INTRODUCTION

A ring \((S, +, \cdot)\) is said to be a regular ring if the multiplicative semigroup \((S, \cdot)\) is a regular semigroup. That is, for each \(x \in S\), there is an \(x' \in S\) such that \(xx'x = x\). These rings are also called Von Neumann regular rings. These regular rings were introduced by Von Neumann in his joint work with Murray on algebras of operators on a Hilbert space (cf. [12]). This class was extensively studied by ring theorists and functional analysts (cf. von Neumann [19], Goodearl [4], Tuganbaev [18]). Study of the multiplicative semigroup of rings is carried out in semigroup theory by several authors like R E Peinado [15], Steve Ligh [11], M Satyanarayana [17].

In the present work we use theory of regular semigroups for the study of regular rings. In semigroup theory, the structure of regular semigroups has been studied extensively and several approaches have been developed. Notable among them are those developed by Nambooripad [13] and [14], Grillet [5], Hall [6], Meakin [9], etc. In the present work we try to make use of the theory of regular semigroups especially the structure theory using cross connections ([14]) to study regular rings.

Theory of cross connections first appeared in the work of Grillet (cf. [5]) in 1974. He gave a characterisation of a regular semigroup \(S\) using the partially ordered sets \(I = S/\mathcal{R}\) and \(\Lambda = S/\mathcal{L}\) of \(\mathcal{R}\)-classes and \(\mathcal{L}\)-classes called regular partially ordered sets. The relation connecting the two partially ordered sets were given in terms of a pair of maps called cross-connection. This cross connection provided structure theory for fundamental regular semigroups.

Nambooripad [14] gave a generalisation of Grillet's cross-connection to describe the structure of regular semigroups which are not necessarily fundamental. Here the regular partially ordered sets are replaced by normal categories (see Definition 1.4.14). Further certain functors between them de-
scribes the cross connection. In the present work we introduce the concept of additive normal category (cf Definition. 3.1.1) in place of normal category in Nambooripad's theory. The additive normal category introduced here characterises the category of principal left ideals of a regular ring. It has been shown that an additive normal category generates a regular ring(cf. Theorem 4.2.3) and that every additive normal category arises as the category of principal left ideals of a regular ring. More precisely we first show that every additive normal category is a normal category and so generates a regular semigroup. We further introduce an addition on this semigroup so that this becomes a regular ring.

This thesis is divided into five chapters. The the first chapter contains the basic concepts and results used in the sequel. The different sections are based on lattices, regular semigroups, categories and category with subobjects. This chapter also serves to fix the terminological conventions followed in this work.

In the second chapter we describe certain properties of regular rings. Here we note many properties of the set \( L(S) \) of all principal left ideals of a regular ring \( S \). These properties motivated the definition of an additive normal category.

In the third chapter we introduce the concept of an additive normal category. If \( S \) is a regular ring, then the category \( L(S) \) of principal left ideals of \( S \) is an additive normal category and dually \( R(S) \), the category of principal right ideals is also an additive normal category. It is shown that every additive normal category is a normal category. Several properties of additive normal category are discussed here.

In the fourth chapter we construct a regular ring from an additive normal category. The construction follows from that of regular semigroups in [14].
The semigroup of normal cones $T^C$ corresponding to an additive normal category $C$ is equipped with an addition which makes it a regular ring.

The fifth chapter contains special cases including the ring of finite rank operators, regular ring with unity and division rings. The corresponding additive normal categories are characterised here.

For any additive normal category $C$ and for the corresponding regular ring $S$ generated by $C$, it is shown that $C$ is isomorphic to the category $L(S)$ of principal left ideals of $S$. The characterisation of the category $R(S)$ of principal right ideals is carried out for the case when the ring $S$ contains a unity. The category which represents $R(S)$ is generally called the dual of $C$. The dual of $C$ is characterised for the special case when the ring contains unity. The case of division rings provides a simple structure for the additive normal category which we described in Theorem 5.3.3. Direct product of division rings is an interesting class of regular rings. In the last section we describe the construction of a regular ring which is the direct product of two division rings.