CHAPTER 1

Some Preliminaries
Chapter 1

Some preliminaries

1.1 Introduction

The object of the present chapter is to introduce basic definitions, preliminary notions and some key results which we shall require for the development of the subject matter in the present thesis. The knowledge of some elementary concepts like groups, rings, ideals, fields, modules, homomorphism etcetera have been pre-assumed. Throughout the thesis, unless otherwise mentioned, \( R \) will denote an associative ring (may be without unity) containing at least two elements. Most of the material included in this chapter occurs in standard literature like, Ambrose [26], Beidar et al. [50], Bonsall and Duncan [85], Herstein [124], Jacobson [137], McCoy [173] and Rudin [186].

1.2 Some definitions and examples

In the present section we give a brief exposition of some important terminology in the theory of rings and algebras. Examples and counter examples are also included in this section to make the matter presented in the section self explanatory and to give a clear sketch of the various notions. We start our discussion with the following definition:

Definition 1.2.1 (Prime ideal). A proper ideal \( P \) of \( R \) is called a prime ideal of \( R \) if for any two ideals \( A \) and \( B \) of \( R \), \( AB \subseteq P \) implies \( A \subseteq P \) or \( B \subseteq P \).

Remark 1.2.1. If \( P \) is an ideal in a ring \( R \), then the following conditions are equivalent:

(i) \( P \) is a prime ideal of \( R \).

(ii) If \( a, b \in P \) such that \( aRb \subseteq P \), then \( a \in P \) or \( b \in P \).
(iii) If \((a)\) and \((b)\) are principal ideals in \(R\) such that \((a)(b) \subseteq P\), then \(a \in P\) or \(b \in P\).

(iv) If \(U\) and \(V\) are left (right) ideals in \(R\) such that \(UV \subseteq P\), then \(U \subseteq P\) or \(V \subseteq P\).

**Definition 1.2.2** (Prime ring). A ring \(R\) is said to be a **prime ring** if and only if the zero ideal is a prime ideal in \(R\).

**Remark 1.2.2.** Equivalently, a ring \(R\) is a prime ring if and only if any one of the following holds:

(i) If \(A\) and \(B\) are ideals in \(R\) such that \(AB = (0)\), then \(A = (0)\) or \(B = (0)\).

(ii) If \(a, b \in R\) such that \(aRb = (0)\), then \(a = 0\) or \(b = 0\).

**Definition 1.2.3** (Semiprime ideal). An ideal \(P\) in a ring \(R\) is said to be a **semiprime ideal** in \(R\) if for every ideal \(I\) of \(R\), \(I^2 \subseteq P\) implies \(I \subseteq P\).

**Remark 1.2.3.** (i) A prime ideal is necessarily semiprime, but the converse need not be true in general.

(ii) Intersection of prime (semiprime) ideals is semiprime. Thus, in the ring \(\mathbb{Z}\) of integers, ideal \((2) \cap (3) = (6)\) is semiprime which is not prime.

**Definition 1.2.4** (Semiprime ring). A ring \(R\) which has no nonzero nilpotent ideal is said to be a **semiprime ring**.

**Remark 1.2.4.** (i) A ring \(R\) is semiprime if and only if for any \(a \in R\), \(aRa = (0)\) implies that \(a = 0\).

(ii) The **radical** of \(R\), denoted by \(\text{rad}(R)\), is the intersection of all maximal ideals of \(R\).

(iii) If \(\text{rad}(R) = (0)\), then \(R\) is called semisimple.

**Definition 1.2.5** (Dense right (left) ideal). A right (resp. left) ideal \(I\) of \(R\) is said to be dense right (resp. left) ideal of \(R\) if for any \(0 \neq r_1 \in R\), \(r_2 \in R\) there exists \(r \in R\) such that \(r_1 r \neq 0\) and \(rr_2 \in I\) (resp. \(rr_1 \neq 0\) and \(rr_2 \in I\)).

The collection of all dense right ideal of \(R\) will be denoted by \(D(R)\).

**Remark 1.2.5.** Let \(I, J, S \in D(R)\) and let \(f : I \to R\) be homomorphism of right \(R\)-modules. Then
(i) \( f^{-1}(J) = \{ a \in I \mid f(a) \in J \} \in D(R) \).

(ii) \( I \cap J \in D(R) \).

(iii) \( IJ \in D(R) \).

**Definition 1.2.6 (Maximal right ring of quotients).** Let \( R \) be a semiprime ring, \( \mathfrak{S} \) be the set of all pairs \((U, f)\) where \( U \neq (0) \) is a dense right ideal of \( R \) and \( f : U \rightarrow R \) is a right \( R \)-module map of \( U \) into \( R \). Define a relation ' \( \sim \) ' on \( \mathfrak{S} \) such that \((U, f) \sim (V, g)\) if \( f = g \) on some dense right ideal \( W \neq (0) \) of \( R \), where \( W \subseteq U \cap V \). It can be easily check that \( \sim \) is an equivalence relation on \( \mathfrak{S} \). Let \( Q_r(R) \) be the set of equivalence classes of \( \mathfrak{S} \). Denote the equivalence class determined by \((U, f)\) as \( \bar{f} \). For \( \bar{f} = cl(U, f) \), \( \bar{g} = cl(V, g) \in Q_r(R) \), define addition and multiplication on \( Q_r(R) \) as \( \bar{f} + \bar{g} = cl(U \cap V, f + g) \) and \( \bar{f} \bar{g} = cl(g^{-1}(U), fg) \). Thus \( Q_r(R) \) forms an associative ring with identity relative to above defined operations known as maximal right ring of quotients or right Utumi quotient ring of \( R \).

**Remark 1.2.6.** Let \( R \) be a semiprime ring. Then \( Q_r(R) \) satisfies:

(i) \( R \) is a subring of \( Q_r(R) \).

(ii) For all \( q \in Q_r(R) \) there exists \( I \in D(R) \) such that \( qI \subseteq R \).

(iii) For all \( q \in Q_r(R) \) and \( I \in D(R) \), \( qI = 0 \) if and only if \( q = 0 \).

(iv) For all \( I \in D(R) \) and \( f : I_R \rightarrow R_R \) there exists \( q \in Q_r(R) \) such that \( f(x) = qx \) for all \( x \in I \).

Furthermore, properties (i)-(iv) characterize ring \( Q_r(R) \) up to isomorphism.

**Definition 1.2.7 (Symmetric ring of quotients).** Let \( R \) be a semiprime ring and \( \mathbb{I} = I(R) = \{ I \mid I \text{ is an ideal of } R \text{ and } l(I) = 0 \} \). We note that \( \mathbb{I} \) is closed under products and finite intersections. Then the symmetric ring of quotients of \( R \) denoted by \( Q_s(R) \) is defined as follows:

\[
Q_s(R) = \{ q \in Q_r(R) \mid qJ \cup Jq \subseteq R \text{ for some } J \in \mathbb{I} \}.
\]

**Definition 1.2.8 (Center of ring).** The center of a ring \( R \) is the set of all those elements of \( R \) which commute with every element of \( R \) and is denoted as \( Z(R) \) i.e., \( Z(R) = \{ x \in R \mid xr = rx \text{ for all } r \in R \} \).
Thus, a ring $R$ is commutative if and only if $Z(R) = R$.

**Remark 1.2.7.** (i) The center of a prime ring is free from zero divisors.

(ii) The center of a semiprime ring contains no nonzero nilpotent element.

**Definition 1.2.9** (Extended centroid). The center $C$ of $Q_r(R)$ is known as extended centroid of $R$.

**Remark 1.2.8.** If $R$ is a prime ring, then extended centroid of $R$ is a field.

**Definition 1.2.10** (Central closure). Let $R$ be a semiprime ring. Then the subring $RC$ of $Q_r(R)$ is said to be the central closure of $R$. Further, $R$ is called centrally closed if it coincides with its central closure i.e., $R = RC$.

**Definition 1.2.11** (Characteristic of a ring). Let $R$ be a ring. If there exists a positive integer $n$ such that $nx = 0$ for all $x \in R$, then the smallest positive integer with this property is called the characteristic of the ring $R$ and is denoted by $\text{char}(R)$. If no such positive integer exists, then $R$ is said to be of characteristic zero.

**Definition 1.2.12** (Torsion free element). An element $x \in R$ is called $n$-torsion free if $nx = 0$ implies $x = 0$.

If $nx = 0$ implies $x = 0$ for all $x \in R$, we say that the ring $R$ is $n$-torsion free.

**Definition 1.2.13** (Lie and Jordan Structures). Let $R$ be a ring. Then using its operations, two new products can be induced as follows:

(i) for all $x, y \in R$, the Lie product $[x, y] = xy - yx$,

(ii) for all $x, y \in R$, the Jordan product $x \circ y = xy + yx$.

**Remark 1.2.9.** For any $x, y, z \in R$, the following identities are obvious,

(i) $[xy, z] = x[y, z] + [x, z]y$,

(ii) $[x, yz] = [x, y]z + y[x, z]$,

(iii) $[[x, y], z] + [[y, z], x] + [[x, z], y] = 0$, (Jacobi’s Identity)

(iv) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$, 

4
(v) \((xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]\).

**Definition 1.2.14** (Lie (Jordan) subring). A nonempty subset \(A\) of \(R\) is said to be a **Lie** (resp. **Jordan**) **subring** of \(R\) if \(A\) is an additive subgroup of \(R\) and for any \(a, b \in A\), implies that \([a, b]\) (resp. \((a \circ b)\)) is also in \(A\).

**Definition 1.2.15** (Lie (Jordan) ideal). A nonempty subset \(U\) of \(R\) is said to be a **Lie** (resp. **Jordan** ideal) of \(R\) if \(U\) is an additive subgroup of \(R\) and whenever \(u \in U\) and \(r \in R\), then \([u, r] \in U\) (resp. \((u \circ r) \in U\)).

**Definition 1.2.16** (Derivation and Jordan derivation). An additive mapping \(d : R \to R\) is said to be a **derivation** (resp. **Jordan derivation**) on \(R\) if \(d(xy) = d(x)y + xd(y)\) (resp. \(d(x^2) = d(x)x + xd(x)\)) holds for all \(x, y \in R\).

**Example 1.2.1.** The most natural example of a non trivial derivation is the usual differentiation on the ring \(F[x]\) of polynomials defined over a field \(F\).

**Definition 1.2.17** (Inner derivation). For a fixed \(a \in R\), define \(d_a : R \to R\) such that \(d_a(x) = [a, x]\) for all \(x \in R\). Then \(d\) is called an **inner derivation** of \(R\) associated with \(a\) and usually denoted by \(I_a\).

It is obvious to see that every inner derivation on a ring \(R\) is a derivation. But the converse need not be true in general.

**Example 1.2.2.** Let \(R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}\). Define a mapping \(d : R \to R\) as follows:

\[
d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}
\]

for all \(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in R\).

It can be easily seen that \(d\) is a derivation on \(R\) which is not an inner derivation on \(R\).

**Remark 1.2.10.** If \(d\) is a derivation on \(R\) and \(r \in Z(R)\), then \(d(r) \in Z(R)\).

**Definition 1.2.18** (Jordan triple derivation). An additive mapping \(d : R \to R\) is said to be a **Jordan triple derivation** on \(R\) if \(d(xyx) = d(x)yx + xd(y)x + xyd(x)\) holds for all \(x, y \in R\).
Definition 1.2.19 ((α, β)-derivation). Let α and β be endomorphisms of \( R \). An additive mapping \( d : R \to R \) is said to be an \((α, β)\)-derivation on \( R \) if \( d(xy) = d(x)α(y) + β(x)d(y) \) holds for all \( x, y \in R \).

Definition 1.2.20 (Jordan (α, β)-derivation). Let α and β be endomorphisms of \( R \). An additive mapping \( d : R \to R \) is said to be a Jordan (α, β)-derivation on \( R \) if \( d(x^2) = d(x)α(x) + β(x)d(x) \) holds for all \( x \in R \).

Definition 1.2.21 (Jordan triple (α, β)-derivation). Let α and β be endomorphisms of \( R \). An additive mapping \( d : R \to R \) is said to be a Jordan triple (α, β)-derivation on \( R \) if \( d(xyx) = d(x)α(yx) + β(x)d(y)α(x) + β(yx)d(y) \) holds for all \( x, y \in R \).

Definition 1.2.22 (Centralizer). An additive mapping \( T : R \to R \) is called a left (resp. right) centralizer if \( T(xy) = T(x)y \) (resp. \( T(xy) = xT(y) \)) holds for all \( x, y \in R \). Also, \( T \) is a centralizer if it is both a left as well as a right centralizer.

Remark 1.2.11. (i) If \( T \) is a centralizer on a semiprime ring \( R \), then there exists an element \( λ ∈ C \), the extended centroid of \( R \) such that \( T(x) = λx \) for all \( x ∈ R \).

(ii) If \( R \) is a ring with identity, then \( T \) is a left centralizer of \( R \) if and only if \( T(x) = ax \) for all \( x ∈ R \) and some fixed element \( a \) of \( R \).

Definition 1.2.23 (Generalized derivation). An additive mapping \( F : R \to R \) is said to be a generalized derivation on \( R \) if there exists a derivation \( d : R \to R \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y ∈ R \).

Definition 1.2.24 (Generalized Jordan derivation). An additive mapping \( F : R \to R \) is said to be a generalized Jordan derivation on \( R \) if there exists a Jordan derivation \( d : R \to R \) such that \( F(x^2) = F(x)x + xd(x) \) for all \( x ∈ R \).

Definition 1.2.25 (Generalized Jordan triple derivation). An additive mapping \( F : R \to R \) is said to be a generalized Jordan triple derivation on \( R \) if there exists a Jordan triple derivation \( d : R \to R \) such that \( F(xyx) = F(x)yx + xd(y)x + xyd(x) \) for all \( x, y ∈ R \).

Clearly, generalized derivation covers the concept of derivation and left centralizer. If we take \( F = d \), then generalized derivation becomes a derivation and if we take \( d = 0 \) then it becomes a left centralizer. By the following example, we can easily see that every generalized derivation need not be a derivation.
Example 1.2.3. Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \). Define \( F, d : R \to R \) such that

\[
F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}
\]

for all \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R \).

Then, \( F \) is a generalized derivation of \( R \) with associated derivation \( d \), but not a derivation of \( R \).

Definition 1.2.26 (Generalized \((\alpha, \beta)\)-derivation). Let \( \alpha \) and \( \beta \) be endomorphisms of \( R \). An additive mapping \( F : R \to R \) is said to be a generalized \((\alpha, \beta)\)-derivation on \( R \) if there exists an \((\alpha, \beta)\)-derivation \( d : R \to R \) such that \( F(xy) = F(x)\alpha(y) + \beta(x)d(y) \) for all \( x, y \in R \).

Definition 1.2.27 (Generalized Jordan \((\alpha, \beta)\)-derivation). Let \( \alpha \) and \( \beta \) be endomorphisms of \( R \). An additive mapping \( F : R \to R \) is said to be a generalized Jordan \((\alpha, \beta)\)-derivation on \( R \) if there exists a Jordan \((\alpha, \beta)\)-derivation \( d : R \to R \) such that \( F(x^2) = F(x)\alpha(x) + \beta(x)d(x) \) for all \( x \in R \).

Definition 1.2.28 (Generalized Jordan triple \((\alpha, \beta)\)-derivation). Let \( \alpha \) and \( \beta \) be endomorphisms of \( R \). An additive mapping \( F : R \to R \) is said to be a generalized Jordan triple \((\alpha, \beta)\)-derivation on \( R \) if there exists a Jordan triple \((\alpha, \beta)\)-derivation \( d : R \to R \) such that \( F(xyx) = F(x)\alpha(yx) + \beta(yd(y)\alpha(x) + \beta(xy)d(x) \) for all \( x, y \in R \).

Definition 1.2.29 (Involution). An involution on a ring \( R \) is a map \( * : R \to R \) satisfying the following conditions:

\[
(i) \quad (x + y)^* = x^* + y^*, \\
(ii) \quad (xy)^* = y^*x^*, \\
(iii) \quad (x^*)^* = x \quad \text{for all} \quad x, y \in R.
\]

A ring equipped with an involution is called a \(*\)-ring or ring with involution.
Example 1.2.4. Let $M_{n \times n}(\mathbb{R})$ be the set of all $n \times n$ matrices over the real field $\mathbb{R}$. Take $A \in M_{n \times n}$ and define $*: M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$ such that $A^* = A^T$, the transpose of $A$. Then $*$ is an involution on $M_{n \times n}(\mathbb{R})$ and hence $M_{n \times n}(\mathbb{R})$ is a ring with involution.

Definition 1.2.30 (Symmetric element). An element $x$ of a $*$-ring $R$ is said to be symmetric if $x^* = x$.

Definition 1.2.31 (Skew symmetric element). An element $x$ of a $*$-ring $R$ is said to be skew symmetric if $x^* = -x$.

The set of all symmetric and skew-symmetric elements of $R$ are denoted by $H(R)$ and $S(R)$, respectively. Therefore we have

$$H(R) = \{x \in R \mid x^* = x\} \text{ and } S(R) = \{x \in R \mid x^* = -x\}.$$  

Definition 1.2.32 (Normal ring). A ring $R$ with involution $*$ is said to be normal if $xx^* = x^*x$ for all $x \in R$. Equivalently, $R$ is said to be normal if $hk = kh$ for all $h \in H(R)$ and $k \in S(R)$.

Example 1.2.5. Let $R$ be the ring of real quaternions. Then the mapping $*: R \to R$ defined by $x^* = \bar{x}$, where $\bar{x}$ denotes the conjugate of $x$, is an involution on $R$ and $xx^* = x^*x$ for all $x \in R$. That is, $R$ is normal.

Remark 1.2.12. (i) Let $R$ be a ring with involution $*$. The set $H(R)$ of symmetric elements of $R$ form a Jordan subring of $R$ and the set $S(R)$ of skew symmetric elements of $R$ form a Lie ideal of $R$.

(ii) The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case, $S(R) \cap Z(R) \neq \{0\}$.

(iii) If $R$ is a 2-torsion free $*$-ring, then every element $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

(iv) Let $R$ be a simple ring with involution $*$ of characteristic different from two. In this case, since $2R$ is an ideal of $R$, so must be $R$, for any $x \in R$ $\frac{x}{2}$ makes sense and so, since

$$x = \frac{x + x^*}{2} + \frac{x - x^*}{2}, \quad R = H(R) + S(R), \quad H(R) \cap S(R) = \{0\}.$$
Definition 1.2.33 (Algebra). Let $A$ be a nonempty set on which there are defined binary operations of addition and multiplication and also a scalar multiplication by elements of a field $F$. Then $A$ is an algebra over the field $F$ if the following conditions are satisfied:

(i) $A$ is a vector space over $F$ with respect to the operations of addition and scalar multiplication;

(ii) $A$ is a ring with respect to the operations of addition and multiplication;

(iii) if $x, y \in A$ and $\alpha \in F$, then $(\alpha x)y = x(\alpha y) = \alpha(xy)$.

Remark 1.2.13. The field $F$ is called the scalar field of $A$. If $F = \mathbb{R}$, field of real numbers, then $A$ is called a real algebra. Moreover, if $F = \mathbb{C}$, field of complex numbers, then $A$ is called a complex algebra.

Definition 1.2.34 (Normed space). A vector space $A$ is said be a normed space if for every $x \in A$ there is associated a nonnegative real number $\|x\|$, called the norm of $x$, in such a way that

(i) $\|x\| \geq 0$ for all $x \in A$;

(ii) $\|x\| = 0 \iff x = 0$;

(iii) $\|\alpha x\| = |\alpha|\|x\|$ if $x \in A$ and $\alpha$ is a scalar;

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in A$.

Definition 1.2.35 (Hilbert space). A complete inner product space is called a Hilbert space.

Definition 1.2.36 (Normed algebra). A normed algebra $A$ is a normed space which is an algebra such that

$\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

Definition 1.2.37 (Semisimple algebra). An algebra $A$ is called a semisimple algebra if rad($A$) = (0).

Definition 1.2.38 (Banach algebra). A complete normed algebra is called a Banach algebra.

Remark 1.2.14. Let $p(t) = \sum_{r=0}^{n} b_{r} t^{r}$ is a polynomial in real variable $t$ for infinite values of $t$ and each $b_{r} \in A$. If $p(t) \in Z(A)$, then each $b_{r}$ lies in $Z(A)$. 

9
1.3 Some known results

In this section we write some well-know results which will be used in the subsequent chapters.

Lemma 1.3.1. [8, Theorem 1] Let \( R \) be an 2-torsion free semiprime ring, and \( \alpha, \beta \) be surjective automorphisms of \( R \). Let \( d : R \rightarrow R \) be surjective mapping. Then the following conditions are equivalent:

(i) \( d \) is a Jordan \((\alpha, \beta)^t\)-derivation;

(ii) \( d(xy) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x) + \beta(xy)d(x) \) for all \( x, y \in R \).

Lemma 1.3.2. [12, Proposition 2.2] Let \( R \) be a 2-torsion free semiprime ring with involution \('\ast'\). If an additive mapping \( f \) of \( R \) into itself is such that \( [f(x), x^*] \in Z(R) \) for all \( x \in R \), then \( [f(x), x^*] = 0 \) for all \( x \in R \).

Lemma 1.3.3. [12, Lemma 2.1] Let \( R \) be a prime ring with involution \('('') such that \( char(R) \neq 2 \). If \( S(R) \cap Z(R) \neq (0) \) and \( R \) is normal, then \( R \) is commutative.

Lemma 1.3.4. [35, Theorem 4.4] Let \( R \) be a prime ring such that \( char(R) \neq 2 \) and \( I \) be a nonzero ideal of \( R \). If \( R \) admits a derivation \( \delta \) such that \( \delta(x) \circ \delta(y) = x \circ y \) for all \( x, y \in I \), then \( R \) is commutative.

Lemma 1.3.5. [39, Theorem 2.3] Let \( R \) be a prime ring with \( char(R) \neq 2 \) and \( U \) be a noncommutative Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \). Suppose that \( \theta, \phi \) are endomorphism on \( R \) such that \( \theta \cdot \) is one-one, onto and \( d \) is a \((\theta, \phi)\)-derivation of \( R \). If \( F : R \rightarrow R \) is generalized Jordan \((\theta, \phi)\)-derivation on \( U \), then \( F \) is generalized \((\theta, \phi)\)-derivation on \( U \).

Lemma 1.3.6. [57, Lemma 1.2] Let \( R \) be a prime ring satisfying an identity \( q(X) = 0 \), where \( q(X) \) is a polynomial in a finite number of non-commuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime \( p \) for which the ring of \( 2 \times 2 \) matrices over \( GF(p) \) satisfies \( q(X) = 0 \), then \( R \) is commutative.

Lemma 1.3.7. [57, Theorem 2.2] Let \( n > 1 \), and let \( R \) be a prime ring. If \( R \) admits a nonzero derivation \( d \) such that \( d(x^n) \in Z(R) \) for all \( x \in R \), then \( R \) is infinite and commutative.
Lemma 1.3.8. [57, Theorem 2.3] Let $R$ be a prime ring with $\text{char}(R) \neq 2$. If $R$ admits a nonzero derivation $d$ such that $d([x,y]) \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.

Lemma 1.3.9. [59, Theorem 1] Let $R$ be semiprime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a derivation $d$ of $R$ such that $[d(x), d(y)] = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$.

Lemma 1.3.10. [64, Lemma 4] If $U \not\subseteq Z(R)$ is a Lie ideal of a prime ring with $\text{char}(R) \neq 2$ and $a, b \in R$ such that $aUb = 0$, then either $a = 0$ or $b = 0$.

Lemma 1.3.11. [64, Lemma 2] Let $R$ be a 2-torsion free prime ring and $U$ be a Lie ideal of $R$. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.

Lemma 1.3.12. [65, Lemma 4] Let $R$ be a semiprime ring and let $a, b \in R$. If for all $x \in R$ the relation

$$axb + bxa = 0$$

holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.

Lemma 1.3.13. [85, Proposition 13, p.91] Let $A$ be a Banach algebra and $a(ab - ba) = (ab - ba)a$ for all $a, b \in A$. Then $r(ab - ba) = 0$.

Lemma 1.3.14. [86, Theorem 1] Let $I$ be an ideal of prime ring $R$ such that and $A$ be an additive subgroup (of $RC$) generated by $\{p(a_1, ..., a_n) \mid a_1, ..., a_n \in I\}$. Then either $p(x_1, ..., x_n)$ is central valued or $A$ contains a proper Lie ideal of $R$, except in the only one case where $R$ is the ring of all $2 \times 2$ matrices over $GF(2)$, the integers modulo 2.

Lemma 1.3.15. [87, Theorem 2] Assume that $R$ is a prime ring and $U$ its Utumi quotient ring. For any rational submodule $M$ of $U$, the GPI's satisfied by $M$ is same as GPI's satisfied by $U$.

Lemma 1.3.16. [100, Theorem 1] Let $R$ be a noncommutative prime ring and $k, m, n, r$ are fixed positive integers. If there exist a generalized derivation $g$ of $R$ such that $[g(x^r)x^r, x^r]_k = 0$ for all $x \in R$, then there exist an element $a \in U$ such that $g(x) = xa$ for all $x \in R$.

Lemma 1.3.17. [106, Theorem 2.2] If $A$ is a closed prime algebra over $\phi$ and $F$ is an extension field of $\phi$, then $A \otimes_\phi F$ is a closed prime algebra over $F$. 

11
Lemma 1.3.18. [107, Theorem] Every simple ring having a minimal one-sided ideal is locally simple matrix ring over a division ring $D$ i.e., there exist a division ring $D$ such that each finite subset $B$ of $S$ is contained in a subring of $S$ which is isomorphic to $D_n$, for some natural number $n$.

Lemma 1.3.19. [112, Theorem 1.1] Let $R$ be a semiprime ring and $m, n$ be fixed positive integers larger than 1, such that $R$ satisfies one of the following conditions (a) 
$[x^n, y^m] \in Z(R)$, (b) $x^n \circ y^m \in Z(R)$, where $x, y \in R$, then $R$ is commutative.

Lemma 1.3.20. [124, Sublemma] Let $R$ be ring having no nonzero nilpotent ideals in which $2x = 0$ implies $x = 0$. If $a \in R$ commutes with all $ax - xa, x \in R$, then $a$ is in center of $R$.

Lemma 1.3.21. [125, Lemma 2] Let $R$ be a 2-torsion free semiprime ring and $U$ be a Lie ideal of $R$. Suppose that $t \in R$ commutes with every element of $[U, U]$. Then $t$ commutes with every element of $U$.

Lemma 1.3.22. [126, Theorem 2.1.2] Let $R$ be any ring and suppose that $A$ is both a subring of $R$ and a Lie ideal of $R$. Then $A$ contains $R[A, A]R$ of $R$ and a Lie ideal of $R$. In particular, if $R$ is semiprime, then:

(i) If $A$ is not commutative, then $A$ contains a nonzero ideal of $R$.

(ii) If $A$ is commutative, then if $a \in A$ we have $a^2 \in Z(R)$.

(iii) If $A$ is commutative and $R$ is 2-torsion free, then $A \subset Z(R)$.

Lemma 1.3.23. [129, Theorem 1] Let $R$ be a prime ring and let $d \neq 0$ be a derivation of $R$. Suppose that $a \in R$ such that $ad(x) = d(x)a$ for all $x \in R$. Then:

(i) If $R$ is not of characteristic 2, $a$ must be in $Z(R)$.

(ii) If $R$ is of characteristic 2, then $a^2 \in Z(R)$.

Lemma 1.3.24. [132, Lemma 1] Let $R$ be 2-torsion free semiprime ring, $I$ a nonzero ideal of $R$, and $a \in R$. If $[a, [x, y]] \in Z(R)$, then $a \in Z(R)$.

Lemma 1.3.25. [134, Theorem 3] Let $n > 1$ be a fixed integer and $R$ be a prime ring with characteristic not equal to 2, ..., $n - 1$. If $f : R \to R$ is a generalized derivation such that $f(x)^n = 0$ for all $x \in R$, then $f = 0$. 

12
Lemma 1.3.26. [138, Theorem] Let $A$ be semi-simple Banach algebra over $F$. Let $D$ be an additive derivation on $A$. Then $A$ contains a central idempotent $e$ such that $eA$ and $(1-e)A$ are closed under $D$, $D((1-e)A$ is continuous and $eA$ is finite dimensional.

Lemma 1.3.27. [140, Theorem] Let $R$ be a prime ring, $d$ a nonzero derivation and $I$ a nonzero two-sided ideal of $R$. Let $f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$ be a differential identity in $I$, that is

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \text{ for all } r_1, \ldots, r_n \in I.$$

Then one of the following holds:

(i) Either $d$ is an inner derivation in $Q$, the Martindale quotient ring of $R$, in the sense that there exist $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$, and $I$ satisfies the generalized polynomial identity

$$f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0$$

or

(ii) $I$ satisfies the generalized polynomial identity

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Lemma 1.3.28. [147, Theorem 2] Let $d \neq 0$ be a derivation of a prime ring $R$ and $a$ be an element of $R$ such that $[a, d(U)] \subseteq Z(R)$, then either $a \in U$ or $U \subseteq Z(R)$.

Lemma 1.3.29. [148, Proposition] Let $A$ be an algebra over infinite field $k$ and $K$ be a field extension over $k$. If $A$ satisfy a GPI $p(X_1, \ldots, X_m)$, so does $A \otimes_k K$.

Lemma 1.3.30. [149, Theorem 3] Let $R$ be a semiprime ring, $U$ its maximal right quotient ring and $I_R$ a dense right $R$-submodule of $U_R$. Then $I$ and $U$ satisfies same differential identities.

Lemma 1.3.31. [153, Theorem 1] Let $R$ be a noncommutative prime ring with derivation $d$ and $\delta$ such that

$$[d(x^m)x^n - x^p\delta(x^n), x^R]_k \text{ for all } x \in R,$$
where $m, n, p, q, r, k$ are fixed positive integers. Then $d = 0$ and $\delta = 0$.

**Lemma 1.3.32.** [167, Theorem 3] Let $R$ be a prime ring and let $S = RC$ be the central closure of $R$. Then $S$ satisfies a generalized polynomial identity over $C$ if and only if $S$ contains a minimal right ideal $eS$ (hence $S$ is primitive) and $eSe$ is a finite dimensional division algebra over $C$.

**Lemma 1.3.33.** [177, Lemma 3] Let $R$ be a prime ring, and $d$ a nonzero derivation of $R$ such that $[d(x), x] = 0$ for all $x \in R$. Then $R$ is commutative.

**Lemma 1.3.34.** [182, Theorem 4] Let $R$ be a prime ring and $n$ be a fixed positive integer larger than 1 such that $[x^n, y]$ is central for all $x, y \in R$. Then $R$ is commutative.

**Lemma 1.3.35.** [183, Lemma 2.6] Let $R$ be a 2-torsion free prime ring and $U$ be a Lie ideal of $R$. If $U$ is a commutative Lie ideal of $R$, then $U \subseteq Z(R)$.

**Lemma 1.3.36.** [183, Corollary 3.2] Let $R$ be a prime ring. If $R$ admits a nonzero generalized derivation with $d$ such that $[F(x), x] = 0$ for all $x \in R$, and if $d \neq 0$, then $R$ is commutative.

**Lemma 1.3.37.** [214, Theorem 1] Suppose that there are non-empty open subsets $G_1$ and $G_2$ of $A$ such that for each $x \in G_1$ and $y \in G_2$ there is an integer $n = n(x, y) > 1$ where either $(xy)^n - x^ny^n$ or $(yx)^n - y^nx^n$ lies in $M$. Then $[x, y] \in M$ for all $x, y \in A$.

**Lemma 1.3.38.** [215, Theorem 2] Suppose that there are non-empty open subsets $G_1$ and $G_2$ of $A$ such that for each $x \in G_1$ and $y \in G_2$ there are positive integers $n = n(x, y)$, $m = m(x, y)$ depending on $x$ and $y$, $n > 1$, $m > 1$, such that either $[x^m, y^n] \in Z(A)$ or $x^m \cdot y^n \in Z(A)$. Then $A$ is commutative if $A$ is semiprime.