CHAPTER-2

On Commutativity of Prime Rings with

Involution involving Derivations
Chapter 2

On commutativity of prime rings with involution involving derivations

2.1 Introduction

A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. Pursuit of this line of inquiry was inspired by celebrated Jacobson's theorem that any ring in which every element $x$ satisfies an equation of the form $x^{n(x)} = x$, where $n(x) \in \mathbb{N} \setminus \{1\}$, must be commutative [136], a result which generalized the famous Wedderburn theorem that every finite division ring is commutative as well as the theorem that every Boolean ring is commutative. There are now more than hundred papers in which conditions are given that determine commutativity for a ring or a special type of ring. Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work [117–119]. A significant contributor has been Bell (see for instance [51], [53–56] and reference therein) who individually, or with co-authors has written more than five dozen articles. Other strong contributors have been Ashraf and Yaqub with a variety of co-authors (viz.; [1], [36–38], [44], [139], [180–182], [194] and [211], where further references can be found).

Another technique for investigating commutativity of rings (algebras) is the use of additive maps like derivations and automorphisms of the ring $R$. We say a function $f : R \to R$ is commuting on a nonempty subset $S$ of $R$ if $f(x)x = xf(x)$ for all $x \in S$, and centralizing on $S$ if $xf(x) - f(x)x \in Z(R)$ for all $x \in S$. The study of such mappings was initiated by Posner (Posner second theorem). In [177, Theorem 2], Posner proved that if a prime ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. The analogous result for
centralizing automorphisms on prime rings was obtained by Mayne [172]. A number of authors have extended these theorems of Poener and Mayne; and have shown that derivations, automorphisms, and some related maps cannot be centralizing on certain subsets of noncommutative prime (and some other) rings. For these kind of results we refer the reader to [35], [40], [60], [62], [67], [70], [94] and [145], where further references can be found. There has been a great deal of work recently concerning the relationship between the commutativity of a ring \(R\) and the existence of certain specified additive maps like derivations and automorphisms of \(R\). Chung, Herstein, Ikeda, Koč, Lui, Martindale, Procesei, Putcha, Richoux, Schacher, Wilson and Yaqub (viz.; [89–91], [120], [127], [131], [135], [165] and [178]) have studied conditions on commutators which imply the commutativity of rings.

An additive mapping \(d : R \to R\) is said to be a derivation of \(R\) if \(d(xy) = d(x)y + xd(y)\) for all \(x, y \in R\). A derivation \(d\) is said to be inner if there exists \(a \in R\) such that \(d(x) = ax - xa\) for all \(x \in R\). An additive mapping \(F : R \to R\) is said to be a generalized derivation of \(R\) if \(F(xy) = F(x)y + xd(y)\) holds for all \(x, y \in R\), where \(d : R \to R\) is a derivation. The concept of generalized derivations was introduced by Brešar [68] and covers both concepts of derivations and left centralizers i.e., an additive map \(T : R \to R\) satisfying \(T(xy) = T(x)y\) for all \(x, y \in R\) (see [216] for details). The definition of a right centralizer is self explanatory. A significant example is a map of the form \(F(x) = ax + xb\) for some \(a, b \in R\); such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example [134] and [152], where further references can be found).

Following [59], A map \(f : R \to R\) is called strong commutativity preserving map (SCP) on \(S\) (where \(S\) is a nonempty subset of \(R\)) if it satisfy the condition \([f(x), f(y)] = [x, y]\) for all \(x, y \in S\). Section 2.2 is devoted to the study of strong commutativity preserving maps (SCP) in the setting of prime rings with involution involving generalized derivations. We study the result of Bell and Daif [59, Theorem 1] in the setting of prime rings with involution. Moreover, we explore the commutativity of prime rings with involution which admits a derivation \(d\) satisfying the condition \(d(x) \circ d(x^*) = x \circ x^*\) for all \(x \in R\).

The history of commuting and centralizing mappings goes back to 1995 when Divinsky [104] proved that a simple artinian ring is commutative if it has a commuting
non-trivial automorphism. Two years later, Posner [177] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Several other papers studying commutativity of prime and semiprime rings admitting derivations or generalized derivations satisfying certain identities can be found in [29], [40], [62], [63], [133], [182], where further references can be looked.

In Section 2.3, we shall continue the similar study and extend Posner’s second theorem [177], and the main result proved in [12] in the setting of prime rings with involution by replacing derivation with generalized derivation. Precisely, we prove the following result: Let $R$ be a prime ring with involution of the second kind such that $\text{char}(R) \neq 2$. If $R$ admits a nonzero generalized derivation $F$ such that $[F(x), x^*] \in Z(R)$ for all $x \in R$, then $R$ is commutative.

Section 2.4 deals with the study of commutativity of prime rings with involution. We generalize Herstein’s theorem [128, Theorem 2] for generalized derivations. In fact, we prove that if a prime ring $R$ with involution of the second kind such that $\text{char}(R) \neq 2$ admits a nonzero generalized derivation $F$ such that $[F(x), F(x^*)] = 0$ for all $x \in R$, then $R$ is commutative.

In Section 2.5, we study the result due to Bell and Daif [60, Theorem 3] in the setting of prime rings with involution by replacing derivation by generalized derivation.

Finally, in Section 2.6 suitable examples have been provided at places to demonstrate that the restriction of the second kind involution imposed on the hypotheses of various results are not superfluous.

### 2.2 The condition $[d(x), d(x^*)] = [x, x^*]$}

We say that a map $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The problems of characterizing maps that preserves certain subsets or relations had been investigated on various rings and algebras. One of the most studied problems is to describe bijective additive (or linear) maps preserving commutativity (see for example [48], [49], [89], [76], [77], [159], [160], [161] [164], [190], [209], [210] and references therein). In [39], Bell and Daif initiated the study of a certain kind of commutativity preserving maps as follows: Let $S$ be a nonempty subset of $R$. A map $f : R \to R$ is called strong commutativity preserving (SCP) on $S$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. More precisely, they proved that $R$ must be commutative if
$R$ is a prime ring and admits a derivation or a non-identity endomorphism which is SCP on a right ideal of $R$. Furthermore, the analogous results for semiprime rings are also obtained. Later, Brešar and Miers [75] characterized an additive map $f : R \to R$ which is SCP on the entire semiprime ring $R$ and showed that $f$ must be of the form $f(x) = \lambda x + \mu(x)$, where $\lambda \in C, \lambda^2 = 1$ and $\mu : R \to C$ is an additive map, where $C$ is the extended centroid of $R$. Recently, Deng and Ashraf [98] proved that if $R$ is a prime ring of characteristic not 2 and there exists a non-identity endomorphism $\theta$ of $R$ such that $[\theta(x), \theta(y)] - [x, y] \in Z(R)$ for all $x, y$ in some essential right ideal of $R$, then $R$ is commutative. In [157], Lin and Liu characterized strong commutativity preserving maps of noncentral Lie ideals on prime rings.

The main objective of the present section is to study a more general concept than SCP mappings. More precisely, we consider the situation when an additive mapping $f : R \to R$ satisfies the condition $[f(x), f(x^*)] = [x, x^*]$ for all $x \in R$. In fact, we investigate the commutativity of a prime ring with involution, when the mapping $f$ is assumed to be a derivation of $R$. In fact, we prove the following result:

**Theorem 2.2.1.** Let $R$ be a prime ring with involution of the second kind such that $\text{char}(R) \neq 2$. If $d$ is a nonzero derivation of $R$ such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, then $R$ is commutative.

**Proof.** By the assumption, we have

$$[d(x), d(x^*)] = [x, x^*] \quad (2.2.1)$$

for all $x \in R$. A linearization of (2.2.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = [x, y^*] + [y, x^*] \quad (2.2.2)$$

for all $x, y \in R$. Replacing $y$ by $xx^*$ in (2.2.2), we get

$$d(x)[d(x), x^*] + x[d(x), d(x^*)] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)]$$

$$+ [d(x), d(x^*)]x^* + [x, d(x^*)]d(x^*) = x[x, x^*] + [x, x^*]x^*$$

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for all $x \in R$. Application of (2.2.1) yields that

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \quad (2.2.3)$$

for all $x \in R$. Replacing $x$ by $x + h'$, where $h' \in H(R) \cap Z(R)$, we obtain

$$d(h')[d(x), x^*] + [d(x), x]d(h') + d(h')[x^*, d(x^*)] + [x, d(x^*)]d(h') = 0.$$

This can be further written as

$$d(h')[[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]] = 0$$

for all $h' \in H(R) \cap Z(R)$ and $x \in R$. In view of Remark 1.2.7 (i) we are forced to conclude that $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Now suppose that

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \quad (2.2.4)$$

Replacing $h'$ by $(k')^2$ in (2.2.4), where $k' \in S(R) \cap Z(R)$, we get

$$0 = d((k')^2) = d(k')k' + k'd(k') = 2d(k')k'.$$

Since $\text{char}(R) \neq 2$, we arrive at

$$d(k')k' = 0 \text{ for all } k' \in S(R) \cap Z(R).$$

For each $k' \in S(R) \cap Z(R)$, the last expression yields that either $d(k') = 0$ or $k' = 0$. Since $k' = 0$ implies $d(k') = 0$, we may write

$$d(k') = 0 \text{ for all } k' \in S(R) \cap Z(R). \quad (2.2.5)$$

Let $x \in Z(R)$. In view of Remark 1.2.12 (iii), we have $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = 0$ for all $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. Since $\text{char}(R) \neq 2$, we
get

\[ d(x) = 0 \text{ for all } x \in Z(R). \quad (2.2.6) \]

Replacing \( y \) by \( k'y \) in (2.2.2), where \( k' \in S(R) \cap Z(R) \) and using (2.2.6), we arrive at

\[ k'(-[d(x), d(y^*)] + [d(y), d(x^*)] + [x, y^*] - [y, x^*]) = 0 \]

for all \( k' \in S(R) \cap Z(R) \) and \( x, y \in R \). Using the primeness of \( R \) and the fact that \( S(R) \cap Z(R) \neq (0) \), we get

\[ -[d(x), d(y^*)] + [d(y), d(x^*)] = -[x, y^*] + [y, x^*] \quad (2.2.7) \]

for all \( x, y \in R \). On comparing (2.2.2) and (2.2.7), we obtain \( 2[d(x), d(y^*)] = 2[x, y^*] \) for all \( x, y \in R \). Replacing \( y \) by \( y^* \) and using the fact that \( \text{char}(R) \neq 2 \), we conclude that \([d(x), d(y)] = [x, y]\) for all \( x, y \in R \). Therefore in view of Lemma 1.3.9, \( R \) is commutative. Now we consider the case

\[ [d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0 \]

for all \( x \in R \). Replacing \( x \) by \( h + k \), where \( h \in H(R) \) and \( k \in S(R) \), we obtain \( 4[d(k), h] = 0 \). Since \( \text{char}(R) \neq 2 \), we are force to conclude that

\[ [d(k), h] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \quad (2.2.8) \]

Replacing \( h \) by \( k_0k' \), where \( k_0 \in S(R) \) and \( k' \in S(R) \cap Z(R) \), we arrive at \( ([d(k), k_0])k' = 0 \). Since \( R \) is prime and \( S(R) \cap Z(R) \neq (0) \), we get

\[ [d(k), k_0] = 0 \text{ for all } k, k_0 \in S(R). \quad (2.2.9) \]

Since \( \text{char}(R) \neq 2 \), every \( x \in R \) can be uniquely represented as \( 2x = h + k \) for all \( h \in H(R) \) and \( k \in S(R) \), so in view of relations (2.2.8) and (2.2.9), we are force to conclude that

\[ [d(k), x] = 0 \text{ for all } k \in S(R) \text{ and } x \in R. \quad (2.2.10) \]
for all \( k \in S(R) \) and \( x \in R \). That is, \( d(k) \in Z(R) \) for all \( k \in S(R) \). First we assume that \( d(S(R)) = (0) \). Then we have \( d(x - x^*) = 0 \) for all \( x \in R \). That is, \( d(x) = d(x^*) \) for all \( x \in R \). Now for \( k \in S(R) \) and \( x \in R \), we have \( 0 = d(kx + x^*k) = kd(x) + d(x^*)k = kd(x) + d(x)k \) for all \( x \in R \). This further implies that \( k^2d(x) = d(x)k^2 \) for all \( x \in R \). Thus, by Lemma 1.3.23, we conclude that \( k^2 \in Z(R) \) for all \( k \in Z(R) \).

Since \( S(R) \cap Z(R) \neq (0) \), let \( 0 \neq k_0 \in S(R) \cap Z(R) \) and let \( k \) be an arbitrary element of \( S(R) \). Then \( (k + k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R) \) and hence \( 2kk_0 \in Z(R) \). Since \( char(R) \neq 2 \), we get \( kk_0 \in Z(R) \) for all \( k \in S(R) \) and \( k_0 \in S(R) \cap Z(R) \). This further implies that \( k \in Z(R) \) for all \( k \in S(R) \) and hence \( R \) is normal. Thus \( R \) is commutative in view of Lemma 1.3.3. Now suppose \( d(S(R)) \neq (0) \). For \( k_0 \in S(R) \) with \( d(k_0) \neq 0 \) and \( k \in [S(R), S(R)] \), we have \( d(k_0kk_0) \in Z(R) \). The last expression can be written as \( d(k_0)kk_0 + k_0kd(k_0) \in Z(R) \), since \( d([S(R), S(R)]) = (0) \). Thus \( d(k_0)(k_0k + kk_0) \in Z(R) \) and hence \( k_0k + kk_0 \in Z(R) \) for all \( k \in [S(R), S(R)] \). This implies that \( d(k_0k + kk_0) \in Z(R) \) and hence \( 2d(k_0)k \in Z(R) \). Since \( char(R) \neq 2 \) and \( R \) is prime, the above relation yields that \( k \in Z(R) \). That is, \( [S(R), S(R)] \subseteq Z(R) \).

Suppose \( [S(R), S(R)] \neq (0) \) and \( k, k_0 \in S(R) \) such that \( [k, k_0] \neq 0 \). Since \( kk_0k \in S(R) \), we have \( [k, kk_0k] = k[k, k_0]k = k^2[k, k_0]k \in Z(R) \). This implies that \( k^2 \in Z(R) \) and hence \( k \in Z(R) \) for all \( k \in S(R) \) as proved earlier. Therefore, \( R \) is commutative in view of Lemma 1.3.3. Now suppose \( [S(R), S(R)] = (0) \). Since \( S(R) \) is both a Lie ideal and a commutative subring of \( R \), by Lemma 1.3.22, \( k^2 \in Z(R) \) for all \( k \in S(R) \) and hence \( k \in Z(R) \) for all \( k \in S(R) \). Thus, \( R \) is normal and hence \( R \) is commutative by Lemma 1.3.3. This completes the proof of the theorem. \( \square \)

If we replace commutator by anti-commutator in Theorem 2.2.1, the corresponding result also holds.

**Theorem 2.2.2.** Let \( R \) be a prime ring with involution of the second kind such that \( char(R) \neq 2 \). If \( d \) is a nonzero derivation of \( R \) such that \( d(x) \circ d(x^*) = x \circ x^* \) for all \( x \in R \), then \( R \) is commutative.

**Proof.** By the given hypothesis, we have \( d(x) \circ d(x^*) = x \circ x^* \) for all \( x \in R \). This can be further written as

\[
d(x)d(x^*) + d(x^*)d(x) = xx^* + x^*x
\]  

(2.2.11)
for all \( x \in R \). A linearization of (2.2.11) yields that
\[
d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) + d(y^*)d(x) = xy^* + yx^* + x^*y + y^*x
\]
(2.2.12)
for all \( x, y \in R \). Replacing \( y \) by \( h'x \) in (2.2.12), where \( h' \in H(R) \cap Z(R) \), we get
\[
d(h')d(x)x^* + d(x)d(x^*)h' + d(h')x^*d(x) + d(x)d(x^*)h' + h'd(x)d(x^*) + d(h')d(x^*)x + d(x)d(x^*)h' + d(h')x^*d(x) + h'd(x)d(x^*) = 2h'xx^* + 2h'x^*x
\]
for all \( x \in R \) and \( h' \in H(R) \cap Z(R) \). In view of (2.2.11), the above expression reduces to
\[
d(h')d(x)x^* + d(h')x^*d(x) + d(h')d(x^*)x + d(h')x^*d(x) = 0
\]
for all \( x \in R \) and \( h' \in H(R) \cap Z(R) \). That is, \( d(h')d(x \circ x^*) = 0 \) for all \( h' \in H(R) \cap Z(R) \) and \( x \in R \). In view of Remark 1.2.7 (i), we have either \( d(h') = 0 \) for all \( h' \in H(R) \cap Z(R) \) or \( d(x \circ x^*) = 0 \) for all \( x \in R \). Suppose on one hand that \( d(h') = 0 \) for all \( h' \in H(R) \cap Z(R) \). This further implies that \( d(x) = 0 \) for all \( x \in Z(R) \). Replacing \( y \) by \( k'y \) in (2.2.12), where \( k' \in S(R) \cap Z(R) \) and using the fact that \( d(x) = 0 \) for all \( x \in Z(R) \), we obtain
\[
k'(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = k'(-xy^* + yx^* + x^*y - y^*x)
\]
for all \( x, y \in R \) and \( k' \in S(R) \cap Z(R) \). This implies that
\[
k'(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x) + xy^* - yx^* - x^*y + y^*x) = 0
\]
for all \( x, y \in R \) and \( k' \in S(R) \cap Z(R) \). Since \( R \) is prime and \( S(R) \cap Z(R) \neq (0) \), we obtain
\[
-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x) = xy^* + yx^* + x^*y - y^*x
\]
(2.2.13)
for all \( x, y \in R \). Comparing (2.2.12) and (2.2.13), we get \( 2(d(x)d(y^*) + d(y^*)d(x)) = 2(xy^* + y^*x) \) for all \( x, y \in R \). This implies that \( d(x)d(y^*) + d(y^*)d(x) = xy^* + y^*x \) for
all $x, y \in R$. Replace $y$ by $y^*$ to get $d(x) \circ d(y) = x \circ y$ for all $x, y \in R$. Hence, $R$ is commutative in view of Lemma 1.3.4. On the other hand, we assume that $d(x \circ x^*) = 0$ for all $x \in R$. The above equation can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) = 0$$  \hspace{1cm} (2.2.14)

for all $x \in R$. Replacing $x$ by $h \in H(R) \cap Z(R)$ in (2.2.14), and using the fact that $char(R) \neq 2$, we obtain

$$d(h)h = 0 \text{ for all } h \in H(R) \cap Z(R).$$

Application of Remark 1.2.7 (i) yields that $d(h) = 0$ or $h = 0$. Since $h = 0$ also implies $d(h) = 0$, so we may write $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(x) = 0$ for all $x \in Z(R)$. Linearizing (2.2.14), we obtain

$$d(x)y^* + d(y)x^* + xd(y^*) + yd(x^*) + d(x^*)y$$
$$+ y^*d(x) + x^*d(y) + y^*d(x) = 0$$  \hspace{1cm} (2.2.15)

for all $x, y \in R$. Replacing $y$ by $y_0 \in Z(R)$ in (2.2.15) and using the fact that $d(x) = 0$ for all $x \in Z(R)$, we get

$$d(x)y_0^* + y_0d(x^*) + d(x^*)y_0 + y_0^*d(x) = 0$$  \hspace{1cm} (2.2.16)

for all $y_0 \in Z(R)$ and $x \in R$. In particular, taking $y_0 = h_0 \in H(R) \cap Z(R)$ in (2.2.16), we get $2(d(x)h_0 + d(x^*)h_0) = 0$ for all $x \in R$ and $h_0 \in H(R) \cap Z(R)$. Since $char(R) \neq 2$, we obtain $d(x)h_0 + d(x^*)h_0 = 0$ for all $x \in R$ and $h_0 \in H(R) \cap Z(R)$. This can be further written as

$$d(x + x^*)h_0 = 0$$  \hspace{1cm} (2.2.17)

for all $x \in R$ and $h_0 \in H(R) \cap Z(R)$. In view of Remark 1.2.7 (i), we get either $d(x + x^*) = 0$ or $H(R) \cap Z(R) = (0)$. But $H(R) \cap Z(R) = (0)$ implies that $S(R) \cap Z(R) = (0)$, which gives a contradiction since we have assumed $S(R) \cap Z(R) \neq (0)$. Therefore, we are left with the case $d(x + x^*) = 0$ for all $x \in R$. Replacing $x$ by $h + k$ in the last
equation, we get $2d(h) = 0$. This implies that $d(h) = 0$ for all $h \in H(R)$. Further, $d(x + x^*) = 0$ implies that $d(x) = -d(x^*)$ for all $x \in R$. Replacing $x$ by $xh$, where $h \in H(R)$ in the last expression we get $d(x)h = -hd(x^*)$, since $d(h) = 0$. This further implies that $d(x)h = hd(x)$ for all $x \in R$. Therefore in view of Lemma 1.3.23, we conclude that $h \in Z(R)$ for all $h \in H(R)$. Hence, $R$ is commutative in view of Lemma 1.3.3. Thereby completing the proof of the theorem. \qed

2.3 The condition $[F(x), x^*] \in Z(R)$

Following [12], a mapping $f : R \to R$ is called $*$-centralizing if $[f(x), x^*] \in Z(R)$ holds for all $x \in S$ (where $S$ is a nonempty subset of $R$), and is called $*$-commuting if $[f(x), x^*] = 0$ holds for all $x \in S$. The study of such mappings was initiated by Ali and Dar in [14]. Moreover, they studied $*$-centralizing and $*$-commuting mappings in prime rings with involution. Further, they characterized normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers. Furthermore, in [13] they obtained some commutativity results in the setting of prime rings with involution involving derivations. In the year 2014, besides characterising $*$-centralizing and $*$-commuting mappings, Ali and Dar [12] established a $*$-version of Posner's second theorem. Precisely, they proved the following result:

**Theorem 2.3.1.** Let $R$ be a prime ring with involution '$*$' such that char$(R) \neq 2$. If $d$ is a nonzero derivation of $R$ such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$, then $R$ is commutative.

Thus it is natural to question whether the analogue of the above theorem holds for generalized derivations. Theorem 2.3.3 answers to this question in affirmative. We begin our investigation with the following theorem:

**Theorem 2.3.2.** Let $R$ be a prime ring with involution of the second kind such that char$(R) \neq 2$. If $R$ admits a nonzero left centralizer $T : R \to R$ such that $[T(x), x^*] \in Z(R)$ for all $x \in R$, then $R$ is commutative.

**Proof.** By the assumption, we have

$$[T(x), x^*] \in Z(R) \quad \text{for all} \quad x \in R. \quad (2.3.1)$$

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In view of Lemma 1.3.2, we conclude that

\[ [T(x), x^*] = 0 \text{ for all } x \in R. \]  \hspace{1cm} (2.3.2)

Linearization of relation (2.3.2) yields that

\[ [T(x), y^*] + [T(y), x^*] = 0 \text{ for all } x, y \in R. \]  \hspace{1cm} (2.3.3)

Putting \( y = k_1 \in S(R) \cap Z(R) \) in (2.3.3), we get

\[ [T(k_1), x^*] = 0 \text{ for all } x \in R \text{ and } k_1 \in S(R) \cap Z(R). \]  \hspace{1cm} (2.3.4)

This implies that \( T(k_1) \in Z(R) \) for all \( k_1 \in S(R) \cap Z(R) \). Next replacing \( x \) by \( h + k \) in (2.3.2), where \( h \in H(R) \) and \( k \in S(R) \), we obtain

\[ [T(h), k] - [T(k), h] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \]  \hspace{1cm} (2.3.5)

Substituting \( k_0 k_1 \) for \( h \) in (2.3.5), where \( k_0 \in S(R) \) and \( k_1 \in S(R) \cap Z(R) \), we find that

\[ ([T(k_0), k] - [T(k), k_0]) k_1 = 0 \]  \hspace{1cm} (2.3.6)

for all \( k, k_0 \in S(R) \) and \( k_1 \in S(R) \cap Z(R) \). This implies that

\[ [T(k_0), k] - [T(k), k_0] = 0 \text{ for all } k, k_0 \in S(R). \]  \hspace{1cm} (2.3.7)

In view of Remark 1.2.12 (iii) and relation (2.3.5), (2.3.7) we find that

\[
2([T(y), k] - [T(k), y]) = [T(2y), k] - [T(k), 2y] \\
= [T(h + k_0), k] - [T(k), h + k_0] \\
= [T(h), k] - [T(k), h] + [T(k_0), k] - [T(k), k_0] \\
= 0 \text{ for all } h \in H(R), k, k_0 \in S(R) \text{ and } y \in R.
\]

Since \( \text{char}(R) \neq 2 \), the above relation yields that

\[ [T(y), k] - [T(k), y] = 0 \text{ for all } k \in S(R) \text{ and } y \in R. \]  \hspace{1cm} (2.3.8)
Next, replacing \(k\) by \(hk_1\) in (2.3.5), where \(h \in H(R)\) and \(k_1 \in S(R) \cap Z(R)\), and proceeding as above, we obtain

\[
[T(h), x] - [T(x), h] = 0 \text{ for all } x \in R \text{ and } h \in H(R).
\]

This can be further written as

\[
[T(x), h] - [T(h), x] = 0 \text{ for all } x \in R \text{ and } h \in H(R).
\] (2.3.9)

Using similar approach as we obtained relation (2.3.8), we find that

\[
[T(x), y] - [T(y), x] = 0 \text{ for all } x, y \in R.
\] (2.3.10)

Taking \(y = x^2\) in (2.3.10), we get

\[
[T(x), x]x + x[T(x), x] - [T(x), x]x = 0 \text{ for all } x \in R.
\] (2.3.11)

This implies that

\[
x[T(x), x] = 0 \text{ for all } x \in R.
\] (2.3.12)

Substituting \(x\) by \(x + k_1\) in (2.3.12), where \(k_1 \in S(R) \cap Z(R)\), we obtain

\[
[T(x), x]k_1 = 0 \text{ for all } x \in R \text{ and } k_1 \in S(R) \cap Z(R).
\] (2.3.13)

Application of Remark 1.2.7 \((t)\) yields that

\[
[T(x), x] = 0 \text{ for all } x \in R.
\] (2.3.14)

Replacing \(x\) by \(x + y\), we get

\[
[T(x), y] + [T(y), x] = 0 \text{ for all } x, y \in R.
\] (2.3.15)

Combining (2.3.10) and (2.3.15), we obtain \(2([T(y), x]) = 0\) for all \(x, y \in R\). Since \(\text{char}(R) \neq 2\), the last expression yields that \([T(y), x] = 0\) for all \(x, y \in R\). Replacing \(y\) by \(yz\), we get \(T(y)[z, x] = 0\) for all \(x, y, z \in R\). This implies that \(T(y)R[z, x] = 0\).
for all $x, y, z \in R$. Hence by the primeness of $R$ either $T = 0$ or $R$ is commutative. Since we have assumed $T \neq 0$, so $R$ must be commutative. This proves the theorem completely.

Now we are ready to prove our main theorem of the present section:

**Theorem 2.3.3.** Let $R$ be a prime ring with involution of the second kind such that $\text{char}(R) \neq 2$. If $R$ admits a nonzero generalized derivation $F : R \to R$ such that $[F(x), x^*] \in Z(R)$ for all $x \in R$, then $R$ is commutative.

**Proof.** By the given hypothesis, we have

$$[F(x), x^*] \in Z(R) \text{ for all } x \in R. \quad (2.3.16)$$

Application of Lemma 1.3.2 yields that

$$[F(x), x^*] = 0 \text{ for all } x \in R. \quad (2.3.17)$$

Substituting $x + y$ for $x$ in (2.3.17), we obtain

$$[F(x), y^*] + [F(y), x^*] = 0 \text{ for all } x, y \in R. \quad (2.3.18)$$

Replacing $y$ by $xx^*$ in (2.3.18), we get

$$[F(x), x^* + x[F(x), x^*] + [F(x), x^*]x^* + [x, x^*]d(x^*) + x[d(x^*), x^*] = 0 \text{ for all } x \in R. \quad (2.3.19)$$

In view of (2.3.17), the above relation reduces to

$$[F(x), x]x^* + [x, x^*]d(x^*) + x[d(x^*), x^*] = 0 \text{ for all } x \in R. \quad (2.3.19)$$

Again replacing $y$ by $x^*x$ in (2.3.18) and using (2.3.17), we get

$$x^*[F(x), x] + [F(x^*), x^*]x + F(x^*)[x, x^*] + x^*[d(x), x^*] = 0 \quad (2.3.20)$$

for all $x \in R$. Taking $x = h$ in (2.3.19), where $h \in H(R)$, we get

$$h[d(h), h] = 0 \text{ for all } h \in H(R). \quad (2.3.21)$$

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Substituting $h + h_1$ for $h$ in (2.3.21), where $h_1 \in H(R) \cap Z(R)$, we obtain

$$h_1[d(h), h] = 0 \text{ for all } h \in H(R) \text{ and } h_1 \in H(R) \cap Z(R).$$  \hfill (2.3.22)

In view of Remark 1.2.7 (i), the last relation yields that either $h_1 = 0$ or $[d(h), h] = 0$ for all $h \in H(R)$. Suppose $h_1 = 0$ for all $h_1 \in H(R) \cap Z(R)$. Since for $k_1 \in S(R) \cap Z(R)$, $(k_1)^2 \in H(R) \cap Z(R)$, so we have $(k_1)^2 = 0$ and hence $k_1 = 0$. That is, $S(R) \cap Z(R) = \{0\}$, which gives a contradiction. Therefore, we are left with the case

$$[d(h), h] = 0 \text{ for all } h \in H(R).$$  \hfill (2.3.23)

A linearization of (2.3.23) yields that

$$[d(h), h_0] + [d(h_0), h] = 0 \text{ for all } h, h_0 \in H(R).$$  \hfill (2.3.24)

Which can be further written as

$$[d(h_0), h] = [h_0, d(h)] \text{ for all } h, h_0 \in H(R).$$  \hfill (2.3.25)

Substituting $h^2$ for $h$ in above expression, we obtain

$$[d(h_0), h^2] = [h_0, d(h)]h + h[h_0, d(h)] + d(h)[h_0, h] + [h_0, h]d(h)$$  \hfill (2.3.26)

for all $h, h_0 \in H(R)$. Also, we have

$$[d(h_0), h^2] = [d(h_0), h]h + h[d(h_0), h] = [h_0, d(h)]h + h[h_0, d(h)]$$  \hfill (2.3.27)

for all $h, h_0 \in H(R)$. Combining (2.3.26) and (2.3.27), we obtain

$$d(h)[h_0, h] + [h_0, h]d(h) = 0 \text{ for all } h, h_0 \in H(R).$$  \hfill (2.3.28)

Now, taking $h_0 = kk_1$ in (2.3.28), where $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, and using Remark 1.2.7 (i), we arrive at

$$d(h)[k, h] + [k, h]d(h) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$  \hfill (2.3.29)
Replacing \( h \) by \( h + h_1 \) in (2.3.29), where \( h_1 \in H(R) \cap Z(R) \), we get

\[
d(h)[k, h] + d(h_1)[k, h] + [k, h]d(h) + [k, h]d(h_1) = 0 \tag{2.3.30}
\]

for all \( h \in H(R) \) and \( k \in S(R) \). In view of (2.3.29), the last relation reduces to

\[
2[k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R).
\]

Since \( \text{char}(R) \neq 2 \), the above expression gives us

\[
[k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R). \tag{2.3.31}
\]

Using the primeness of \( R \), we get either \( [k, h] = 0 \) or \( d(h_1) = 0 \). If \( [k, h] = 0 \), then \( R \) is normal and hence in view of Lemma 1.3.3, \( R \) is commutative. Now assume that \( d(h_1) = 0 \) for all \( h_1 \in H(R) \cap Z(R) \). This further implies that \( d(k_1) = 0 \) and hence \( d(k_1) = 0 \) for all \( k_1 \in S(R) \cap Z(R) \). Replacing \( y \) by \( k_1y \) in (2.3.18), where \( k_1 \in S(R) \cap Z(R) \), we obtain

\[
-[F(x), y^*]k_1 + F(k_1)[y, x^*] + [F(k_1), x^*]y + k_1[d(y), x^*] = 0 \tag{2.3.32}
\]

for all \( x, y \in R \) and \( k_1 \in S(R) \cap Z(R) \). Taking \( y = k_1 \) in (2.3.18), where \( k_1 \in S(R) \cap Z(R) \), we obtain

\[
[F(k_1), x^*] = 0 \text{ for all } x \in R \text{ and } k_1 \in S(R) \cap Z(R). \tag{2.3.33}
\]

Application of expressions (2.3.18) and (2.3.33) in (2.3.32) yields that

\[
[F(y), x^*]k_1 + F(k_1)[y, x^*] + k_1[d(y), x^*] = 0 \tag{2.3.34}
\]

for all \( x, y \in R \) and \( k_1 \in S(R) \cap Z(R) \). Replacing \( y \) by \( x^* \) in (2.3.34), we obtain

\[
([F(x^*), x^*] + d(x^*, x^*])k_1 = 0 \tag{2.3.35}
\]

for all \( x \in R \) and \( k_1 \in S(R) \cap Z(R) \). Since \( k_1 \in S(R) \cap Z(R) \neq (0) \), so we are force to
conclude that

$$[F(x^*), x^*] = [d(x^*), x^*]$$

(2.3.36)

for all $x \in R$. Now, replacing $x$ by $x + k_1$ in (2.3.19), where $k_1 \in S(R) \cap Z(R)$ and using Lemma 1.3.3, we get

$$[F(x), x] = [d(x^*), x^*]$$

(2.3.37)

for all $x \in R$. Making use of (2.3.36) and (2.3.37) in (2.3.20), we obtain

$$x^*[d(x^*), x^*] = [d(x^*), x^*]x + F(x^*)[x, x^*] + x^*[d(x), x^*] = 0$$

(2.3.38)

for all $x \in R$. Substituting $x + k_1$ for $x$ in (2.3.38), where $k_1 \in S(R) \cap Z(R)$, we get

$$x^*[d(x^*), x^*] - [d(x^*), x^*]x = [d(x^*), x^*]k_1$$

(2.3.39)

$$+ F(x^*)[x, x^*] - F(k_1)[x, x^*] + x^*[d(x), x^*] - k_1[d(x), x^*] = 0$$

for all $x \in R$ and $k_1 \in S(R) \cap Z(R)$. On combining (2.3.38) and (2.3.39), we arrive at

$$2[d(x^*), x^*]k_1 + F(k_1)[x, x^*] + k_1[d(x), x^*] = 0$$

(2.3.40)

for all $x \in R$ and $k_1 \in S(R) \cap Z(R)$. Next, replacing $y$ by $x$ in (2.3.34), we get

$$F(k_1)[x, x^*] + k_1[d(x), x^*] = 0$$

(2.3.41)

for all $x \in R$ and $k_1 \in S(R) \cap Z(R)$. From (2.3.40) and (2.3.41), we obtain $2[d(x^*), x^*]k_1 = 0$ for all $x \in R$ and $k_1 \in S(R) \cap Z(R)$. Replacing $x$ by $x^*$ and using the fact that $\text{char}(R) \neq 2$, we have $[d(x), x]k_1 = 0$ for all $x \in R$ and $k_1 \in S(R) \cap Z(R)$. Since $R$ is prime, so either $S(R) \cap Z(R) = (0)$ or $[d(x), x] = 0$. If $S(R) \cap Z(R) = (0)$, we get a contradiction. Therefore the remaining possibility is $[d(x), x] = 0$ for all $x \in R$. Hence by Lemma 1.3.33, we conclude that either $d = 0$ or $R$ is commutative. Suppose that $d = 0$, then $F(xy) = F(x)y$ for all $x, y \in R$. That is, $F$ is a nonzero left centralizer of $R$ and hence $R$ is commutative by Theorem 2.3.2. This finishes the proof. \(\square\)
2.4 The condition \([F'(x), F'(x^*)] = 0\)

Long ago Herstein [128], proved that if \(R\) is a prime ring of characteristic different from two which admits a nonzero derivation \(d\) such that \([d(x), d(y)] = 0\) for all \(x, y \in R\), then \(R\) is commutative. Further, Dafif [94] generalized the above mentioned result for semiprime rings and the condition \([d(x), d(y)] = 0\) is merely satisfied on an ideal of the ring. Motivated by the above study, very recently Ali and Dar [11] studied the classical result due to Herstein mentioned above in the setting of prime rings with involution. Precisely, they proved the following result:

**Theorem 2.4.1.** Let \(R\) be a prime ring with involution of the second kind such that \(\text{char}(R) \neq 2\). If \(d\) is a nonzero derivation of \(R\) such that \([d(x), d(x^*)] = 0\) for all \(x \in R\), then \(R\) is commutative.

The aim of this section is to extend the Theorem 2.4.1 for generalized derivations of prime rings with involution. In fact, we prove the following theorem:

**Theorem 2.4.2.** Let \(R\) be a prime ring with involution of the second kind such that \(\text{char}(R) \neq 2\). If \(R\) admits a nonzero generalized derivation \(F : R \to R\) such that \([F(x), F(x^*)] = 0\) for all \(x \in R\), then \(R\) is commutative.

**Proof.** By the given hypothesis, we have

\[
[F(x), F(x^*)] = 0 \quad \text{for all } x \in R. \tag{2.4.1}
\]

Replacing \(x\) by \(h + k\) in (2.4.1), where \(h \in H(R)\) and \(k \in S(R)\), we have

\[
[F(k), F(h)] = 0 \quad \text{for all } h \in H(R) \text{ and } k \in S(R). \tag{2.4.2}
\]

Taking \(h = k_1^2\) in (2.4.2), where \(k_1 \in S(R) \cap Z(R)\), we get

\[
[F(k), F(k_1)]k_1 = 0 \quad \text{for all } k \in S(R) \text{ and } k_1 \in S(R) \cap Z(R). \tag{2.4.3}
\]

In view of Remark 1.2.7 (i), the above relation yields that

\[
[F(k), F(k_1)] = 0 \quad \text{for all } k \in S(R) \text{ and } k_1 \in S(R) \cap Z(R). \tag{2.4.4}
\]
Replacing $k$ by $h_0k_1$ in (2.4.4), where $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$, we get

$$[F(h_0), F(k_1)]h_1 + [h_0, F(k_1)]d(k_1) = 0$$

for all $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Application of (2.4.2) gives

$$[h_0, F(k_1)]d(k_1) = 0 \quad \text{for all} \quad h_0 \in H(R) \quad \text{and} \quad k_1 \in S(R) \cap Z(R).$$

Since $R$ is prime, the last relation forces that $[h_0, F(k_1)] = 0$ or $d(k_1) = 0$. Assume that $[h_0, F(k_1)] = 0$ for all $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Replacing $h_0$ by $h_0k_1$ in the last expression, where $h_0 \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, we obtain $[h_0, F(k_1)]k_1 = 0$ for all $h_0 \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. This further implies that $[k_0, F(k_1)] = 0$ for all $k_0 \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. In view of Remark 1.2.12 (iii), we have $2[y, F(k_1)] = [2y, F(k_1)] = [h_0 + k_0, F(k_1)] = [h_0, F(k_1)] + [k_0, F(k_1)] = 0$. Since $\text{char}(R) \neq 2$, the last expression yields that $[y, F(k_1)] = 0$ for all $y \in R$ and $k_1 \in S(R) \cap Z(R)$. That is, $F(k_1) \in Z(R)$ for all $k_1 \in S(R) \cap Z(R)$. Next, substituting $k$ for $h_0k_1$ in (2.4.2), where $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$, we get $[F(h_0k_1), F(h)] = [F(h)k_1, F(h)] + [h, F(h)]d(k_1) = 0$ for all $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Application of Remark 1.2.7 (i) forces that either $[h, F(h)] = 0$ for all $h \in H(R)$ or $d(k_1) = 0$ for all $k_1 \in S(R) \cap Z(R)$. Assume that $[h, F(h)] = 0$ for all $h \in H(R)$. Taking $h = kk_1$ for all $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$ and using $F(k_1) \in Z(R)$, we get $[k, d(k)]k_1^2 = 0$ for all $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. By Remark 1.2.7 (i), we conclude that $[k, d(k)] = 0$ for all $k \in S(R)$. Next, replacing $k$ by $hk_1$ in the last relation, we get $[h, d(h)]k_1^2 = 0$ for all $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. This implies that

$$[h, d(h)] = 0 \quad \text{for all} \quad h \in H(R).$$

On linearizing (2.4.7), we obtain

$$[d(h), h_0] + [d(h_0), h] = 0 \quad \text{for all} \quad h, h_0 \in H(R).$$

Which can be further written as

$$[d(h_0), h] = [h_0, d(h)] \quad \text{for all} \quad h, h_0 \in H(R).$$

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Substituting $h^2$ for $h$ in above expression, we get

\[ [d(h_0), h^2] = [h_0, d(h)]h + h[h_0, d(h)] + d(h)[h_0, h] = h[h_0, d(h)] + [h_0, h]d(h) \quad (2.4.10) \]

for all $h, h_0 \in H(R)$. Also, we have

\[ [d(h_0), h^2] = [d(h_0), h]h + h[d(h_0), h] = [h_0, d(h)]h + h[h_0, d(h)] \quad (2.4.11) \]

for all $h, h_0 \in H(R)$. Combining (2.4.10) and (2.4.11), we obtain

\[ d(h)[h_0, h] + [h_0, h]d(h) = 0 \text{ for all } h, h_0 \in H(R). \quad (2.4.12) \]

Now, taking $h_0 = kk_1$ in (2.4.12), where $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, and using the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

\[ d(h)[k, h] + [k, h]d(h) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \quad (2.4.13) \]

Replacing $h$ by $h + h_1$ in (2.4.13), where $h_1 \in H(R) \cap Z(R)$, we get

\[ d(h)[k, h] + d(h_1)[k, h] + [k, h]d(h) + [k, h]d(h_1) = 0 \quad (2.4.14) \]

for all $h \in H(R)$ and $k \in S(R)$. In view of (2.4.13), the last relation reduces to

\[ 2[k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R). \]

Since $\text{char}(R) \neq 2$, the above expression yields that

\[ [k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R). \quad (2.4.15) \]

Since $R$ is prime, we find that either $[k, h] = 0$ or $d(h_1) = 0$. If $[k, h] = 0$, then $R$ is normal and hence in view of Lemma 1.3.3, $R$ is commutative. Now we assume that $d(h_1) = 0$ for all $h_1 \in H(R) \cap Z(R)$. This further implies that $d(k_1^2) = 0$ and hence $d(k_1) = 0$ for all $k_1 \in S(R) \cap Z(R)$. Replacing $k$ by $h_0k_1$ in (2.4.2), where $h_0 \in H(R)$
and $k_1 \in S(R) \cap Z(R)$ and using $d(k_1) = 0$, we get

$$[F(h_0), F(h)]k_1 = 0$$

(2.4.16)

for all $h_0, h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. This further implies that

$$[F(h_0), F(h)] = 0 \text{ for all } h_0, h \in H(R).$$

(2.4.17)

In view of Remark 1.2.12 (iii) and (2.4.2), (2.4.17) we find that $2([F(x), F(h)]) = [F(2x), F(h)] = [F(k + h_0), F(h)] = [F(k), F(h)] + [F(h_0), F(h)] = 0$ for all $h, h_0 \in H(R), k \in S(R)$ and $x \in R$. Since $\text{char}(R) \neq 2$ the last expression yields that $[F(x), F(h)] = 0$ for all $x \in R$ and $h \in H(R)$. Proceeding as above, we arrive at

$$[F(x), F(y)] = 0 \text{ for all } x, y \in R.$$  

(2.4.18)

Substituting $xy$ for $y$ in above expression, we get

$$F(x)[F(x), y] + [F(x), x]d(y) + x[F(x), d(y)] = 0 \text{ for all } x, y \in R.$$ 

(2.4.19)

Replacing $x$ by $x + k_1$ in (2.4.19), where $k_1 \in S(R) \cap Z(R)$, we get

$$F(k_1)[F(x), y] + [F(x), x]d(y) + k_1[F(x), d(y)] = 0$$  

(2.4.20)

for all $x, y \in R$ and $k_1 \in S(R) \cap Z(R)$. Substituting $k_1y$ for $y$ in (2.4.18), where $k_1 \in S(R) \cap Z(R)$ and using $F(k_1) \in Z(R)$, we obtain

$$F(k_1)[F(x), y] + k_1[F(x), d(y)] = 0$$  

(2.4.21)

for all $x, y \in R$ and $k_1 \in S(R) \cap Z(R)$. On combining (2.4.20) and (2.4.21), we obtain

$$[F(x), x]d(y) = 0 \text{ for all } x, y \in R.$$  

(2.4.22)

This implies that either $[F(x), x] = 0$ or $d(y) = 0$. If $[F(x), x] = 0$ for all $x \in R$, then by Lemma 1.3.32, $R$ is commutative. Next, suppose that $d(y) = 0$ for all $y \in R$, this implies that $F(xy) = F(x)y$ for all $x, y \in R$. Replacing $y$ by $yz$ in (2.4.18), we find
that $F(y)[F(x), z] = 0$ for all $x, y, z \in R$. This implies that $F(y)R[F(x), z] = (0)$. Hence by the primeness of $R$ either $F = 0$ or $[F(x), z] = 0$ for all $x, z \in R$. But we have assumed that $F$ is nonzero. Thus the remaining possibility is $[F(x), z] = 0$ for all $x, z \in R$. Substituting $xw$ for $x$, we obtain $F(w)[x, z] = 0$ for all $w, x, z \in R$. This further implies that either $F = 0$ or $R$ is commutative. Henceforth, we conclude that $R$ must be commutative. Thereby the proof of the theorem is completed. \hfill \square

It is natural to enquire that what happens if the condition $[F(x), F(x^*)] = 0$ for all $x \in R$ in Theorem 2.4.2 is replaced by $F(x) \circ F(x^*) = 0$ for all $x \in R$. The following theorem gives an answer:

**Theorem 2.4.3.** Let $R$ be a prime ring with involution of the second kind such that $\text{char}(R) \neq 2$. If $R$ admits a generalized derivation $F : R \to R$ such that $F(x) \circ F(x^*) = 0$ for all $x \in R$, then $F = 0$.

**Proof.** We are given that $F$ is a generalized derivation of $R$ such that

$$F(x) \circ F(x^*) = 0 \quad \text{for all } x \in R. \quad (2.4.23)$$

This implies that

$$F(x)F(x^*) + F(x^*)F(x) = 0 \quad \text{for all } x \in R. \quad (2.4.24)$$

On linearizing, we get

$$F(x)F(y^*) + F(y)F(x^*) + F(y^*)F(x) + F(x^*)F(y) = 0 \quad (2.4.25)$$

for all $x, y \in R$. Replacing $y$ by $yh_1$ in (2.4.25), where $h_1 \in H(R) \cap Z(R)$, we get

$$0 = F(x)F(y^*)h_1 + F(x)y^*d(h_1) + F(y)F(x^*)h_1$$
$$+ d(h_1)yF(x^*) + F(x^*)F(y)h_1$$
$$+ F(x^*)yd(h_1) + F(y^*)F(x)h_1$$
$$+ d(h_1)yF(x) \quad \text{for all } x, y \in R \text{ and } h_1 \in H(R) \cap Z(R).$$

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In view of (2.4.25), we get

\[(F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x))d(h_1) = 0 \quad (2.4.26)\]

for all \(x, y \in R\) and \(h_1 \in H(R) \cap Z(R)\). Application of Remark 1.2.7 (i) yields that either \(F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x) = 0\) or \(d(h_1) = 0\). Suppose that \(d(h_1) = 0\) for all \(h_1 \in H(R) \cap Z(R)\). This further implies that \(d(h_1) = 0\) for all \(h_1 \in S(R) \cap Z(R)\).

Replacing \(y\) by \(y h_1\), in (2.4.25), where \(h_1 \in S(R) \cap Z(R)\) and using the fact that \(S(R) \cap Z(R) \neq \{0\}\), we obtain

\[-F(x)F(y^*) + F(y)F(x^*) - F(y^*)F(x) + F(x^*)F(y) = 0 \quad (2.4.27)\]

for all \(x, y \in R\). On combining (2.4.25) and (2.4.27), we get

\[F(x)F(y^*) + F(y^*)F(x) = 0 \text{ for all } x, y \in R. \quad (2.4.28)\]

Taking \(y = x^*\) and using the fact that \(\text{char}(R) \neq 2\), we obtain

\[(F(x))^2 = 0 \text{ for all } x \in R. \quad (2.4.29)\]

In view of Lemma 1.3.25, we conclude that \(F = 0\). Next, we suppose that

\[F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x) = 0 \text{ for all } x, y \in R. \quad (2.4.30)\]

Taking \(y = h_1\), where \(h_1 \in H(R) \cap Z(R)\) and using the fact that \(\text{char}(R) \neq 2\), we obtain

\[F(x + x^*)h_1 = 0 \text{ for all } x \in R \text{ and } h_1 \in H(R) \cap Z(R). \quad (2.4.31)\]

This further implies that

\[F(x + x^*) = 0 \text{ for all } x \in R. \quad (2.4.32)\]
That is,

\[ F(x) = -F(x^*) \text{ for all } x \in R. \]  \hspace{1cm} (2.4.33)

This reduces (2.4.23) into

\[ (F(x))^2 = 0 \text{ for all } x \in R. \]  \hspace{1cm} (2.4.34)

Application of Lemma 1.3.25 yields that \( F = 0 \). Thereby the proof of the theorem is completed. \( \Box \)

2.5 The condition \( F([x, x^*]) = 0 \)

In the year 1995, Bell and Daif [60] showed that if \( R \) is a prime ring admitting a nonzero derivation \( d \) such that \( d([x, y]) = 0 \) for all \( x, y \in R \), then \( R \) is commutative. This result was extended for semiprime rings in [94] by Daif. Further, for semiprime rings, Andima and Pajoohesh [25] showed that an inner derivation satisfying the above mentioned condition on a nonzero ideal of \( R \) must be zero on that ideal. Moreover, for semiprime rings with identity, they generalized this result to inner derivations of powers of \( x \) and \( y \) in [25]. In [4], Albaş and Arğaç established same result for generalized derivations in the setting of prime rings. Very Recently, Ali et al. [19], studied the above mentioned result in the setting of prime rings with involution by replacing \( y \) by \( x^* \). Precisely, they proved the following theorem:

**Theorem 2.5.1.** Let \( R \) be a prime ring with involution such that \( \text{char}(R) \neq 2 \). If \( d \) is a nonzero derivation of \( R \) such that \( d([x, x^*]) = 0 \) for all \( x \in R \) and \( S(R) \cap Z(R) \neq (0) \), then \( R \) is commutative.

It was also remarked by the authors that the assumption of \( S(R) \cap Z(R) \neq (0) \) in the hypotheses of the above theorem may be avoided. We have succeeded in removing this restriction from the hypotheses. In fact, the above theorem has been extended for generalized derivations in rings with involution as follows:

**Theorem 2.5.2.** Let \( R \) be a prime ring with involution of the second kind such that \( \text{char}(R) \neq 2 \). If \( R \) admits a nonzero generalized derivation \( F : R \to R \) such that \( F([x, x^*]) = 0 \) for all \( x \in R \), then \( R \) is commutative.
Proof. In view of our hypothesis, we have

\[ F([x, x^*]) = 0 \text{ for all } x \in R. \] (2.5.1)

Replacing \( x \) by \( h + k \) in (2.5.1), where \( h \in H(R) \) and \( k \in S(R) \), we get

\[ F([h, k]) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \] (2.5.2)

Taking \( h = k_0k_1 \) in above expression, where \( k_0 \in S(R) \) and \( k_1 \in S(R) \cap Z(R) \), we get

\[ F([k_0, k])k_1 + [k_0, k]d(k_1) = 0 \] (2.5.3)

for all \( k_0, k \in S(R) \) and \( k_1 \in S(R) \cap Z(R) \). Substituting \( h_0k_1 \) for \( k \) in (2.5.3), where \( h_0 \in H(R) \) and \( k_1 \in S(R) \cap Z(R) \), we obtain

\[ F([k_0, h_0])k_1^2 + 2[k_0, h_0]d(k_1)k_1 = 0 \] (2.5.4)

for all \( k_0 \in S(R) \), \( h_0 \in H(R) \) and \( k_1 \in S(R) \cap Z(R) \). In view of (2.5.2), we have

\[ 2[k_0, h_0]d(k_1)k_1 = 0 \text{ for all } k_0 \in S(R), \; h_0 \in H(R) \text{ and } k_1 \in S(R) \cap Z(R). \]

Since \( \text{char}(R) \neq 2 \), so the last expression yields that

\[ [k_0, h_0]d(k_1)k_1 = 0 \] (2.5.5)

for all \( k_0 \in S(R) \), \( h_0 \in H(R) \) and \( k_1 \in S(R) \cap Z(R) \). In view of Remark 1.2.7 (i), we conclude that either \([k_0, h_0] = 0\) or \(d(k_1)k_1 = 0\). First assume that \([k_0, h_0] = 0\) for all \( k_0 \in S(R) \), \( h_0 \in H(R) \) and hence \( R \) is commutative by Lemma 1.3.3. On the other hand, we assume that \(d(k_1)k_1 = 0\) for all \( k_1 \in S(R) \cap Z(R) \). This further implies that \( d(k_1) = 0 \) or \( k_1 = 0 \). Since \( k_1 = 0 \) also implies that \( d(k_1) = 0 \) for all \( k_1 \in S(R) \cap Z(R) \). Thus the relation (2.5.3) reduces to

\[ F([k_0, k])k_1 = 0 \text{ for all } k, k_1 \in S(R) \text{ and } k_1 \in S(R) \cap Z(R). \] (2.5.6)
Application of Remark 1.2.7 (i) yields that

\[ F([k_0, k]) = 0 \text{ for all } k_0, k \in S(R). \]  \hspace{1cm} (2.5.7)

Since \( \text{char}(R) \neq 2 \), every \( x \in R \) can be uniquely represented as \( 2x = h + k \) for all \( h \in H(R) \) and \( k \in S(R) \), so in view of relations (2.5.2) and (2.5.7), we are forced to conclude that

\[ F([x, k]) = 0 \text{ for all } k \in S(R) \text{ and } x \in R. \]

Similarly taking \( k = hk_1 \), in the above expression, where \( h \in H(R) \) and \( k_1 \in S(R) \cap Z(R) \), and proceeding as above we arrive at \( F([x, y]) = 0 \) for all \( x, y \in R \). This gives

\[ 0 = F([x, yz]) = [x, y]d(x) \text{ for all } x, y \in R. \]

This implies that \( [x, y]Rd(x) = (0) \) for all \( x, y \in R \). Hence, by the primeness of \( R \), we have either \( R \) is commutative or \( d(x) = 0 \) for all \( x \in R \). Thus we assume that \( d(x) = 0 \) for all \( x \in R \). This implies that

\[ F(xy) = F(x)y \text{ for all } x, y \in R. \]

Hence \( 0 = F([x, y]) = F([x, y]z + y[x, z]) = F(y)[x, z] \) for all \( x, y, z \in R \) and hence \( F(y)R[x, z] = (0) \) for all \( x, y, z \in R \). Primeness of \( R \) forces that either \( F = 0 \) or \( R \) is commutative. Since \( F \neq 0 \), so \( R \) must be commutative. This proves the theorem completely.

\[ \square \]

We now prove another theorem in this vein that is,

**Theorem 2.5.3.** Let \( R \) be a prime ring with involution of the second kind such that \( \text{char}(R) \neq 2 \). If \( R \) admits a generalized derivation \( F : R \to R \) such that \( F(x \circ x^*) = 0 \) for all \( x \in R \), then \( F = 0 \).

**Proof.** By the given hypothesis, we have

\[ F(x \circ x^*) = 0 \text{ for all } x \in R. \]  \hspace{1cm} (2.5.8)

This can be further written as

\[ F(x)x^* + xd(x^*) + F(x^*)x + x^*d(x) = 0 \text{ for all } x \in R. \]  \hspace{1cm} (2.5.9)

Replacing \( x \) by \( h_1 \) in (2.5.9), where \( h_1 \in H(R) \cap Z(R) \), we get

\[ (F(h_1) + d(h_1))h_1 = 0 \text{ for all } h_1 \in H(R) \cap Z(R). \]  \hspace{1cm} (2.5.10)
In view of Remark 1.2.7 (i), we conclude that either \( F(h_1) + d(h_1) = 0 \) or \( h_1 = 0 \). Since \( h_1 = 0 \) implies that \((F + d)h_1 = 0 \) i.e., \( F(h_1) + d(h_1) = 0 \) for all \( h_1 \in H(R) \cap Z(R) \). Substituting \( k_1^2 \) for \( h_1 \) in the above expression, where \( k_1 \in S(R) \cap Z(R) \), we obtain

\[
F(k_1)k_1 + 3d(k_1)k_1 = 0 \quad \text{for all} \quad k_1 \in S(R) \cap Z(R).
\] (2.5.11)

Now taking \( x = k_1 \) in (2.5.9), where \( k_1 \in S(R) \cap Z(R) \) and using the fact that \( \text{char}(R) \neq 2 \), we get

\[
F(k_1)k_1 + d(k_1)k_1 = 0 \quad \text{for all} \quad k_1 \in S(R) \cap Z(R).
\] (2.5.12)

Combining (2.5.11) and (2.5.12), we obtain

\[
2d(k_1)k_1 = 0 \quad \text{for all} \quad k_1 \in S(R) \cap Z(R).
\] (2.5.13)

Since \( \text{char}(R) \neq 2 \), the last expression yields that

\[
d(k_1)k_1 = 0 \quad \text{for all} \quad k_1 \in S(R) \cap Z(R).
\] (2.5.14)

This further implies that \( d(k_1) = 0 \) for all \( k_1 \in S(R) \cap Z(R) \). Thus (2.5.12) reduces to \( F(k_1)k_1 = 0 \) and hence \( F(k_1) = 0 \) for all \( k_1 \in S(R) \cap Z(R) \). On linearizing (2.5.9), we find that

\[
0 = F(x)y^* + F(y)x^* + xd(y^*)
+ yd(x^*) + F(x^*)y + F(y^*)x
+ x^*d(y) + y^*d(x) \quad \text{for all} \quad x, y \in R.
\] (2.5.15)

Substituting \( k_1 \) for \( x \) in (2.5.15), where \( k_1 \in S(R) \cap Z(R) \), we get

\[
(F(y^* - y) + d(y^* - y))k_1 = 0 \quad \text{for all} \quad y \in R \text{ and } k_1 \in S(R) \cap Z(R).
\] (2.5.16)

The above relation forces that

\[
F(y^* - y) + d(y^* - y) = 0 \quad \text{for all} \quad y \in R.
\] (2.5.17)
Substituting $h - k$ for $y$ in (2.5.17), where $h \in H(R)$ and $k \in S(R)$ and using the fact that $\text{char}(R) \neq 2$, we get $F(k) + d(k) = 0$ for all $k \in S(R)$. Now replacing $k$ by $k_1 h$ in the last expression, where $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$ and making use of $F(k_1) = d(k_1) = 0$, we obtain $d(h)k_1 = 0$ for all $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$ and hence $d(h) = 0$ for $h \in H(R)$. Since $kk_1 \in H(R)$, where $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, so we obtain $d(kk_1) = 0$. Hence, $d(k) = 0$ for all $k \in S(R)$. Application of Remark 1.2.12 (iii) yields that $d(x) = 0$ for all $x \in R$. Thus in view of (2.5.15), we have

\[ F(x)y^* + F(y)x^* + F(x^*)y + F(y^*)x = 0 \quad \text{for all } x, y \in R. \tag{2.5.18} \]

Taking $y = k_1$ in (2.5.18), where $k_1 \in S(R) \cap Z(R)$, we get

\[ F(x^* - x)k_1 = 0 \quad \text{for all } x \in R \text{ and } k_1 \in S(R) \cap Z(R). \tag{2.5.19} \]

Again application of Remark 1.2.7 (i) yields that $F(x^* - x) = 0$ for all $x \in R$. Substituting $h - k$ for $x$ in the last expression, we arrive at $F(k) = 0$ for all $k \in S(R)$. This further yields that $F(h) = 0$ for all $h \in H(R)$. By Remark 1.2.7 (iii), we conclude that $F(x) = 0$ for all $x \in R$. This proves the theorem completely.

\[ \square \]

### 2.6 Some examples

We begin this section by the following examples which shows that the condition of the second kind involution i.e., $S(R) \cap Z(R) \neq (0)$ in the hypothesis of Theorems 2.2.1, 2.3.3, 2.4.2 and 2.5.2 is not superfluous.

**Example 2.6.1.** Let

\[ R = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in Z \right\}. \]

Of course, $R$ is a prime ring with matrix addition and matrix multiplication. Define mappings $d : R \rightarrow R$, $F : R \rightarrow R$ and $* : R \rightarrow R$ such that

\[ d \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 0 & -b \\ c & 0 \end{array} \right) \quad \text{for all } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in R, \]

\[ F \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & -b \\ c & a \end{array} \right) \quad \text{for all } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in R, \]

\[ * \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a + c & b + d \\ c & d \end{array} \right) \quad \text{for all } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in R. \]

\[ \]
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R
\]
and
\[
F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.
\]

Obviously, \( Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z} \right\} \). Then \( x^* = x \) for all \( x \in Z(R) \), and hence \( Z(R) \subseteq H(R) \), which shows that the involution is of the first kind. This implies that \( S(R) \cap Z(R) = \{0\} \). It is easy to see that the mappings \( d \) and \( F \) are nonzero derivation and generalized derivation on \( R \). Further, the following conditions: (i) \( [d(x), d(x^*)] - [x, x^*] = 0 \), (ii) \( F([x, x^*]) = 0 \) for all \( x \in R \), are satisfied. However, \( R \) is not commutative. Hence, the condition of the second kind involution is crucial in Theorem 2.2.1 and Theorem 2.5.2.

**Example 2.6.2.** Consider the ring \( R \) as in Example 2.6.1 and take \( F \) to be the identity map on \( R \). Then \( F \) is a generalized derivation with an associated derivation \( d = 0 \), and \( F \) satisfies the identity \( [F(x), x^*] = 0 \) for all \( x \in R \). However, \( R \) is not commutative. This shows that the hypothesis of the second kind involution is essential in Theorem 2.3.3.

**Example 2.6.3.** Consider the ring \( R \) as in Example 2.6.1. Define a map \( F : R \rightarrow R \) by
\[
F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.
\]
Then, \( F \) is a nonzero generalized derivation and \( F \) satisfies the identity \( [F(x), F(x^*)] = 0 \) for all \( x \in R \). However, \( R \) is not commutative. Hence, in Theorem 2.4.2, the hypothesis of the second kind involution can not be omitted.