CHAPTER 3

On Jordan $^*$-derivations in Rings with Involution
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3.1 Introduction

In the year 1957, Herstein [121] introduced the notion of a function which he called Jordan derivation and defined as: an additive mapping $d : R \rightarrow R$ is said to be a Jordan derivation of $R$ if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. Every derivation is obviously a Jordan derivation but the converse need not be true in general (see [41, Example 3.2.1]). A classical result due to Herstein [121] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein’s theorem can be found in [79]. Cusack [92] generalized Herstein’s result to 2-torsion free semiprime rings (see [65] for an alternative proof). This famous result was further generalized by many authors in various directions (viz.; [33], [46], [47], [66], [82], [110], where further references can be found).

Let $R$ be a ring with involution. According to Brešar and Vukman [80], an additive mapping $d : R \rightarrow R$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where $a$ is a fixed element in $R$, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. One might expect that any Jordan $*$-derivation on a 2-torsion free semiprime $*$-ring is a $*$-derivation, but this is not the case. It is easy to prove that there exist nonzero $*$-derivations on noncommutative prime $*$-rings (see [80] for details). The study of Jordan $*$-derivations has been motivated by the problem of the representability of quadratic forms by bilinear forms (for results concerning this problem we refer to [143], [187–189], [196] and references therein).
It turns out that the question whether each quadratic form can be represented by some bilinear form is intimately connected with the question whether every Jordan \(\ast\)-derivation is inner, as shown by Šemrl [188]. In [80], Bršebar and Vukman studied some algebraic properties of Jordan \(\ast\)-derivations. Following [9] and [110], an additive mapping \(F : R \to R\) is called a generalized \(\ast\)-derivation if there exists a \(\ast\)-derivation \(d : R \to R\) such that \(F(xy) = F(x)y^\ast + xd(y)\) holds for all \(x, y \in R\). An additive mapping \(F : R \to R\) is called a generalized Jordan \(\ast\)-derivation if there exists a Jordan \(\ast\)-derivation \(d : R \to R\) such that \(F(x^3) = F(x)x^\ast + xd(x)\) holds for all \(x \in R\). An additive mapping \(d\) of a \(\ast\)-ring \(R\) into itself is called a Jordan triple \(\ast\)-derivation if \(d(xyz) = d(x)y^\ast x^\ast + xd(y)x^\ast + xyd(x)\) is fulfilled for all \(x, y \in R\). One can easily prove that every Jordan \(\ast\)-derivation on a 2-torsion free \(\ast\)-ring is a Jordan triple \(\ast\)-derivation. However, the converse of this statement is not true in general (see [15, Example 2.4] for \(\alpha = \beta = I_R\), the identity mapping on \(R\)). In [204], Vukman showed that the converse holds if \(R\) is 6-torsion free semiprime \(\ast\)-ring. Recently, Fošner and Ilišević [110] proved that every Jordan triple \(\ast\)-derivation on a 2-torsion free semiprime \(\ast\)-ring must be a Jordan \(\ast\)-derivation.

An additive mapping \(d : R \to R\) is called \((\theta, \phi)^\ast\)-derivation (resp. Jordan \((\theta, \phi)^\ast\)-derivation) if \(d(xy) = d(x)\theta(y^\ast) + \phi(x)d(y)\) (resp. \(d(x^3) = d(x)\theta(x^\ast) + \phi(x)d(x)\)) for all \(x, y \in R\), where \(\theta, \phi\) are endomorphisms of \(R\). Very recently, Ali and Dar [10] proved the following result: Let \(R\) be a 2-torsion free semiprime \(\ast\)-ring and let \(d : R \to R\) be an additive mapping satisfying the relation \(d(xyz) = d(xy)x^\ast + yxd(x)\) for all \(x, y \in R\). In this case, \(d\) is a \(\ast\)-derivation. Motivated by the above mentioned result, in Section 3.2, we prove that on a 2-torsion free semiprime ring with involution, any additive mapping \(d : R \to R\) satisfying the relation \(d(xyz) = d(xy)\theta(x^\ast) + \phi(xy)d(x)\) for all \(x, y \in R\) must be a \((\theta, \phi)^\ast\)-derivation.

In Section 3.3, we not only introduce the notion of symmetric Jordan \((\theta, \phi)^\ast\)-biderivations and symmetric Jordan triple \((\theta, \phi)^\ast\)-biderivations, but also establish a set of conditions under which the two concepts are equivalent.

Section 3.4 is devoted to the study of additive mapping satisfying some \(\ast\)-identities on \(\ast\)-closed Lie ideal of prime \(\ast\)-rings involving \((\theta, \phi)\)-derivations. In fact, we prove the following result: Let \(R\) be a prime ring with involution such that \(\text{char}(R) \neq 2\) and \(U\) be a noncommutative \(\ast\)-closed Lie ideal of \(R\) such that \(u^2 \in U\) for all \(u \in U\). Suppose that \(\theta, \phi\) are endomorphisms of \(R\) such that \(\theta\) is an automorphism of \(R\).
If there exists an additive mapping $F : R \rightarrow R$ associated with a nonzero $(\theta, \phi)$-derivation $d$ of $R$ such that $F(uv^*) = F(u)\theta(v^*) + \phi(u)d(v^*)$ holds for all $u \in U$, then $F(uv) = F(u)\theta(v) + \phi(u)d(v)$ for all $u, v \in U$. Moreover, some related results have also been obtained.

3.2 Jordan $(\theta, \phi)^*$-derivations on semiprime $*$-rings

Let $R$ be a ring with involution, and let $\theta, \phi$ be endomorphisms of $R$. According to Ali and Fošner [15], an additive mapping $d : R \rightarrow R$ is called a $(\theta, \phi)^*$-derivation (resp. Jordan $(\theta, \phi)^*$-derivation) if $d(xy) = d(x)\theta(y^*) + \phi(x)d(y)$ (resp. $d(x^2) = d(x)\theta(x^*) + \phi(x)d(x)$) holds for all $x, y \in R$. An additive mapping $d : R \rightarrow R$ is called a Jordan triple $(\theta, \phi)^*$-derivation if $d(xy^x) = d(x)\theta(y^x^*) + \phi(x)d(y^x) + \phi(xy)d(x)$ holds for all $x, y \in R$. Clearly, every $(\theta, \phi)^*$-derivation on a ring with involution is a Jordan triple $(\theta, \phi)^*$-derivation but the converse is in general not true (see [15, Example 2.4]). Recently, Ali and Fošner [15] proved that on a 6-torsion free semiprime $*$-ring $R$, every Jordan triple $(\theta, \phi)^*$-derivation is a Jordan $(\theta, \phi)^*$-derivation. Further in [8], Ali improved this result by removing 3-torsion free restriction. Motivated by the above study very recently, Ali and Dar [10] proved the following result:

**Theorem 3.2.1.** Let $R$ be a 2-torsion free semiprime $*$-ring, and let $d : R \rightarrow R$ be an additive mapping satisfying the relation $d(xy^x) = d(xy)x^* + xyd(x^*)$ for all $x, y \in R$. In this case, $d$ is a $*$-derivation.

In the present section, we generalize the above mentioned result as follows:

**Theorem 3.2.2.** Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta, \phi$ are endomorphisms of $R$ such that $\phi$ is an automorphism of $R$ and commutes with $*$. If there exists an additive mapping $d : R \rightarrow R$ such that

$$d(xy^x) = d(xy)\theta(x^*) + \phi(xy)d(x)$$

holds for all pairs $x, y \in R$. In this case, $d$ is a $(\theta, \phi)^*$-derivation.

In order to develop the proof of the above theorem, we need the following lemma:

**Lemma 3.2.1.** Let $R$ be a semiprime $*$-ring, and let $f : R \rightarrow R$ be an additive mapping and $\theta$ be an automorphism of $R$ such that $\theta$ commutes with $*$. Suppose that either $f(x)\theta(x^*) = 0$ or $\theta(x^*)f(x) = 0$ holds for all $x \in R$. In both the cases, $f = 0$. 45
Proof. Assume that \( f(x)\theta(x) = 0 \), for all \( x \in R \). Replacing \( x \) by \( x + y \), we have
\[
f(x)\theta(y) + f(y)\theta(x) = 0 \tag{3.2.1}
\]
for all \( x, y \in R \). Substituting \( y^2 \) for \( y \) in (3.2.1), we get
\[
f(x)\theta((y^2)^2) + f(y^2)\theta(x) = 0 \tag{3.2.2}
\]
for all \( x, y \in R \). Right multiplying (3.2.1) by \( \theta(y^*) \), we obtain
\[
f(x)\theta((y*)^2) + f(y^2)\theta(x)\theta(y^*) = 0 \tag{3.2.3}
\]
for all \( x, y \in R \). Comparing (3.2.2) and (3.2.3), we get
\[
f(y^2)\theta(x) - f(y)\theta(x)\theta(y^*) = 0 \tag{3.2.4}
\]
for all \( x, y \in R \). Substituting \( \theta^{-1}(f(y^*))x \) for \( x \) in (3.2.4) and using the fact that \( \theta \) commutes with \( * \), we obtain
\[
f(y^2)\theta(x^*)f(y) - f(y)\theta(x^*)f(y)\theta(y^*) = 0
\]
for all \( x, y \in R \). In view of our hypothesis, we conclude that
\[
f(y^2)\theta(x^*)f(y) = 0 \tag{3.2.5}
\]
for all \( x, y \in R \). Right multiplying (3.2.4) by \( f(y) \) and making use of (3.2.5), we obtain
\[
f(y)\theta(x^*)\theta(y^*)f(y) = 0
\]
for all \( x, y \in R \). Right multiplying above expression by \( \theta(y^*) \), we find that \( \theta(y^*)f(y)\theta(x^*) \theta(y^*)f(y) = 0 \) for all \( x, y \in R \). This implies that \( \theta(y^*)f(y)R\theta(y^*)f(y) = 0 \) for all \( y \in R \). The semiprimeness of \( R \) yields that
\[
\theta(y^*)f(y) = 0 \tag{3.2.6}
\]
for all \( y \in R \). Right multiplying (3.2.1) by \( f(x) \) gives because of (3.2.6) that \( f(x)\theta(y^*) = 0 \)
$f(x) = 0$ for all $x, y \in R$. Since $\theta$ is an automorphism of $R$, the last relation can be written as $f(x)Rf(x) = (0)$ for all $x \in R$. Thus by the semiprimeness of $R$, we conclude that $f(x) = 0$ for all $x \in R$. □

**Proof of Theorem 3.2.2.** In view of the hypothesis, we have

$$d(xy) = d(xy)\theta(x^*) + \phi(xy)d(x)$$  \hspace{1cm} (3.2.7)

for all $x, y \in R$. Replacing $z$ by $x + z$ in (3.2.7), we get

$$d(xyz) + d(xyz + zyx) + d(zyz)$$
$$= d(xy)\theta(x^*) + \phi(xy)d(x) + d(xy)\theta(x^*) + d(zy)\theta(x^*)$$
$$+ \phi(xy)d(x) + \phi(xy)d(x) + d(xy)\theta(x^*) + \phi(zy)d(x)$$

for all $x, y, z \in R$. In view of (3.2.7), we find that

$$d(xyz + zyx) = d(xy)\theta(x^*) + d(zy)\theta(x^*) + \phi(xy)d(x) + \phi(zy)d(x)$$ \hspace{1cm} (3.2.8)

for all $x, y, z \in R$. Taking $x^2$ for $z$, we get

$$d(xyx^2 + x^2yx) = d(xy)\theta((x^2)^*) + d(x^2y)\theta(x^*)$$
$$+ \phi(xy)d(x^2) + \phi(x^2y)d(x)$$ \hspace{1cm} (3.2.9)

for all $x, y \in R$. Substituting $xy + yx$ for $y$ in (3.2.7), we get

$$d(xy^2 + x^2y) = d(xy^2)\theta(x^*) + d(xy)\theta((x^2)^*)$$
$$+ \phi(xy)d(x^2) + \phi(x^2y)d(x)$$
$$+ \phi(xy)d(x)$$ \hspace{1cm} (3.2.10)

for all $x, y \in R$. Comparing (3.2.9) and (3.2.10), we get

$$\phi(xy)A(x) = 0,$$ \hspace{1cm} (3.2.11)

for all $x, y \in R$, where $A(x) = d(x^2) - d(x)\theta(x^*) - \phi(x)d(x)$ for all $x \in R$. Right multiplying (3.2.11) by $\phi(x)$ and left multiplying by $A(x)$, we get $A(x)\phi(xy)A(x)\phi(x) = 47$
0 for all \( x, y \in R \). Since \( \phi \) is an automorphism of \( R \), the above expression can be written as \( A(x)\phi(x)RA(x)\phi(x) = 0 \) for all \( x \in R \). The semiprimeness of \( R \) yields that

\[
A(x)\phi(x) = 0 \tag{3.2.12}
\]

for all \( x \in R \). Taking \( \phi^{-1}(A(x))yx \) for \( y \) in (3.2.11), we get \( \phi(x)A(x)\phi(yx)A(x) = 0 \) for all \( x, y \in R \). That is, \( \phi(x)A(x)R\phi(x)A(x) = 0 \) for all \( x \in R \). Semiprimeness of \( R \) forces that

\[
\phi(x)A(x) = 0 \tag{3.2.13}
\]

for all \( x \in R \). Replacing \( x \) by \( x + y \) in (3.2.12), we get

\[
B(x, y)\phi(x) + A(x)\phi(y) + B(x, y)\phi(y) + A(y)\phi(x) = 0
\]

for all \( x, y \in R \), where \( B(x, y) = d(xy) - d(x)d(y) - d(y)d(x) - d(y)d(x) - d(x)d(y) - d(x)d(y) \). Taking \(-x\) for \( x \), we obtain

\[
B(x, y)\phi(x) + A(x)\phi(y) - B(x, y)\phi(y) - A(y)\phi(x) = 0
\]

for all \( x, y \in R \). Combining the above two expressions and using 2-torsion freeness of \( R \), we get

\[
B(x, y)\phi(x) + A(x)\phi(y) = 0
\]

for all \( x, y \in R \). Right multiplying in above expression by \( A(x) \) and making use of (3.2.13), we obtain \( A(x)\phi(y)A(x) = 0 \) for all \( x, y \in R \). This implies that \( A(x)RA(x) = 0 \) for all \( x \in R \). Semiprimeness of \( R \) forces that \( A(x) = 0 \) for all \( x \in R \). That is, \( d(x^2) = d(x)d(x^*) + \phi(x)d(x) \) for all \( x \in R \). Which shows that \( d \) is a Jordan \((\theta, \phi)^\ast\)-derivation of \( R \). In view of Lemma 1.3.1, we have

\[
d(xy) = d(x)\theta(y^*x^*) + \phi(x)d(y)\theta(x^*) + \phi(xy)d(x) \tag{3.2.14}
\]

for all \( x, y \in R \). Combining (3.2.7) and (3.2.14), we get

\[
(d(xy) - d(x)\theta(y^*) - \phi(x)d(y))\phi(x^*) = 0
\]
for all \( x, y \in R \). For any fixed \( y \) we have an additive mapping \( x \mapsto d(xy) - d(x)\theta(y^*) - \phi(x)d(y) \). Application of Lemma 3.2.1 yields that \( d(xy) - d(x)\theta(y^*) - \phi(x)d(y) = 0 \) for all \( x, y \in R \). That is, \( d \) is a \((\theta, \phi)^*\)-derivation. The proof of the theorem is completed.

**Theorem 3.2.3.** Let \( R \) be a 2-torsion free semiprime \(*\)-ring. Suppose that \( \theta, \phi \) are endomorphisms of \( R \) such that \( \theta \) is an automorphism of \( R \) and commutes with \(*\). If there exists an additive mapping \( d : R \rightarrow R \) such that

\[
d(xy) = d(x)\theta(y^*) + \phi(x)d(y^*)
\]

holds for all pairs \( x, y \in R \). In this case, \( d \) is a \((\theta, \phi)^*\)-derivation.

**Proof.** Using the similar approach as we have used in the proof of Theorem 3.2.2, we get the required result. \( \square \)

As an immediate consequence of the above theorem, we have the following result:

**Corollary 3.2.1.** [10, Theorem 4.3] Let \( R \) be a 2-torsion free semiprime \(*\)-ring and let \( d : R \rightarrow R \) be an additive mapping satisfying the relation \( d(xy) = d(x)y^*x^* + xd(yz) \) for all \( x, y \in R \). In this case, \( d \) is a \(*\)-derivation.

### 3.3 Jordan \((\theta, \phi)^*\)-biderivations on prime \(*\)-rings

A symmetric biadditive map \( B : R \times R \rightarrow R \) is called a symmetric biderivation if \( B(xy, z) = B(x, z)y + xB(y, z) \) is fulfilled for all \( x, y, z \in R \). The concept of a symmetric biderivation was introduced by Maksai in [169] (see also [170], where an example can be found). A symmetric biadditive map \( B : R \times R \rightarrow R \) is said to be a symmetric Jordan biderivation if \( B(x^2, z) = B(x, z)x + xB(x, z) \) holds for all \( x, z \in R \). Following [18], a symmetric biadditive map \( B : R \times R \rightarrow R \) is called a symmetric \(*\)-biderivation if \( B(xy, z) = B(x, z)y^* + xB(y, z) \) holds for all \( x, y, z \in R \), where \( R \) is a ring with involution. In [10], Ali and Dar introduced the concept of symmetric Jordan \(*\)-biderivation and symmetric Jordan triple \(*\)-biderivation as follows:

a symmetric biadditive map \( D : R \times R \rightarrow R \) is said to be a symmetric Jordan \(*\)-biderivation if \( D(x^2, z) = D(x, z)x^* + xD(x, z) \) holds for all \( x, z \in R \). A symmetric biadditive map \( D : R \times R \rightarrow R \) is called a symmetric Jordan triple \(*\)-biderivation if \( D(xy, z) = D(x, z)y^*x^* + xD(y, z)x^* + xyD(x, z) \) holds for all \( x, y, z \in R \). Motivated by
the definition of Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation we introduce the concept of symmetric Jordan $(\theta, \phi)^*$-biderivation and symmetric Jordan triple $(\theta, \phi)^*$-biderivation as follows: a symmetric biadditive map $D : R \times R \to R$ is said to be a symmetric Jordan $(\theta, \phi)^*$-biderivation if $D(x^2, z) = D(x, z)\theta(x^*) + \phi(x)D(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $D : R \times R \to R$ is called a symmetric Jordan triple $(\theta, \phi)^*$-biderivation if $D(xy, z) = D(x, z)\theta(y^*x^*) + \phi(x)D(y, z)\theta(x^*) + \phi(xy)D(x, z)$ holds for all $x, y, z \in R$. Note that a Jordan triple $(I_R, I_R)^*$-biderivation is just a Jordan triple $*$-biderivation, where $I_R$ is the identity map on $R$. Clearly, this notion includes the notion of Jordan triple $*$-biderivation when $\theta = \phi = I_R$, where $I_R$ is the identity map on $R$. It is obvious to see that every symmetric Jordan $(\theta, \phi)^*$-biderivation on a 2-torsion free ring with involution is a symmetric Jordan triple $(\theta, \phi)^*$-biderivation. But the converse need not be true in general. In the present section, our aim is to establish a set of conditions under which every symmetric Jordan triple $(\theta, \phi)^*$-biderivation on a ring with involution is a symmetric Jordan triple $(\theta, \phi)^*$-biderivation. More precisely, we prove that on a 2-torsion free prime ring with involution, every symmetric Jordan triple $(\theta, \phi)^*$-biderivation is a symmetric Jordan $(\theta, \phi)^*$-biderivation.

We begin our discussion with the following lemma:

Lemma 3.3.1. Let $R$ be a prime ring with involution and $\theta, \phi$ be automorphisms of $R$. For $a \in R$, if $\theta(x)a\phi(x^*) = 0$ for all $x \in R$, then $a = 0$.

Proof. We have

$$\theta(x)a\phi(x^*) = 0 \text{ for all } x \in R. \quad (3.3.1)$$

Replacing $x$ by $x^* + y$, we get

$$\theta(y)a\phi(x) + \theta(x^*)a\phi(y^*) = 0 \text{ for all } x, y \in R. \quad (3.3.2)$$

This can be further written as

$$\theta(y)a\phi(x) = -\theta(x^*)a\phi(y^*) \text{ for all } x, y \in R. \quad (3.3.3)$$

Consider for any $x, z \in R$

$$a\theta(x)a\theta(z)a\phi(x)a = a(\theta(x)a\theta(z))a\phi(x)a$$
This implies that
\[ a\theta(x)aRa\phi(x)a = (0) \text{ for all } x \in R. \]

Primeness of \( R \) forces that either \( a\theta(x)a = 0 \) or \( a\phi(x)a = 0 \) for all \( x \in R \). Since \( \theta \) and \( \phi \) are automorphisms of \( R \), so we are force to conclude that \( aRa=(0) \) and hence \( a = 0 \). This proves the lemma.

**Lemma 3.3.2.** Let \( R \) be a 2-torsion free ring with involution and \( \theta, \phi \) be endomorphisms of \( R \). If \( d : R \times R \to R \) is a symmetric Jordan \((\theta, \phi)^*\)-biderivation of \( R \), then the following hold:

(i) \[ d(xy+yx, z) = d(x, z)\theta(y^*) + d(y, z)\theta(x^*) + \phi(x)d(y, z) + \phi(y)d(x, z) \text{ for all } x, y, z \in R; \]

(ii) \[ d(xy, z) = d(x, z)\theta(y^*x^*) + \phi(x)d(y, z) + \phi(xy)d(x, z) \text{ for all } x, y, z \in R; \]

(iii) \[ d(xy + tyx, z) = d(x, z)\theta(y^*t^*) + \phi(x)d(y, z)\theta(t^*) + \phi(xy)d(t, z) + d(t, z)\theta(y^*x^*) + \phi(t)d(y, z)\theta(x^*) + \phi(ty)d(x, z) \text{ for all } t, x, y, z \in R. \]

**Proof.** (i) We are given that \( d : R \times R \to R \) is a symmetric Jordan \((\theta, \phi)^*\)-biderivation of \( R \) i.e.,

\[ d(x^2, z) = d(x, z)\theta(x^*) + \phi(x)d(x, z) \]

for all \( x, z \in R \). Replacing \( x \) by \( x + y \) in above expression, we obtain

\[ d(xy + yx, z) = d((x + y)^2, z) - d(x^2, z) - d(y^2, z) \]

\[ = d(x^2, z) + d(xy + yx, z) + d(y^2, z) - d(x^2, z) - d(y^2, z) \]

\[ = d(xy + yx, z) = d(xy, z) + d(yx, z) \]

\[ = d(x, z)\theta(y^*) + d(y, z)\theta(x^*) + \phi(x)d(y, z) + \phi(y)d(x, z). \]

for all \( x, y, z \in R. \]
(ii) Replacing $y$ by $xy + yz$ in (i), we get

$$d((xy + yz)x, z) + (xy + yz)x, z) = d((xy + yz)x, z) + d((xy + yz)x, z)$$

$$+ \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

$$= d((xy + yz)x, z) + d((xy + yz)x, z) + d(x, z)\theta(\theta(x^*y^* + y^*x^*))$$

$$+ d(x, z)\theta(y^*x^*) + \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

$$+ \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

$$= d((xy + yz)x, z) + d((xy + yz)x, z) + d(x, z)\theta(\theta(x^*y^*))$$

$$+ d(y, z)\theta((x^*)^2 + \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

$$+ \phi(x)d(x, z)\theta(y^*) + \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

$$+ \phi(x)d((xy + yz)x, z) + \phi(yz)x, z)$$

for all $x, y, z \in R$. On the other hand, we have

$$d((xy + yz)x, z) = d(x^2y + yx^2, z) + 2d((xy + yz)x, z)$$

$$= d(x^2y, z) + d(yx^2, z) + 2d((xy + yz)x, z)$$

$$= d((xy + yz)x, z) + d((xy + yz)x, z) + d(y, z)\theta((x^*)^2)$$

$$+ \phi(x^2)d(y, z) + \phi(yz)x, z)\theta(y^*) + \phi(yz)x, z)$$

$$+ 2d((xy + yz)x, z)$$

for all $x, y, z \in R$. Comparing (3.3.7) and (3.3.8) and using the fact that $R$ is 2-torsion free, we get the required result.

(iii) Putting $x + t$ instead of $x$ in (ii), we get

$$d((x + t)y(x + t), z) = d((x + t)y(x + t), z)\theta(y^*) + \phi(x + t)d(y, z)\theta(x^* + t^*)$$

$$+ \phi(x + t)\phi(y)d(x + t, z)$$

$$= d((x + t)y(x + t), z)\theta(y^*x^*) + d((x + t)y(x + t), z)\theta(y^*x^*) + d(t, z)\theta(y^*x^*)$$

$$+ \phi(x)d(y, z)\theta(x^*) + \phi(x)d(y, z)\theta(t^*) + \phi(t)d(y, z)\theta(x^*) + \phi(t)d(y, z)\theta(t^*)$$

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\[ d(x + t)y(x + t), z) \]
\[ = d(xyz, z) + d(tyt, z) + d(xyt + tyx, z) \]
\[ = d(x, z)\theta(y^*x^*) + \phi(x)d(y, z)\theta(x^*) + \phi(xy)d(x, z) \]
\[ + d(t, z)\theta(y^*t^*) + \phi(t)d(y, z)\theta(t^*) + \phi(ty)d(t, z) + d(xyt + tyx, z) \]

for all \( t, x, y, z \in R \). Comparing so obtained relations we get the desired result. \( \square \)

We are now ready to prove the main result of the present section.

**Theorem 3.3.1.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \) and \( \theta, \phi \) be automorphisms of \( R \). Then every symmetric Jordan triple \( (\theta, \phi)^* \)-biderivation \( d : R \times R \rightarrow R \) is a symmetric Jordan \( (\theta, \phi)^* \)-biderivation.

**Proof.** By the given assumption, we have

\[ d(xyx, z) = d(x, z)\theta(y^*x^*) + \phi(x)d(y, z)\theta(x^*) + \phi(xy)d(x, z) \tag{3.3.9} \]

for all \( x, y, z \in R \). In view of Lemma 3.3.2 (iii), we have

\[ d(xyt + txy, z) = d(x, z)\theta(y^*t^*) + \phi(x)d(y, z)\theta(t^*) + \phi(xy)d(t, z) \]
\[ + d(t, z)\theta(y^*x^*) + \phi(t)d(y, z)\theta(x^*) + \phi(ty)d(t, z) \]

for all \( t, x, y, z \in R \). Thus, we obtain

\[ d((xy)^2, z) = d(xyx, y) = d(xy(xy) + (xy)yx - xy^2x, z) \]
\[ = d(xyx, y) + (xy)yx, z) - d(xy^2x, z) \]
\[ = d(x, z)\theta((y^*)^2\theta(x^*) + \phi(x)d(y, z)\theta(y^*x) + \phi(xy)d(xy, z) \]
\[ + d(xy, z)\theta(y^*x^*) + \phi(xy)d(y, z)\theta(x^*) + \phi(xy^2)d(x, z) \]
\[ - d(x, z)\theta((y^*)^2\theta(x^*) - \phi(x)d(y^2, z)\theta(x^*) - \phi(xy^2)d(x, z) \]

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for all $x, y, z \in R$. It follows that

$$
0 = d((xy^2), x) - d(xy, z)\theta(y^*x^*) - \phi(xy)d(x, y) - \phi(x)d(y, z)\theta(y^*)
$$

(3.3.10)

for all $x, y, z \in R$. Therefore relation (3.3.10) can be written as

$$
\Delta(xy) + \phi(x)\Delta(y)\theta(x^*) = 0
$$

(3.3.11)

for all $x, y \in R$, where

$$
\Delta(x) = d(x^2, z) - d(x, z)\theta(x^*) - \phi(x)d(x, z)
$$

for all $x, z \in R$. In view of relation (3.3.11), we find that

$$
2\phi(ty)\Delta(x)\theta(t^*z^*) = \phi(ty)\Delta(x)\theta(y^*t^*) + \phi(ty)\Delta(x)\theta(y^*z^*)
$$


= $-\phi(t)\Delta(yx)\theta(t^*) - \Delta(tyx)

= $-\phi(t)\Delta(yx)\theta(t^*) - \Delta(tyx)

= $\Delta(tyx) - \Delta(tyx)

= 0

for all $x, y, t \in R$. Thus $2\phi(ty)\Delta(x)\theta(y^*z^*) = 0$ for all $x, y, t \in R$. Since $\text{char}(R) \neq 2$, the above relation yields that $\phi(ty)\Delta(x)\theta(y^*z^*) = 0$ for all $x, y, t \in R$. Hence, application of Lemma 3.3.1 twice yields that $\Delta(x) = 0$ for all $x \in R$. That is, $d(x^2, z) - d(x, z)\theta(x^*) - \phi(x)d(x, z) = 0$ for all $x, z \in R$. Hence, $d$ is a symmetric Jordan $(\theta, \phi^*)$-biderivation on $R$. Thereby the proof is completed.

\[\square\]

3.4 Generalized Jordan $(\theta, \phi)$-derivations on $*$-closed Lie ideals of prime rings

Following [39], an additive mapping $F : R \to R$ is called a generalized $(\theta, \phi)$-derivation if there exists a $(\theta, \phi)$-derivation $d$ of $R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$, where $\theta, \phi$ are endomorphisms of $R$. An additive mapping $F : R \to R$ is called a generalized Jordan $(\theta, \phi)$-derivation if there exists a $(\theta, \phi)$-derivation $d$ of $R$ such that
$F(x^2) = F(x)\theta(x) + \phi(x)\psi(x)$ for all $x \in R$. According to [162], an additive mapping $F : R \to R$ is called a generalised Jordan triple $(\theta, \phi)$-derivation of $R$ if there exists a Jordan triple $(\theta, \phi)$-derivation $d$ of $R$ such that $F(xyz) = F(x)\theta(yz) + \phi(x)d(y)\theta(x) + \phi(xy)d(x)$ holds for all $x, y \in R$. It is obvious to see that every generalised $(\theta, \phi)$-derivation of a ring $R$ is a generalised Jordan $(\theta, \phi)$-derivation on $R$ (see [39] for $U = R$). Recently, Vukman and Kosi-Ulubul [207] showed that on a 2-torsion free semiprime ring with involution, an additive mapping $T : R \to R$ such that $T(x^2) = T(x)x^*$ holds for all $x \in R$, must be a left centralizer i.e., $T(xy) = T(x)y$ for all $x, y \in R$. Motivated by this result, Daif and El-Sayid [97] proved the following result: Let $R$ be a 2-torsion free semiprime ring with involution and $F : R \to R$ be an additive mapping with an associated derivation $d : R \to R$ such that $F(x^2) = F(x)x^* + 2d(x^*)$ holds for all $x \in R$. Then, $F$ is a generalised Jordan $*$-derivation on $R$. In [45], Ahsraf et al. extended this result for generalised $(\theta, \phi)$-derivation in the setting of semiprime rings with involution. In fact, they proved the following theorem:

**Theorem 3.4.1.** Let $R$ be a 2-torsion free semiprime $*$-ring. Suppose that $\theta, \phi$ are endomorphisms of $R$ such that $\theta$ is an automorphism of $R$. If there exists an additive mapping $F : R \to R$ associated with a $(\theta, \phi)$-derivation $d$ of $R$ such that $F(x^2) = F(x)x^* + \phi(x)d(x^*)$ holds for all $x \in R$, then $F(xy) = F(x)(y) + \phi(x)d(y)$ for all $x, y \in R$.

In the present section, we extend the above mentioned result on $*$-closed Lie ideal of prime rings with involution. Precisely, we prove the following result:

**Theorem 3.4.2.** Let $R$ be a prime ring with involution such that char($R$) $\neq 2$ and $U$ be a noncentral $*$-closed Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. Suppose that $\theta, \phi$ are endomorphisms of $R$ such that $\theta$ is an automorphism of $R$. If there exists an additive mapping $F : R \to R$ associated with a nonzero $(\theta, \phi)$-derivation $d$ of $R$ such that $F(u^2) = F(u)(u^*) + \phi(u)d(u^*)$ holds for all $u \in U$, then $F$ is a generalised $(\theta, \phi)$-derivation on $U$.

We begin our discussion with following lemma which will be extensively useful to prove our results.

**Lemma 3.4.1.** Let $R$ be a prime ring with involution such that char($R$) $\neq 2$, and $U$ be a noncentral $*$-closed Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. Suppose that $\theta$
is an automorphism of \( R \). If there exists an element \( a \in U \) such that \( a \theta(u^*) = a \theta(u) \) holds for all \( u \in U \), then \( a \in Z(R) \).

**Proof.** In view of the hypothesis, we have

\[
a \theta(u^*) = a \theta(u) \quad \text{for all} \quad u \in U.
\]  

(3.4.1)

Replacing \( u \) by \( 2u^*v \) and using the fact that \( \text{char}(R) \neq 2 \), we obtain

\[
a \theta([u,v]) = 0 \quad \text{for all} \quad u,v \in U.
\]  

(3.4.2)

Substituting \( \theta^{-1}(v) \) for \( v \) in (3.4.2), we get

\[
a \theta^{-1}(u), a] = 0 \quad \text{for all} \quad u \in U.
\]  

(3.4.3)

Replacing \( u \) by \( 2uv \) in (3.4.3) and using the fact that \( \text{char}(R) \neq 2 \), we find that

\[
a \theta(v)[\theta(u), a] = 0 \quad \text{for all} \quad u,v \in U.
\]  

(3.4.4)

Now, putting \( 2uv \) for \( v \) in (3.4.4) and using the fact that \( \text{char}(R) \neq 2 \), we obtain

\[
a \theta(v)\theta(u)[\theta(u), a] = 0 \quad \text{for all} \quad u,v \in U.
\]  

(3.4.5)

Left multiplication by \( \theta(u) \) to the relation (3.4.4) yields that

\[
\theta(u)a \theta(v)[\theta(u), a] = 0 \quad \text{for all} \quad u,v \in U.
\]  

(3.4.6)

Comparing (3.4.5) and (3.4.6), we have

\[
[\theta(u) , a][\theta(v) , a] = 0 \quad \text{for all} \quad u,v \in U.
\]  

(3.4.7)

Since \( \theta \) is surjective, the last expression yields that

\[
[\theta(u) , a]U[\theta(u) , a] = (0) \quad \text{for all} \quad u \in U.
\]  

(3.4.8)

Lemma 1.3.10 forces that \( [\theta(u) , a] = 0 \) for all \( u \in U \). Since \( \theta \) is onto, we can write the
latter relation as \([U,a] = (0)\). Further it can be written as \([a,[a,r]] = 0\) for all \(r \in R\). Then by Lemma 1.3.20, \(a \in Z(R)\). This proves the lemma. \(\square\)

**Proof of Theorem 3.4.2.** In view of the hypothesis, we have

\[ F(uu^*) = F(u)\theta(u^*) + \phi(u)d(u^*) \text{ for all } u \in U. \]

Linearization of above relation yields that

\[ F(uv^* + vu^*) = F(u)\theta(v^*) + F(v)\theta(u^*) + \phi(v)d(u^*) \]
\[ + \phi(u)d(v^*) \text{ for all } u, v \in U. \tag{3.4.9} \]

Replacing \(u^*\) for \(v\) in (3.4.9), we get

\[ F(uu + u^*u^*) = F(u)\theta(u) + F(u^*)\theta(u^*) + \phi(u^*)d(u^*) \]
\[ + \phi(u)d(u) \text{ for all } u \in U. \tag{3.4.10} \]

Further, this can be written as

\[ \eta(u) + \eta(u^*) = 0 \text{ for all } u \in U. \tag{3.4.11} \]

Where \(\eta(u) = F(u^2) - F(u)\theta(u) - \phi(u)d(u)\). Now substituting \(2(uv^* + vu^*)\) for \(v\) in (3.4.9), we get

\[ 2F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = 2(F(u)\theta(uv^* + vu^*) + F(uv^* + vu^*)\theta(u^*) \]
\[ + \phi(uv^* + vu^*)d(u^*) + \phi(u)d(uv^* + vu^*)) \]

for all \(u, v \in U\). Since \(\text{char}(R) \neq 2\), we find that

\[ F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = F(u)\theta(uv^* + vu^*) + F(uv^* + vu^*)\theta(u^*) \]
\[ + \phi(uv^* + vu^*)d(u^*) + \phi(u)d(uv^* + vu^*) \]

for all \(u, v \in U\). This implies that

\[ F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = F(u)\theta(uv^*) + F(u)\theta(vu^*) + F(u)\theta(v^*)\theta(u^*) \]

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\[ + F(v)\theta((u^*)^2) + \phi(u)d(v^*)\theta(u^*) \\
+ \phi(v)d(u^*)\theta(u^*) + \phi(uv^*)d(u^*) \\
+ \phi(vu^*)d(u^*) + \phi(u)d(u)\theta(v^*) \\
+ \phi(u^2)d(v^*) + \phi(u)d(v)\theta(u^*) \\
+ \phi(uv)d(u^*) \]  
(3.4.12)

for all \(u, v \in U\). On the other hand, we have

\[
F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = F(u^2v^* + v(u^*)^2) + F(uuu^* + uv^*u^*) \\
= F(u^2)\theta(v^*) + F(v)\theta((u^*)^2) \\
+ \phi(u^*)d(v^*) + \phi(u)d(u^*)\theta(u^*) \\
+ \phi(vu^*)d(u^*) + F(u(v + v^*)u^*) \]  
(3.4.13)

for all \(u, v \in U\). Combining (3.4.12) and (3.4.13), we obtain

\[
F(u(v + v^*)u^*) = -\eta(u)\theta(v^*) + F(u)\theta((v + v^*)u^*) \\
+ \phi(u)d((v + v^*)u^*) \quad \text{for all } u, v \in U. 
(3.4.14)
\]

Replacing \(v\) by \(v - v^*\) in (3.4.14), we get

\[
\eta(u)\theta(v) = \eta(u)\theta(v^*) \quad \text{for all } u, v \in U. 
(3.4.15)
\]

Lemma 3.4.1 implies that \(\eta(u) \in Z(R)\) for all \(u \in U\). Further, replace \(v\) by \(v^*\) in (3.4.9), to get

\[
F(uv + v^*u^*) = F(u)\theta(v) + F(v^*)\theta(u^*) + \phi(v^*)d(u^*) \\
+ \phi(u)d(v) \quad \text{for all } u, v \in U. 
(3.4.16)
\]

Now, substituting \(2uv\) for \(v\) in (3.4.16) and using the fact that \(\text{char}(R) \neq 2\), we obtain

\[
F(u^2v + v^*u^2) = F(u)\theta(uv) + F(v^*u^*)\theta(u^*) + \phi(v^*u^*)d(u^*) \\
+ \phi(u)d(uv) \quad \text{for all } u, v \in U. 
(3.4.17)
\]
On the other hand, taking $u^2$ for $u$ in (3.4.16), we find that

$$F(u^2v + v^*u^2) = F(u^2)\theta(v) + F(v^*)\theta(u^2) + \phi(v^*)d(u^2)$$

$$+ \phi(u^2)d(v) \quad \text{for all } u, v \in U. \tag{3.4.18}$$

In view of (3.4.17) and (3.4.18), we obtain

$$\eta(u)\theta(v) + (F(v^*)\theta(u^*) - F(v^*u^*) + \phi(v^*)d(u^*))\theta(u^*) = 0 \tag{3.4.19}$$

for all $u, v \in U$. Taking $u = v$ in (3.4.19), we get

$$\eta(u)\theta(u) - \eta(u^*)\theta(u^*) = 0 \quad \text{for all } u \in U. \tag{3.4.20}$$

In view of (3.4.11), we have

$$\eta(u)\theta(u + u^*) = 0 \quad \text{for all } u \in U. \tag{3.4.21}$$

Substituting $u$ for $v$ in (3.4.15) to obtain

$$\eta(u)\theta(u - u^*) = 0 \quad \text{for all } u \in U. \tag{3.4.22}$$

Comparing (3.4.21) and (3.4.22) and using the fact that $\text{char}(R) \neq 2$, we get

$$\eta(u)\theta(u) = 0 \quad \text{for all } u \in U. \tag{3.4.23}$$

Since $\eta(u) \in Z(R)$ for all $u \in U$, the last expression implies that $\theta(u)\eta(u) = 0$ for all $u \in U$. Linearization of (3.4.23) yields that

$$\eta(u)\theta(v) + \eta(v)\theta(u) + B(u, v)\theta(u) + B(u, v)\theta(v) = 0 \tag{3.4.24}$$

for all $u, v \in U$. Where $B(u, v) = F(uv + vu) - F(u)\theta(v) - F(v)\theta(u) - \phi(u)d(v) - \phi(v)d(u)$. Taking $u = -u$ in (3.4.24), we get

$$\eta(u)\theta(v) - \eta(v)\theta(u) + B(u, v)\theta(u) - B(u, v)\theta(v) = 0 \tag{3.4.25}$$

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for all \( u, v \in U \). Combining (3.4.24) and (3.4.25) and using the fact that \( \text{char}(R) \neq 2 \), we obtain \( \eta(u)\theta(u) + B(u, v)\theta(u) = 0 \) for all \( u, v \in U \) i.e., \( \eta(u) = F(u^2) - F(u)\theta(u) - \phi(u)d(u) \) for all \( u \in U \). On right multiplying by \( \eta(u) \), we find that \( \eta(u)\theta(u)\eta(u) = 0 \) for all \( u, v \in U \). Since \( \eta \) is onto, we have \( \eta(u)\eta(u) = (0) \) for all \( u \in U \). Lemma 1.3.10 forces that \( \eta(u) = 0 \) for all \( u \in U \) i.e., \( \eta(u) = F(u^2) - F(u)\theta(u) - \phi(u)d(u) \) for all \( u \in U \). Therefore, \( F \) is a generalized Jordan \((\theta, \phi)\)-derivation on \( U \). Further, in view of Lemma 1.3.5 \( F \) is a generalized \((\theta, \phi)\)-derivation on \( U \). This completes the proof of the theorem.

As an immediate consequence of Theorem 3.2.2 we have the following result:

**Corollary 3.4.1.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \). Suppose that \( \theta, \phi \) are endomorphisms of \( R \) such that \( \theta \) is an automorphism of \( R \). If there exists an additive mapping \( F : R \to R \) associated with a nonzero \((\theta, \phi)\)-derivation \( d \) of \( R \) such that \( F(x^*) = F(x)\theta(x^*) + \phi(x)d(x^*) \) holds for all \( x \in R \), then \( F \) is a generalized \((\theta, \phi)\)-derivation on \( R \).

**Theorem 3.4.3.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \), and \( U \) be a noncentral \(*\)-closed Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \). Suppose that \( \theta, \phi \) are endomorphisms of \( R \) such that \( \theta \) is an automorphism on \( R \). If there exists an additive mapping \( F : R \to R \) associated with a \((\theta, \phi)\)-derivation \( d \) of \( R \) such that \( F(\theta^*u) = F(u)\theta(u^*) + \phi(u)d(u^*)\theta(u) + \phi(u^*)d(u) \) for all \( u, v \in U \), then \( F \) is a generalized \((\theta, \phi)\)-derivation on \( U \).

**Proof.** By the hypothesis, we have

\[
F(\theta^*u) = F(u)\theta(u^*) + \phi(u)d(u^*)\theta(u) + \phi(u^*)d(u) \tag{3.4.26}
\]

for all \( u, v \in U \). Linearizing above expression, we get

\[
F((u + w)v^*(u + w)) = F(u)\theta(v^*w) + F(u)\theta(v^*w) + F(w)\theta(v^*w) + F(u)\theta(v^*w) + \phi(u)d(v^*)\theta(u) + \phi(u)d(v^*)\theta(u) + \phi(u^*)d(u) + \phi(u^*)d(u) \tag{3.4.27}
\]

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\[ + \phi(uv^*)d(u) + \phi(uv^*)d(w) \text{ for all } u, v, w \in U. \]

On the other hand, we obtain

\[
F((u + w)v^*(u + w)) = F(uv^*w + uv^*u) + F(u)\theta(v^*u) + F(u)\theta(v^*u) + \phi(u)d(v^*)\theta(u) + \phi(w)d(v^*)\theta(w) + \phi(uv^*)d(w) \text{ for all } u, v, w \in U. \tag{3.4.28}
\]

Comparing (3.4.27) and (3.4.28), we get

\[
F(uv^*w + uv^*u) = F(u)\theta(u) + F(uv^*w + uv^*u) + \phi(u)d(v^*)\theta(u) + \phi(uv^*)d(w) + \phi(uv^*)d(u) \text{ for all } u, v, w \in U. \tag{3.4.29}
\]

Substituting \( u^2 \) for \( w \) in (3.4.29), we obtain

\[
F(uv^*w + uv^*u) = F(u)\theta(u) + F(uv^*w + uv^*u) + \phi(u)d(v^*)\theta(u^2) + \phi(uv^*)d(u) + \phi(uv^*)d(u) \text{ for all } u, v \in U. \tag{3.4.30}
\]

Further, on replacing \( 2(uv^*w + uv^*u) \) for \( v \) in (3.4.26), we obtain

\[
2F(uv^*w^2 + u^2v^*u) = 2F(uv^*w^2 + u^2v^*u) + \phi(u)d(uv^*)\theta(u) + \phi(u)d(uv^*)\theta(u) + \phi(uv^*)d(u) + \phi(uv^*)d(u) \text{ for all } u, v \in U. \tag{3.4.31}
\]

Since \( \text{char}(R) \neq 2 \), we find that

\[
F(uv^*w^2 + u^2v^*u) = F(uv^*w^2 + u^2v^*u) + \phi(u)d(uv^*)\theta(u) + \phi(uv^*)d(u) \text{ for all } u, v \in U. \tag{3.4.31}
\]
On comparing (3.4.30) and (3.4.31), we obtain

\[ F(u^2)\theta(v^*u) - F(u)\theta(u)\theta(v^*u) - \phi(u)d(u)\theta(v^*u) = 0 \quad (3.4.32) \]

for all \( u, v \in U \). If we take \( A(u) = F(u^2) - F(u)\theta(u) - \phi(u)d(u) \), then the relation (3.4.32) reduces to

\[ A(u)\theta(v^*u) = 0 \text{ for all } u, v \in U. \quad (3.4.33) \]

Since \( \theta \) is surjective, (3.4.33) implies that

\[ \theta^{-1}(A(u))v^*u = 0 \text{ for all } u, v \in U. \quad (3.4.34) \]

Substituting \( 2v^*u^* \) for \( v \) in (3.4.34) and using the fact that \( \text{char}(R) \neq 2 \), we find that

\[ \theta^{-1}(A(u))wu = 0 \text{ for all } u, v \in U. \quad (3.4.35) \]

Now taking \( v = 2\theta^{-1}(w)\theta^{-1}(A(u)) \) in (3.4.35) and using the fact that \( \text{char}(R) \neq 2 \), we obtain

\[ \theta^{-1}(A(u))w\theta^{-1}(w)\theta^{-1}(A(u))u = 0 \text{ for all } u, w \in U. \quad (3.4.36) \]

This implies that

\[ A(u)\theta(u)wA(u)\theta(u) = 0 \text{ for all } u, w \in U. \quad (3.4.37) \]

That is,

\[ A(u)\theta(u)UA(u)\theta(u) = (0) \text{ for all } u \in U. \quad (3.4.38) \]

In view of Lemma 1.3.10, we get

\[ A(u)\theta(u) = 0 \text{ for all } u \in U. \quad (3.4.39) \]
Linearization of (3.4.39) gives

\[ A(u)\theta(v) + \gamma(u,v)\theta(u) + A(v)\theta(u) + \gamma(u,v)\theta(v) = 0 \]  
(3.4.40)

for all \( u, v \in U \), where \( \gamma(u,v) = F(uv + vu) - F(u)\theta(v) - \phi(u)d(v) - F(v)\theta(u) - \phi(v)d(u) \)

for all \( u, v \in U \). Replacing \( u \) by \( -u \) in (3.4.40), we obtain

\[ A(u)\theta(v) + \gamma(u,v)\theta(u) - A(v)\theta(u) - \gamma(u,v)\theta(v) = 0 \]  
(3.4.41)

for all \( u, v \in U \). Combining (3.4.40) and (3.4.41) and using the fact that \( \text{char}(R) \neq 2 \), we get

\[ A(u)\theta(v) + \gamma(u,v)\theta(u) = 0 \text{ for all } u, v \in U. \]  
(3.4.42)

On right multiplication of above equation by \( A(u) \), we obtain

\[ A(u)\theta(v)A(u) + \gamma(u,v)\theta(u)A(u) = 0 \text{ for all } u, v \in U. \]  
(3.4.43)

Substitution \( v^* \) for \( v \) in (3.4.33), we find that \( \theta(u)A(u)\theta(v)\theta(u)A(u) = 0 \). That is,

\[ \theta(u)A(u)U\theta(u)A(u) = (0) \text{ for all } u \in U. \]  
(3.4.44)

Again application of Lemma 1.3.10 yields that

\[ \theta(u)A(u) = 0 \text{ for all } u \in U. \]  
(3.4.45)

Using (3.4.45) in (3.4.43) we have \( A(u)\theta(v)A(u) = 0 \) i.e., \( A(u)UA(u) = (0) \) for all \( u \in U \). Again making use of Lemma 1.3.10, we get \( A(u) = 0 \) for all \( u \in U \) i.e., \( F(u^2) - F(u)\theta(u) - \phi(u)d(u) = 0 \) for all \( u \in U \). Therefore, \( F \) is a generalized Jordan \((\theta, \phi)\)-derivation on \( U \). Hence by Lemma 1.3.5, \( F \) is a generalized \((\theta, \phi)\)-derivation on \( U \). \( \square \)

If we take \( d = 0 \) in Theorem 3.4.3, then we get the following result.

**Theorem 3.4.4.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \) and \( U \) be a noncentral \( \ast \)-closed Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \). Suppose that
\( \theta \) is surjective. If there exists an additive mapping \( F : R \to R \) such that \( F(uv^*u) = F(u)\theta(v^*u) \) for all \( u, v \in U \), then \( F(uv) = F(u)\theta(v) \) for all \( u, v \in U \).

Following are immediate consequences of the above theorems:

**Corollary 3.4.2.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \), and \( U \) be a noncentral \(*\)-closed Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \). If there exists an additive mapping \( F : R \to R \) associated with a derivation \( d \) of \( R \) such that \( F(uv^*u) = F(u)v^*u + ud(v^*)u + uv^*d(u) \) for all \( u, v \in U \), then \( F \) is a generalized derivation on \( U \).

**Corollary 3.4.3.** [45, Theorem 2.2] Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \). Suppose that \( \theta, \phi \) are endomorphisms of \( R \) such that \( \theta \) is an automorphism of \( R \). If there exists an additive mapping \( F : R \to R \) associated with a \((\theta, \phi)\)-derivation \( d \) of \( R \) such that \( F(xy^*x) = F(x)\theta(y^*x) + \phi(x)d(y^*)\theta(x) + \phi(xy^*)d(x) \) for all \( x, y \in R \), then \( F \) is a generalized \((\theta, \phi)\)-derivation on \( R \).

**Corollary 3.4.4.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \). Suppose that \( \theta \) is surjective. If there exists an additive mapping \( F : R \to R \) such that \( F(xy^*x) = F(x)\theta(y^*x) \) for all \( x, y \in R \), then \( F(xy) = F(x)\theta(y) \) for all \( x, y \in R \).