \( \text{im} f^0 g = \text{cod} (f^0 g^0) = \text{cod} e_2 u_2 = \text{cod} g^0 \).

**Proposition 1.4.8.** Let \( f, g \) be morphisms having normal factorisations. Suppose \( fg \) is defined. Then \( (fg)^0 = f^0 (jg)^0 \) where \( j = j(\text{im} f, \text{dom} g) \)

Proof. Let \( f = euj \) be a normal factorisation of \( f \). Then \( fg = (euj)g = eu(jg) \). So \( (fg)^0 = eu(jg)^0 = f^0 (jg)^0 \).

**Proposition 1.4.9.** Let \( f \) be a morphism having normal factorisation. Then for any inclusion \( j_1 \), we have \( f^0 = (fj_1)^0 \).

Proof. Let \( f = euj \) be a normal factorisation of \( f \). Then \( fj_1 = eu j_1 \). So \( (fj_1)^0 = eu = f^0 \).

### 1.4.4 Normal Cones

Here it is assumed that \( \mathcal{C} \) is a small category which admits subobjects and every morphism in \( \mathcal{C} \) has normal factorisations in which the inclusion splits.

A cone \( \gamma \) from the base \( \forall \mathcal{C} \) to the vertex \( d \) is a natural transformation from the identity functor \( 1_{\mathcal{C}} \) on the preorder of inclusions \( \forall \mathcal{C} \) to the constant functor with value \( d \). Therefore we have the definition

**Definition 1.4.10.** A cone \( \gamma \) in \( \mathcal{C} \) with vertex \( d \) is a mapping \( \gamma : \forall \mathcal{C} \to d \) such that

(i) \( \gamma(a) \in \mathcal{C}(a, d) \), for all \( a \in \forall \mathcal{C} \).

(ii) if \( a \subseteq b \), then \( j(a, b) \gamma(b) = \gamma(a) \).
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The vertex $d$ of the cone $\gamma$ is usually denoted by $c_\gamma$. A normal cone is a cone in which at least one component $\gamma(c)$ is an isomorphism.

Now will prove that the condition that at least one component of a normal cone is an isomorphism can be replaced by the condition that at least one component is an epimorphism.

**Proposition 1.4.11.** Let $\mathcal{C}$ be a category with subobjects where each morphism has a normal factorisation. Also let $d \in v\mathcal{C}$ and $\gamma : v\mathcal{C} \to d$ be a cone. If there exists $c \in v\mathcal{C}$ such that $\gamma(c) : c \to d$ is an epimorphic component of a morphism then $\gamma$ is a normal cone in $\mathcal{C}$.

**Proof.** Let $\gamma(c)$ be an epimorphic component of a morphism. Then by Definition 1.4.4, $\gamma(c) = eu$ where $e$ is a retraction and $u$ is an isomorphism. Let $e : c \to c_0$. Then, since $c_0$ is a subobject of $c$ by Definition 1.4.10, $\gamma(c_0) = j(c_0, c)\gamma(c) = j(c_0, c)e(c, c_0)u = u$. Thus $\gamma(c_0)$ is an isomorphism and hence $\gamma$ is a normal cone. $\square$

In what follows we say a morphism is an epimorphism to mean that it is the epimorphic component of a morphism.

**Lemma 1.4.12.** (cf. Lemma 1, Ch III, [14]) Let $TC$ denote the set of all normal cones in $\mathcal{C}$. If $\gamma \in TC$ and $f \in \mathcal{C}(c_\gamma, c')$, is an epimorphism define $\gamma * f : v\mathcal{C} \to \mathcal{C}$ by $(\gamma * f)(a) = \gamma(a)f$. Then $\gamma * f \in TC$.

Next we will define a binary operation on $TC$ so that it becomes a semigroup.

For $\gamma_1, \gamma_2 \in TC$, define $\gamma_1 \cdot \gamma_2 = \gamma_1 * (\gamma_2(c_{\gamma_1}))^0$. (1.4)
Theorem 1.4.13. (cf. Theorem 2, Ch III, [14]) Let $\mathcal{C}$ be a small category where each morphism has a normal factorisation. Then we have the following:

(a) $T\mathcal{C}$ is a semigroup with the binary operation defined by Equation (1.4)

(b) $\gamma \in T\mathcal{C}$ is an idempotent if and only if $\gamma(c_\gamma) = 1_c$,

(c) $\gamma \in T\mathcal{C}$ is a regular element in $T\mathcal{C}$ if and only if there is a $\gamma' \in T\mathcal{C}$ such that $\gamma'(c_\gamma)$ is a monomorphism.

Moreover, the set of regular elements of $T\mathcal{C}$ is a subsemigroup of $T\mathcal{C}$.

1.4.5 Normal Categories

Normal category is basic category used in cross connection theory of semigroups. The factorisation property of morphisms in this category give rise to a rich structure for the category.(cf. [14] and [16])

Definition 1.4.14. A small category $\mathcal{C}$ with subobjects is said to be a normal category if

(i) every inclusion in $\mathcal{C}$ splits,

(ii) every $f \in \mathcal{C}$ has a factorisation of the form $f = euj$, where $e$ is a retraction, $u$ is an isomorphism and $j$ is an inclusion and

(iii) for each $A \in \mathcal{C}$, there exists $\gamma \in T\mathcal{C}$ such that $\gamma(A) = 1_A$.

Two normal categories are isomorphic if there exists a category isomorphism between them that preserves inclusions.

In the following for a morphism $f$ with normal factorisation $f = euj$, $im f$ denote the object in $\mathcal{C}$ which is codomain of $f^0 = eu$ and also it is the domain of $j$. 

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Theorem 1.4.15. Let \( \mathcal{C} \) be a normal category and \( f : A \rightarrow B \) in \( \mathcal{C} \). If \( A_0 \subseteq A \), then \( \text{im} j(A_0, A) f \subseteq \text{im} f \).

Proof. Let \( f = euj \) be a normal factorisation of \( f \). Therefore \( \text{im} j(A_0, A) f = \text{im} j(A_0, A) euj \). Let \( g = j(A_0, A) e u \). It also has a normal factorisation, say, \( g = e_1 u_1 j_1 \). Therefore \( j(A_0, A) f = gj = e_1 u_1 j_1 j \). Therefore \( \text{im} j(A_0, A) f = \text{dom} j_1 j = \text{dom} j_1 \subseteq \text{dom} j = \text{im} f \). This completes the proof. \( \square \)

1.4.6 M-set of a normal cone

The isomorphism components of a normal cone are often useful in analysing a cone. This is given by what is called the M-set. For \( \gamma \in T \mathcal{C} \), the M-set of \( \gamma \) is defined by

\[
M_\gamma = \{ c \in T \mathcal{C} : \gamma(c) \text{ is an isomorphism} \} \quad (1.5)
\]

For any subset \( X \) of a semigroup \( S \), \( E(X) \) denotes the set of idempotents of \( S \) contained in \( X \).

Lemma 1.4.16. (cf. Lemma 3, Ch III, [14]) Let \( \gamma \in T \mathcal{C} \) and \( \varepsilon \in E(T \mathcal{C}) \). Then \( \varepsilon \gamma = \gamma \) if and only if there exists a unique epimorphism \( f : c_\varepsilon \rightarrow c_\gamma \) such that \( \gamma = \varepsilon \ast f \).

The following theorem explains how the M-set \( M_\gamma \) is related to the normal cone \( \gamma \).

Proposition 1.4.17. (cf. Proposition 4, Ch III, [14]) For \( \gamma \in T \mathcal{C} \), we have the following.

(a) For each \( c \in M_\gamma \) there is a unique idempotent normal cone \( \varepsilon^c \) whose vertex is \( c \) and an isomorphism \( f : c \rightarrow c_\gamma \) such that \( \gamma = \varepsilon^c \ast f \). Conversely for each
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\( \varepsilon \in E(T\mathcal{C}) \), if there exists an isomorphism \( f : c_\varepsilon \to c_\gamma \) with \( \gamma = \varepsilon \circ f \) then \( c_\varepsilon \in M_\gamma \).

(b) Let \( c \in M_\gamma \). Then \( E(Rc) = \{ c' : c' \leq c \} \)

(c) If \( \gamma' \in T\mathcal{C} \) and if \( \gamma \not\leq \gamma' \) then \( M_\gamma = M_{\gamma'} \)

In particular, \( \gamma \) is regular in \( T\mathcal{C} \) if and only if \( \varepsilon \not\leq \gamma \) for some \( c \in M_\gamma \).

**Proposition 1.4.18.** (cf. Proposition 5, Ch III, [14]) Let \( \mathcal{C} \) be a normal category. If \( \gamma, \gamma' \in T\mathcal{C} \) and if \( L_\gamma \leq L_{\gamma'} \) then \( c_\gamma \leq c_{\gamma'} \). The converse also holds if \( \gamma' \) is regular. In particular, if \( \mathcal{C} \) is normal then the map \( L_\gamma \to c_\gamma \) is an order isomorphism of the partially ordered set of \( \mathcal{L} \) - classes onto the partially ordered set \( v\mathcal{C} \).

Let \( \mathcal{C} \) be a normal category. For each \( \gamma \in T\mathcal{C} \), define \( H(\gamma ; -) : \mathcal{C} \to \text{Set} \) by

\[
H(\gamma; c) = \{ \gamma \ast f^0 : f \in \mathcal{C}(c_\gamma, c) \} \quad \text{and for} \quad g \in \mathcal{C}(c, c')
\]

\[
H(\gamma; g) : \gamma \ast f^0 \to \gamma \ast (fg)^0
\]

Then \( H(\gamma ; -) \) is a functor and \( H(\gamma ; -) \) is representable with \( c_\gamma \) as the representing object. That is, \( H(\gamma; -) \) and \( \mathcal{C}(c_\gamma; -) \) are naturally isomorphic.

In fact a natural isomorphism \( \eta_\gamma \) can be defined whose component at \( c_\gamma \) is \( \eta_\gamma(c_\gamma) : H(\gamma; c_\gamma) \to \mathcal{C}(c_\gamma, c_\gamma) \) sending \( \gamma \to 1_{c_\gamma} \). We denote the full subcategory of \( \mathcal{C}^* \) (see Section 1.3.1)with vertex set \( \{ H(\gamma; -) : \gamma \in T\mathcal{C} \} \) by \( N^*\mathcal{C} \).

Then \( N^*\mathcal{C} \) is also a normal category with inclusion as the inclusion of the set valued functors. Alternatively we have

**Proposition 1.4.19.** (cf. Proposition 7, Ch III, [14]) For \( \gamma, \delta \in T\mathcal{C} \), we have the following.
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(a) \( H(\gamma; -) \subseteq H(\delta; -) \) if and only if there exists a unique epimorphism \( h \) from \( c_\delta \) to \( c_\gamma \) such that \( \gamma = \delta \ast h \). Moreover, \( H(\gamma; -) = H(\delta; -) \) if and only if there is a unique isomorphism \( h : c_\delta \to c_\gamma \) such that \( \gamma = \delta \ast h \).

(b) If \( R_\gamma \leq R_\delta \) then \( H(\gamma; -) \subseteq H(\delta; -) \). The converse also holds if \( \delta \) is regular. In particular when \( \mathcal{C} \) is a normal category, the map \( R_\gamma \to H(\gamma; -) \) is an order embedding of \( T^\mathcal{C} / \mathcal{R} \) into \( vN^*\mathcal{C} \).

It has the following important corollaries

Corollary 1.4.20. (cf. Corollary 8, 9, 10, Ch III, [14])

(i) If \( H(\gamma; -) = H(\delta; -) \) then \( M_\gamma = M_\delta \).

(ii) \( \delta \in T^\mathcal{C} \) is a universal element of \( H(\gamma; -) \) if and only if \( \delta = \gamma \ast h \) for some isomorphism \( h \). In particular the set of universal elements of \( H(\gamma; -) \) is \( R_\gamma \).

(iii) If \( H(\gamma; -) = H(\delta; -) \) and if \( h : c_\delta \to c_\gamma \) is the unique isomorphism such that \( \gamma = \delta \ast h \), then the following diagram of functors and natural transformation commute.

\[
\begin{array}{ccc}
H(\gamma; -) & \xrightarrow{\eta_\gamma} & \mathcal{C}(c_\gamma, -) \\
\| & & \| \\
H(\delta; -) & \xrightarrow{\eta_\delta} & \mathcal{C}(c_\delta, -)
\end{array}
\]  \quad (1.8)

The above results are used to characterise the Greens relations \( \mathcal{L}, \mathcal{R}, \mathcal{D} \) on the semigroup \( T^\mathcal{C} \) where \( \mathcal{C} \) is a normal category. So we have the following theorem.

Theorem 1.4.21. (cf. Theorem 11, Ch III, [14]) Let \( \mathcal{C} \) be a normal category.

Then

(i) \( \gamma \mathcal{L} \delta \iff c_\gamma = c_\delta \)
(ii) $\gamma \circ \delta \iff H(\gamma; -) = H(\delta; -)$

(iii) $\gamma \circ \delta \iff c_\gamma \cong c_\delta$

1.4.7 The Categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$

Given a regular semigroup $S$, a mapping $\rho : S \to S$ such that $(xy)\rho = x(yp)$ for all $x, y \in S$ is called a right translation of $S$ and a mapping $\lambda : S \to S$ such that $\lambda(xy) = (\lambda x)y$ for all $x, y \in S$ is called a left translation of $S$. Also we have the right translation determined by $a$ as the map $x \to xa$ and is denoted by $\rho_a$. Similarly the left translation determined by $a$ is the map $x \to ax$ and is denoted by $\lambda_a$. By a partial right translation we mean a map $\rho : Se \to Sf$ where $\rho = \rho_u|Se$ for some $u \in S$. Suppose that $\rho = \rho_u : Se \to Sf$ is a partial right translation. Then $e \in Se$ and so $e\rho = e\rho_u = eu \in Sf$. Now for every $x \in Se$, $x = ye$ for some $y \in S$. Therefore $x\rho = x\rho_u = ye\rho_u = yeu = ye.eu = x\rho_{eu}$. Thus $\rho_u = \rho_{eu}$ and $eu \in Sf$. In fact $eu \in Sf$. Thus we see that any partial right translation $\rho : Se \to Sf$ can be written as $\rho = \rho_u$ for some $u \in eSf$. For this reason we can always assume that a partial right translation is $\rho = \rho_u|Se$ for some $u \in eSf$.

Definition 1.4.22. We define the category of left ideals $\mathcal{L}(S)$ as the category with vertex set

$$\mathcal{V}\mathcal{L}(S) = \{Se : e \in E(S)\}$$

and morphism set

$$\mathcal{L}(S)(Se, Sf) = \{\rho_u : Se \to Sf : u \in eSf\}$$

We denote the morphism $\rho_u$ by $\rho(e, u, f) : Se \to Sf$. 
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The category $\mathcal{R}(S)$ of right ideals is defined as the category with vertex set

$$\nu \mathcal{R}(S) = \{eS : e \in E(S)\}$$

and morphism set as

$$\mathcal{R}(S)(eS, fS) = \{\lambda_v : eS \rightarrow fS : v \in fSe\}$$

The morphism $\lambda_v$ is denoted by $\lambda(e, v, f) : eS \rightarrow fS$.

Note that for any $a \in S$, $\rho_a : Se \rightarrow Sf$ can be considered as a morphism in $\mathcal{L}(S)$. In this case $\rho_a = \rho_b$ for some $b \in eSf$. So in all representations of a morphism as $\rho_a$, it is not required us $a \in eSf$. Hence $\mathcal{L}(S)$ is the category with vertex set as the set of all principal left ideals and morphism set as the set of all partial right translations. Similarly we have, $\mathcal{R}(S)$ is the category with vertex set as the set of all principal right ideals and morphism set as the set of all partial left translations.

**Lemma 1.4.23.** (cf. Lemma 12, Ch III, [14])

(i) $\mathcal{L}(S)$ is a category with normal factorisation property in which inclusions are the usual set inclusions

(ii) For every $e, f \in E(S)$ and $u \in eSf$, $\rho(e, u, f) \in \mathcal{L}(S)(Se, Sf)$ and the map $\rho(e, u, f) \mapsto u$ is a bijection of $\mathcal{L}(S)(Se, Sf)$ onto $eSf$.

(iii) $\rho(e, u, f) = \rho(e', u', f')$ if and only if $e \mathcal{L} e'$, $f \mathcal{L} f'$, $u \in eSf$, $u' \in e'Sf'$ and $u' = e'u$.

(iv) If $\rho(e, u, f)$ and $\rho(g, v, h)$ are composable morphisms in $\mathcal{L}(S)$, so that $f \mathcal{L} g$, $u \in eSf$, and $v \in gSh$, then

$$\rho(e, u, f) \rho(g, v, h) = \rho(e, uv, h).$$

The following remark is very significant to mention (see Remark 5 and 6 Ch III, [14]).
Remark 1.4.24. If \( S^{op} \) denote the semigroup on the set \( S \) with binary operation \( \circ \) defined by \( x \circ y = yx \) for all \( x, y \in S \), then

\[
R(S) = L(S^{op}) \quad \text{and} \quad L(S) = R(S^{op}).
\]

Therefore to every statement which holds for \( L(S) \) (or \( R(S) \)) there is a corresponding dual statement which holds for \( R(S) \) (respectively \( L(S) \)).

Thus we have the following lemma.

Lemma 1.4.25. \( R(S) \) has the following properties.

(a) For every \( e, f \in E(S) \) and \( u \in fSe, \lambda(e, u, f) \in R(S)(eS, fS) \). Moreover, the map \( \lambda(e, u, f) \mapsto u \) is a bijection of \( R(S)(eS, fS) \) onto \( fSe \).

(b) \( \lambda(e, u, f) = \lambda(e', v, f') \) if and only if \( e \Join e' \), \( f \ Join f' \), \( u \in fSe, f' \in f'Se' \) and \( v = ue' \).

(c) If \( \lambda(e, u, f) \) and \( \lambda(g, v, h) \) are composable morphisms, then

\[
\lambda(e, u, f) \lambda(g, v, h) = \lambda(e, vu, h).
\]

Remark 1.4.26. In compositions left translations act as left operators and right translations act as right operators. That the composite of left translations \( \lambda \lambda' \) is taken with \( \lambda' \) acting first and then \( \lambda \). The composite \( pp' \) of right translations is taken with \( p \) acting first and then \( p' \).

Theorem 1.4.27. (cf. Proposition 13, Ch III, [14]) Let \( \rho = \rho : Se \rightarrow Sf \) be a morphism in \( L(S) \). Then we have the following.

(a) The morphism \( \rho(e, u, f) \) is a monomorphism if and only if \( \rho(e, u, f) \) is injective and this is true if and only if \( e \Join u \). In this case \( x \Join xp \) for all \( x \in Se \).
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(b) The morphism $\rho(e, u, f)$ is a epimorphism if and only if $\rho(e, u, f)$ is surjective and this is true if and only if $u \subseteq f$.

(c) $Se$ and $Sf$ are isomorphic in $\mathcal{L}(S)$ if and only if $e \subseteq f$. In this case there is a bijection between the set of all isomorphisms of $Se$ onto $Sf$ and the $H$ class $R_e \cap L_f$.

(d) If $Se \subseteq Sf$, then $j(se, sf) = \rho(e, e, f)$ and $\rho : Sf \to Se$ is a retraction if and only if $\rho = \rho(f, g, e)$ for some $g \in E(L_a) \cap \omega(f)$. In particular, $\rho(f, fe, e) : Sf \to Se$ is a retraction.

Lemma 1.4.28. (cf. Lemma 15, page 50, [14]) Let $a \in S$ and $f \in E(L_a)$. Then the map

$$\rho^a(Se) = \rho(e, ea, f)$$

is a normal cone in $\mathcal{L}(S)$ with vertex $Sa = Sf$ and such that

$$M\rho^a = \{Se : e \in E(R_a)\}.$$ 

Moreover $\rho^a$ is an idempotent if and only if $a \in E(S)$.

Lemma 1.4.29. (cf. Theorem 16, [14]) Let $S$ be a regular semigroup. Then $\mathcal{L}(S)$ is a normal category. Moreover there exists a homomorphism $\overline{\rho} : S \to TL(S)$ and an injective homomorphism $\phi : S_p \to TL(S)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
S & \xrightarrow{\rho} & S_p \\
\downarrow & & \downarrow \phi \\
S & \xrightarrow{\overline{\rho}} & TL(S)
\end{array}
$$

Here $S_p$ is the semigroup of all right translations $\{\rho_a : a \in S\}$. 

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Theorem 1.4.30. (cf. Theorem 19, [14]) Let $\mathcal{C}$ be a normal category. Define $F$ on objects and morphisms of $\mathcal{C}$ as follows. For $c \in \mathcal{C}$, let

$$vF(c) = (T\mathcal{C})e,$$

where $e \in E(T\mathcal{C})$, with $c_e = c$; and for $f \in \mathcal{C}(c, d)$, let

$$F(f) = \rho(e, e' f^0, e')$$

where $e, e' \in E(T\mathcal{C})$, with $c_e = c, c_e' = d$. Then $F : \mathcal{C} \to \mathcal{L}(T\mathcal{C})$ is an isomorphism of categories.

The following are results based on normal category which are used in later discussion.

Lemma 1.4.31. (cf. Lemma 22, [14]) Let $e, e' \in E(T\mathcal{C})$. Then the map $\lambda(e, \gamma, e') \mapsto \tilde{\gamma}$ where $\gamma \in e'(T\mathcal{C})e$ and $\tilde{\gamma} = \gamma(c_e') \cdot e_{c_e}$ is a bijection of $\mathcal{R}(T\mathcal{C})(e, e')$ onto $\mathcal{C}(c_e, c_e')$.

Lemma 1.4.32. (cf. Lemma 23, [14]) If $\gamma e_1$ and $e'e_1$ and if $\tilde{\gamma} \in \mathcal{C}(c_e, c_e')$ are morphisms corresponding to the representations $\lambda = \lambda(e, \gamma, e') = \lambda(e_1, \gamma_1, e_1')$ of the morphism $\lambda \in \mathcal{R}(T\mathcal{C})(e, e')$, then we have the following commutative diagram.

\[
\begin{array}{ccc}
\ \ c_{e_1} & \xrightarrow{g} & c_{e_1} \\
\downarrow_{\gamma_1(e_1)} & & \uparrow_{\gamma_1(e_1)} \\
 c_{e} & \xrightarrow{\gamma_1} & c_{e}
\end{array}
\]

Lemma 1.4.33. (cf. Lemma 24, [14]) Let $\gamma' \in e'T\mathcal{C}e$ and $e' \in e''T\mathcal{C}e'$. Assume that $\tilde{\gamma}, \tilde{\gamma}'$ and $\gamma''$. $\gamma$ are morphisms as defined in Lemma 1.4.31. Then $\tilde{\gamma} \tilde{\gamma}' = \gamma'' \gamma$

Theorem 1.4.34. (cf. Theorem 25, [14]) Let $\mathcal{C}$ be a normal category. Define $G$ on objects and morphisms of $\mathcal{R}(T\mathcal{C})$ as follows.
1.4. Subobjects in a Category

\[ vG(\varepsilon T\xi) = H(\varepsilon, -) \]

and for \( \lambda = \lambda(\varepsilon, \gamma, \varepsilon') : \varepsilon T\xi \to \varepsilon' T\xi \), let \( G(\lambda) \) be the natural transformation making the following diagram commutative.

\[
\begin{array}{ccc}
H(\varepsilon; -) & \xrightarrow{\eta_\varepsilon} & \xi(\varepsilon, -) \\
\downarrow G(\lambda) & & \downarrow \psi(\varepsilon, -) \\
H(\varepsilon'; -) & \xrightarrow{\eta_{\varepsilon'}} & \xi(\varepsilon', -)
\end{array}
\] (1.9)

Then \( G : \mathcal{R}(T\xi) \to N^*\xi \) is an isomorphism of normal categories.

2.1 Definition and Basic Properties

Definition 2.1.1. A ring \( R \) is regular if for every \( x \in R \) there exists \( y \in R \) such that \( yxy = x \).

Usually regular rings considered in literature are rings with unity. But we consider the general case where the ring may not have an identity. In this case, the property may be slightly different, and for these properties, which continue to hold for rings without unity, the proof of the results may be different. In such cases we provide the relevant proof for the results.