CHAPTER 3

THEORETICAL FRAMEWORK OF THE MODELS UNDER STUDY

3.1. Introduction

Having selected our alternative model to Geometric Brownian Motion after a detailed study of various models in the previous chapter, we now outline the theoretical framework of the models under study. In this chapter we discuss the origin and features of the Geometric Brownian Motion model as well as the alternative model based on the Tsallis Distribution, clearly elaborating the mathematics of the respective models.

The theory of option pricing has had a long history, dating back to Louis Bachelier (1900)\(^{218}\) who invented Brownian motion to model options on French government bonds. This work pre-empted Einstein’s independent use of Brownian motion in physics. While this work almost lay in oblivion for a long time, renewed interest in this model again shot up in the 1960's. This was pioneered by Samuelson (1965)\(^{219}\). Considering long-term equity options, he used Geometric Brownian Motion (also known as economic exponential model) to model the random behaviour of the underlying asset (stock). Nobel winning contributions of Fischer Black and Myron Scholes (1973)\(^{220}\) and Robert Merton

\(^{218}\)Bachelier, L (1900), op cit
\(^{219}\)Samuelson, P.A. (1965), op cit
\(^{220}\)Black, F. and M. Scholes, 1973, op cit
contributed to the formulations of the Option pricing model also known as the Black-Scholes-Merton model. This model has a widespread use in finance today. There are several variants of this model which have been developed.

3.2. Stochastic Processes

An important role in the mathematical treatment of financial instruments is played by stochastic processes. The Brownian motion of market prices, also known in financial mathematics as the Wiener-Bachelier process, Feller (1957), can be traced back to the start of 20th Century when a French mathematician, Louis Bachelier (1900) presented his PhD thesis in which he analysed stock market fluctuations.

A stochastic process $X(t)$ is a family of random variables $\{X(t) \mid t \in T\}$, and defined on a given probability space and indexed by parameter $t$ (which varies over an index set $T$) that serves as a time-line. Essentially, the above definition says that mapping $t \rightarrow X(t)$ is a function at time.

3.3. Brownian Motion (Wiener-Bachelier Processes)

The most interesting class of stochastic process used in financial economics is perhaps Brownian motion. The original Brownian motion, the irregular motion of pollen suspended in liquid, was first observed by Scottish botanist Robert Brown.

\[221\] Merton, R.C. 1976, *op cit*
\[223\] Bachelier, L (1901), *op cit*
(whence its name) in 1827, and was studied further by Albert Einstein (1905). The mathematical theory of Brownian motion was first discussed by Bachelier (1900) in his thesis and was further developed by Norbert Wiener (1923). Accordingly, a stochastic process generating Brownian motion is also referred to as a Wiener process or a Wiener- Bachelier process [Feller (1971); Karlin and Taylor (1975)].

Although Brownian motion was first introduced to model the stock markets by Louis Bachelier (1900) in his thesis, the modern application of Brownian motion to financial markets only began in the late 1960’s and 1970’s with the fundamental work done by, among others, Paul Samuelson (1965) and Robert Merton (1971, 1973). This was applied to option pricing in the celebrated paper by Fischer Black and Myron Scholes (1973).

In the next section, we present an introduction to Brownian motion, as an important class of continuous-time stochastic process.

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225 Bachelier, Louis (1900) op cit
227 Feller W (1971), op cit
229 Bachelier, Louis (1900) op cit
230 Samuelson, P.A. (1965), op cit
231 Merton, R.C. 1976, op cit
232 Black, F. and M. Scholes, 1973, op cit
3.4. Generalised Brownian Motion with Drift

In section 3.3 we defined the standard Brownian motion and mentioned that it can be used to model asset returns. However, one problem is that the standard Brownian motion always has zero mean - that is, the growth rate of the returns is always zero. We thus extend the standard Brownian motion to Brownian motion with non-zero mean (drift). The extension models the dynamics of a pollen grain in a fluid drifting at constant velocity, and also prices in a steady market.

A 1-dimensional generalised Wiener process has $X(t)$ determined by a stochastic differential equation of the form:

$$dX(t) = adt + bdZ(t), \quad X(0) = X_0$$  \[3.4.1\]

where $a$ (drift rate) and $b$ (standard deviation) are constants, $Z(t) = \varepsilon \sqrt{t}$ is a Wiener process and $\varepsilon$ is a standard Wiener process with mean zero and variance one. The mean of $dX(t)$ is given by $adt$ and its variance is given by $b^2 dt$. For interval $[0,t]$ the integral form of (3.4.1) is given as:

$$X(t) = X_0 + a \int_0^t ds + b \int_0^t dZ(s)$$  \[3.4.2\]

where $X_0 = X(0)$
The solution of equation (3.4.2) is given as:

\[ X(t) = X_0 + at + bZ(t) \]  \hspace{1cm} [3.4.3]

with \( Z(0) = Z_0 = 0 \) and \( X_0 = X(0) \)

It is noted that the process \( X(t) \) has a linear trend (or drift) given by \( X_0 + at \) and variance given by \( b^2 dt \).

Louis Bachelier supposed that this type of process can model stock prices, but more recent work suggests that the supposition applies rather to relative prices (known as stock returns) in steady markets [Samuelson (1965)\textsuperscript{233}; Black and Scholes (1973)\textsuperscript{234}; Merton (1973)\textsuperscript{235}].

The process constants or parameters \( a \) and \( b \) in (3.4.3) are called, respectively, drift velocity (usually denoted by \( \mu \)) and volatility (usually denoted by \( \sigma \)) in studies of asset price dynamics.

\textsuperscript{233} Samuelson, P.A. (1965), \textit{op cit}
\textsuperscript{234} Black, F. and M. Scholes, 1973, \textit{op cit}
\textsuperscript{235} Merton, R.C. 1976, \textit{op cit}
Typical trajectories of a generalised Brownian motion depicted in Fig.3.1 below. It can be noticed that $X(t)$ can take both positive and negative values. This is one of the drawbacks with the Bachelier’s model since stock prices can not be negative.

Fig.3.1 Typical trajectories of a generalised 1-D Brownian motion with $\text{drift} = 0.5$ and standard deviation = 1 and 95% envelope (limiting curves) i.e. $\pm 1.96\sigma \sqrt{t}$ envelope about the trend line, $X_0 + at$. One such trajectory is shown in bold.

In the following section we describe further types of generalized Brownian motions that are commonly applied to stochastic processes in financial market dynamics.
3.5. Itô Processes

In this section we consider a broad class of stochastic processes that covers many possible dynamical system behaviours. This class is Itô processes, also known as diffusion processes. Itô processes are the main objects of study in stochastic calculus. They generalise Brownian motion by letting parameters $a$ and $b$ become functions of the underlying variable $X(t)$ and time $t$, such that $a = a(X(t), t)$ and $b = b(X(t), t)$ in equation (3.5.1). Thus a 1-dimensional generalised Itô process is given by:

$$dX(t) = a(X(t), t)dt + b(X(t), t)dZ(t)$$  \[3.5.1\]

Here the process parameters are the instantaneous drift rate, $a(X(t), t)$ (which measures the expected rate of change in $X(t)$) and the instantaneous variance, $b(X(t), t)$ (which measures the amount of random diffusion). Both drift and variance are liable to change over time $t$. The expected drift and variance of equation [3.5.1] are $a(X(t), t)dt$ and $b^2(X(t), t)dt$ respectively. The change over the interval $[0, t], X(t)$ can be expressed in the integral form:

$$X(t) = X_0(t) + a \int_0^t (X(s), s)ds + b \int_0^t (X(s), s)dZ(s)$$  \[3.5.2\]
This type of stochastic process is widely used in finance to model derivatives since they are functions of underlying assets and time.

3.6. Geometric Brownian Motion or Samuelson’s Model

A particular class of Itô process commonly used to model stock returns is the so-called geometric Brownian motion (GBM). This type of stochastic process is also referred to as log-normal diffusions”, “geometric Wiener processes or “economic exponential models” (Samuelson 1965)\(^{236}\). In the derivation of the celebrated Black–Scholes-Merton Option Pricing formula, Black and Scholes (1973)\(^{237}\), and Merton (1973)\(^{238}\) assume that the stock return, \(S(t)\), follows a geometric Brownian motion. These motions are Itô processes with linear coefficients \(a = aX(t)\) and \(b = bX(t)\) in equation (3.5.1) and are represented by the stochastic differential equation:

\[
dX(t) = aX(t)dt + bX(t)dZ(t) ; \quad a > 0, \quad b > 0, \quad X(0) = X_0
\]

[3.6.1]

where \(a\) represents the annual mean of returns \(dX(t)/X(t)\) and \(b\) represents the annual volatility of returns. The particular, for interval \([0,t]\), equation [3.6.1] is expressible as an integral form as:

\(^{236}\) Samuelson, P.A. (1965), op cit
\(^{237}\) Black, F. and M. Scholes, 1973, op cit
\(^{238}\) Merton, R.C. 1976, op cit
\[ X(t) = X_0 + a \int_0^t X(s) \, ds + b \int_0^t X(s) \, dZ(s) \]  

where \( X_0 > 0 \)

Equation (1.6) can also be expressed as a law of asset returns:

\[ \frac{dX(t)}{X(t)} = adt + bdZ(t) ; \quad X(0) = X_0 \]  

which is the relative (percentage) change of the process over the infinitesimally short interval \([0, t]\). If \( X(t) \) is the price of a traded asset, then \( \frac{dX(t)}{X(t)} \) is the rate of return on the asset over the next instant and its solution is given by:

\[ S(t) = \log \left( \frac{X(t)}{X_0} \right) = \int_0^t ads + \int_0^t bdZ(s) \]  

Equation (3.6.3) cannot be solved directly since it is not linear in \( X(t) \) and contains parts that involve standard Wiener process, \( Z(t) \). The integral form (3.6.4), however, shows that the logarithmic stock return executes the relatively simple generalised Wiener process now called ‘geometric Brownian motion’. This process, first introduced in finance by Samuelson (1965)\(^ {239} \) has an advantage over the generalised Wiener process introduced by Bachelier in that \( X(t) = X_0 \exp(S(t)) \) can never assume negative values as shown in Fig.3. 2 which depicts trajectories of Geometric Brownian Motion. This characteristic is

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\(^{239}\) Samuelson, P.A. (1965), *op cit*
important when modelling stock prices in the form of geometric Brownian motion, as stock prices never assume negative values, contrary to those generated by Bachelier’s early generalised Brownian Motion.

![Geometric or Relative Economic Brownian motion paths](image)

**Fig3.2** Typical trajectories of a 1-D geometric Brownian motion, with $X_0 = 1$, drift = 0.5, standard deviation = 1 and envelope (limiting curve), $\pm 1.96\sqrt{\left(e^{0.5}\left(e^t - 1\right)\right)}$ about the expected mean, $X_i e^{0.5t}$ (middle dotted curve).

In their famous Option Pricing theory, (Black and Scholes 1973)$^{240}$ used this Geometric Brownian motion as their model to describe the price movement of the underlying asset. Since our work concentrates on the Geometric Brownian

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$^{240}$ B Black, F. and M. Scholes, 1973, *op cit*
Motion assumption and not the option pricing theory, we are not discussing the
derivation of the same for which one may refer to Wilmot (2000)\textsuperscript{241} and Paul & Baschangel (1999)\textsuperscript{242}.

3.7. Tsallis Distribution

We now discuss the theoretical framework of the Tsallis Distribution, based on
which an alternative option Pricing Model was developed by Borland (2002)\textsuperscript{243}. In
this case also we discuss the properties of Tsallis Distribution and how the Geometric Brownian Motion is modified using the same distribution. For the entire derivation of the option pricing theory one may refer to Borland (2002)\textsuperscript{244}.

3.7.1 Tsallis statistics

3.7.1.1 Pure Tsallis Statistics

Tsallis derived the probability of finding the system at a given point in phase
space $G = (r_N, p_N)$ by extremizing Gibbs Shannon entropy $S_q$ subject to the
constraints

$$\int p_q (\Gamma) d\Gamma = 1 \quad \text{and} \quad \int [p_q (\Gamma)]H(\Gamma) d\Gamma = E_q$$  \[3.7.1.1.1\]

where $H(\Gamma)$ is the system Hamiltonian for $N$ distinguishable particles in $d$
dimensions. The result is

\textsuperscript{242} Paul, Wolfgang, and Jorg Baschangel. \textit{Stochastic Process: from Physics to Finance}. Springer-Verlag, 1999
\textsuperscript{243} Borland, L., A(2002) , op cit
\textsuperscript{244} Ibid
\[ p_q(\Gamma) = \frac{1}{Z_q h^{dN}} \left[ 1 - (q-1)\beta H(\Gamma) \right]^{1\over q-1} \]  \[ \text{[3.7.1.1.2]} \]

where

\[ Z_q = \frac{1}{h^{dN}} \int \left[ 1 - (q-1)\beta H(\Gamma) \right]^{1\over q-1} \]  \[ \text{[3.7.1.1.3]} \]

plays the role of the canonical ensemble partition function. Using the identity

\[ \lim_{n \to 0} (1 + an)^{1/a} = \exp(a) \Gamma, \]  

in the limit that \( q = 1 \Gamma \), the standard probability distribution of classical Gibbs-Boltzmann statistical mechanics

\[ p(\Gamma) = \frac{1}{Z h^{dN}} \exp(-\beta H(\Gamma)) \]  \[ \text{[3.7.1.1.4]} \]

is recovered. However, there is a problem. For certain values of \( q \) and a harmonic potential \( \Gamma \), the distribution \( p_q[\Gamma] \) has infinite variance and higher moments.

### 3.7.1.2 Introduction of the "q-expectation" value

To address this problem, Tsallis derived the statistical probability by extremizing Gibbs Shannon entropy \( S_q \) subject to the modified constraints

\[ \int p_q(\Gamma)d\Gamma = 1 \quad \text{and} \quad \int \left[ p_q(\Gamma) \right]^y H(\Gamma)d\Gamma = E_q \]  \[ \text{[3.7.1.1.5]} \]

where the averaged energy is defined using a "q-expectation" value. The result is

\[ p_q(\Gamma) = \frac{1}{Z_q h^{dN}} \left[ 1 - (1-q)\beta H(\Gamma) \right]^{1\over 1-q} \]  \[ \text{[3.7.1.1.6]} \]
with the "partition function"

\[ Z_q = \frac{1}{h^{dN}} \int [1 - (1 - q)\beta H(\Gamma)]^{\frac{1}{1-q}} \]  

[3.7.1.1.7]

In the limit that \( q = 1 \) the classical canonical density distribution is recovered.

To be consistent, the \( q \)-expectation value is also used to compute the average of an observable \( A \)

\[ \langle A \rangle_{q}^{NN} = \frac{1}{\left( Z_q h^{N} \right)^q} \int A(\Gamma)[1 - (1 - q)\beta H(\Gamma)]^{\frac{q}{1-q}} d\Gamma \]  

[3.7.1.1.8]

However, since the averaging operator is not normalized, in general \( \langle 1 \rangle_{q}^{NN} \neq 1 \)

for \( q \neq 1 \). Moreover, it is necessary to compute \( Z_q \) to determine the average. To avoid this difficulty, a different generalization of the canonical ensemble average was proposed

\[ \langle A \rangle_{q} = \frac{\int A(\Gamma)[1 - (1 - q)\beta H(\Gamma)]^{\frac{q}{1-q}} d\Gamma}{\int [1 - (1 - q)\beta H(\Gamma)]^{\frac{q}{1-q}} d\Gamma} \]  

[3.7.1.1.9]

It is obviously normalized and convenient to apply.

3.7.1.3 The \( q \)-Gaussian function from Tsallis statistics
We briefly review the definition and properties of the q-Gaussian function. First, we introduce the q-logarithm and its inverse, the q-exponential (Yamano 2002)\(^{245}\), as

\[
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q}
\]

[3.7.1.2.1]

And

\[
e^r_q \equiv \begin{cases} [1+(1-q)r]^{\frac{1}{1-r}}, & 1+1(1-q)r \geq 0 \\ 0, & \text{else} \end{cases}
\]

[3.7.1.2.2]

These functions reduce to the usual logarithm and exponential functions when \(q = 1\). The q-Gaussian density is defined for \(-\infty < q < 3\) as

\[
p(x; \overline{\mu}_q, \overline{\sigma}_q) = A_q \sqrt{B_q} \left[ 1 + (q-1)B_q \left( x - \overline{\mu}_q \right)^{q-1} \right]^{\frac{1}{q-1}}
\]

\[
= A_q \sqrt{B_q} e^r_q \left( x - \overline{\mu}_q \right)^q
\]

[3.7.1.2.3]

where the parameters \(\overline{\mu}_q, \overline{\sigma}_q, A_q, B_q\) are defined as follows. First, the q-mean \(\overline{\mu}_q\) is defined analogously to the usual mean, except using the so-called q-expectation value (based on the escort distribution), as follows:

\[
\mu_q \equiv \langle x \rangle_q \equiv \frac{\int x[p(x)]^q dx}{\int [p(x)]^q dx}
\]

[3.7.1.2.4]

Similarly, the q-variance, \(\sigma_q^2\) is defined analogously to the usual second order central moment, as

\[
(y) = 2e^y A_q \sqrt{B_q} \left[ 1 - \frac{1}{2}(1 - q)e^{2y} \right]^{\frac{1}{2}\left(\frac{1+q}{1-q}\right)}
\]  

[3.7.1.2.5]

When \(q = 1\), these expressions reduce to the usual mean and variance. The normalization factor is given by

\[
A_q = \begin{cases} 
\frac{\Gamma\left[\frac{3+q}{1+q}\right]^{\frac{1-q}{1+q}}}{\sqrt{\pi}} & q < 1 \\
\frac{\Gamma\left[\frac{q}{1-q}\right]^{\frac{q-1}{2}}}{\sqrt{\pi}} & q = 1 \\
\frac{\Gamma\left[\frac{q+3}{2\lceil q \rceil}\right]}{\sqrt{\pi}} & 1 < q < 3
\end{cases}
\]

[3.7.1.2.6]

Finally, the width of the distribution is characterized by

\[
B_q = \left[ (3 - q)\sigma_q^2 \right]^{-1}, \quad q \in (-\infty, 3)
\]

[3.7.1.2.7]

Denote a general q-Gaussian random variable \(X\) with q-mean \(\overline{\mu}_q\) and q-variance \(\overline{\sigma}_q^2\) as

\[
X \sim N_q\left(\overline{\mu}_q, \overline{\sigma}_q^2\right)
\]

and call the special case of \(\overline{\mu}_q \equiv 0\) and \(\overline{\sigma}_q^2 \equiv 1\) a standard q-Gaussian, \(Z \sim N_q(0, 1)\). The density of the standard q-Gaussian distribution may then be written as

\[
p(x; \overline{\mu}_q = 0, \overline{\sigma}_q = 1)
\]

\[
= A_q \sqrt{B_q} e^{-\frac{B_q x^2}{2}} = \frac{A_q}{\sqrt{3 - q}} \left[ 1 + \frac{q - 1}{3 - q} x^2 \right]^{\frac{1}{2-q}}
\]

[3.7.1.2.8]
The q-Gaussian distribution reproduces the usual Gaussian distribution then $q = 1$. Its density has compact support for $q < 1$, and decays asymptotically as a power law for $1 < q < 3$. For $3 \geq q$, the form given in [3.7.1.2.8] is not normalisable. The usual variance (second order moment) is finite for $q < 5/3$, and, for the standard q-Gaussian, is given by $\sigma^2 = (3-q)/(5-3q)$. The usual variance of the q-Gaussian diverges for $5/3 \leq q < 3$, however the q-variance remains finite for the full range $-\infty < q < 3$, equal to unity for the standard q-Gaussian.

3.7.2 Borland Model

Borland (2002)$^{246}$ introduce a new model of stock return fluctuations, which derives directly from stochastic processes introduced within the Tsallis (1987) framework. In this setting, they assume that the log returns $Y(t) = \ln S(\tau + t)/\ln S(\tau)$ follow the process

$$dY = \mu dt + \sigma d\Omega$$ \hspace{1cm} [3.7.2.1]

across timescales $t$, where the driving noise is modelled as being drawn from a non-Gaussian distribution. To do this, it is assumed that the noise follows the statistical feedback process [Tsallis (1987)$^{247}$]

$$d\Omega = P(\Omega)^{1-q} d\omega$$ \hspace{1cm} [3.7.2.2]

Here $\omega$ is a zero-mean Gaussian noise process as defined above. And the q-statistics is as per section 3.7.1.2.

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$^{246}$ Borland, L., A (2002), *op cit*

$^{247}$ Tsallis C., (1988), *op cit*
3.8. Discrete version of Geometric Brownian Motion and Borland Model

For our purpose, we do a Monte Carlo simulation of first the classical Geometric Brownian Motion. We use the discrete time version of the model:

\[
\frac{\delta S}{S} = \mu \delta t + \sigma \varepsilon \sqrt{\delta t}
\]  

[3.8.1]

Or

\[
\delta S = \mu S \delta t + \sigma S \varepsilon \sqrt{\delta t}
\]  

[3.8.2]

The variable \( \delta S \) is the change in the stock price \( S \) in a small time interval \( \delta t \) (in our case 1 day), and \( \varepsilon \) is a random drawing from a standardized normal distribution. The parameter \( \mu \) is the expected rate of return per annum and \( \sigma \) is the volatility of the stock expressed annually.

Then we do a Monte Carlo simulation of the Borland proposed model.

\[
\delta S = \mu S \delta t + \sigma S \Omega \sqrt{\delta t}
\]  

[3.8.3]

where \( \Omega \) is a random drawing from q-Gaussian Tsallis distribution, all other parameters being same as equation 3.8.2

We have outlined the theoretical underpinnings of the models under study. In the next chapter, we now elaborate the empirical test methodologies that we have used to capture the dynamics of the model based time series with the original data set from the Indian Market.