ABSTRACT OF THE THESIS ENTITLED

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ABSTRACT

The present thesis is a part of the research work carried out by the author during the last four years concerning the study of derivations and its various generalizations in the setting of rings and algebras. This exposition consists of five chapters and each chapter is subdivided into various sections.

Chapter 1 contains preliminary notions, basic definitions, examples, counter examples and some important well-known results related to our study which may be needed for the development of the subject in subsequent chapters. This chapter is an attempt to make this thesis as self contained as possible. However, the basic knowledge of ring theory has been pre-assumed.

Throughout the discussion all rings are associative unless indicated otherwise and \( Z(R) \) denotes the center of the ring \( R \). For elements \( x \) and \( y \) in a ring \( R \), we shall write \( [x, y] = xy - yx \) and \( x \circ y = xy + yx \). The element \( [x, y] \) is called the Lie product (or the commutator) of elements \( x \) and \( y \), and \( x \circ y \) is called the Jordan product (or the anti-commutator) of \( x \) and \( y \).

Chapter 2 deals with the commutativity of prime rings with involution involving derivations. An additive mapping \( x \mapsto x^* \) is called an involution on a ring \( R \) if \( (xy)^* = y^*x^* \) and \( (x^*)^* = x \) hold for all \( x, y \in R \). A ring equipped with an involution is called a ring with involution or \(*\)-ring. An element \( x \) of a \(*\)-ring \( R \) is said to be symmetric if \( x^* = x \) and is said to be skew symmetric if \( x^* = -x \). The set of all symmetric (resp. skew symmetric) elements of a \(*\)-ring \( R \) is denoted by \( H(R) \) (resp. \( S(R) \)). An involution is said to be of the first kind if \( Z(R) \subseteq H(R) \), otherwise it is said to be of the second kind. In the latter case, \( S(R) \cap Z(R) \neq (0) \). Let \( d : R \to R \) be an additive mapping such that \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). Then, the mapping \( d \) is said to be a derivation on \( R \). Let \( F, d : R \to R \) be two mappings such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \). If \( F \) is additive and \( d \) is a derivation of \( R \), then \( F \) is said to be a generalized derivation of \( R \). The notion of generalized derivation was introduced by Brešar [Glasgow Math. J. 33(1) (1991), 89–93] and primarily studied on operator algebras. While the study
of generalized derivation from the algebraic point of view was made by Hvala [Comm. Algebra 26(4) (1998), 1147–1166] and Lee [Comm. Algebra 27(8) (1999), 4057–4073], respectively. We say that a map \( f : R \to R \) preserves commutativity if \( [f(x), f(y)] = 0 \) whenever \( [x, y] = 0 \) for all \( x, y \in R \). The problems of characterizing maps that preserves commutativity on certain subsets or relations had been investigated on various rings and algebras. One of the most studied problems is to describe bijective additive (or linear) maps preserving commutativity (see for example [Comm. Algebra 25(1) (1997), 247–265], [Trans. Amer. Math. Soc. 335(2) (1993), 525–546], [J. Algebra 284(1) (2005), 102–110], [J. Algebra 301(2) (2006), 803–837], [Linear Algebra Appl. 371 (2003), 361–368], [Monatsh. Math. 166 (2012), 453–465], [Linear Algebra Appl. 426 (2007), 143–148], [Linear Algebra Appl. 288(1-3) (1999), 89–104], [Linear Algebra Appl. 413(2-3) (2006), 364–393], [Linear Algebra Appl. 14(1) (1976), 29–35], [Pacific J. Math. 92(2) (1981), 469–488 and references therein]. In [Canad. Math. Bull. 37(4) (1994), 443–447], Bell and Daif initiated the study of a certain kind of commutativity preserving maps as follows: Let \( S \) be a nonempty subset of \( R \). A map \( f : R \to R \) is called strong commutativity preserving (SCP) on \( S \) if \( [f(x), f(y)] = [x, y] \) for all \( x, y \in S \). More precisely, they proved that \( R \) must be commutative if \( R \) is a prime ring and \( R \) admits a derivation or a non-identity endomorphism which is SCP on a right ideal of \( R \). Furthermore, the analogous results for semiprime rings are also obtained. Later Brešar and Miers [Canad. Math. Bull. 37(4) (1994), 457–460] characterized an additive map \( f : R \to R \) which is SCP on the entire semiprime ring \( R \) and showed that \( f \) must be of the form \( f(x) = \lambda x + \mu(x) \), where \( \lambda \in C, \lambda^2 = 1 \) and \( \mu : R \to C \) is an additive map, where \( C \) is the extended centroid of \( R \). Recently, Deng and Ashraf [Results Math. 30(3-4) (1996), 259–263] proved that if \( R \) is a prime ring of characteristic not 2 and there exists a non-identity endomorphism \( \theta \) of \( R \) such that \( [\theta(x), \theta(y)] = [x, y] \in Z(R) \) for all \( x, y \) in some essential right ideal of \( R \), then \( R \) is commutative. In [Linear Algebra Appl. 428 (2008), 1601–1609], Lin and Liu characterized strong commutativity preserving maps of noncentral Lie ideals on prime rings. In Section 2.2, we extend the study of SCP mappings in the setting of prime rings with involution. In fact, it is shown that a prime
ring $R$ with involution of the second kind admitting a nonzero derivation $d$ such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, must be commutative. A mapping $f : R \to R$ is said to be centralizing on $S$ (where $S$ is a nonempty subset of $R$) if $[f(x), x] \in Z(R)$ for all $x \in S$, and is said to be commuting on $S$ if $[f(x), x] = 0$ for all $x \in S$. Motivated by the definition of centralizing and commuting mappings in rings, Ali and Dar [Georgian Math. J. 21(1) (2014), 25–28] introduced the notions of $*$-centralizing and $*$-commuting mappings in rings with involution as follows: Let $R$ be a ring with involution, and $S$ be a nonempty subset of $R$. A mapping $f : R \to R$ is said to be $*$-centralizing on $S$ if $[f(x), x^*] \in Z(R)$ for all $x \in S$. As a special case, if $[f(x), x^*] = 0$ for all $x \in S$, then $f$ is said to be $*$-commuting on $S$. A classical result due to Posner [Proc. Amer. Math. Soc. 8(1957), 1093–1100] states that a prime ring $R$ admitting a nonzero derivation $d : R \to R$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. The analogous result for centralizing automorphisms on prime rings was obtained by Mayne [Canad. Math. Bull. 19(1) (1976), 113–115]. Further, this result was subsequently refined and extended by a number of authors in several directions (viz.; [Results Math. 42(1-2) (2002), 3–8], [Southeast Asian Bull. Math. 31 (2007), 415–421], [Acta Math. Hungar. 66(4) (1995), 337–343], [Canad. Math. Bull. 30(1) (1987), 92–101], [Proc. Amer. Math. Soc. 111(2) (1991), 501–510], [Int. J. Math. Math. Sci. 21(3) (1998), 471–474], where further references can be found). Very recently, Ali and Dar [Georgian Math. J. 21(1) (2014), 25–28] established a $*$-version of Posner’s second theorem in the setting of prime rings with involution. In Section 2.3, besides proving some other results, we extend Ali and Dar’s [Georgian Math. J. 21(1) (2014), 25–28] result for generalized derivations in rings with involution. In fact, it is shown that a prime ring with involution of the second kind, of characteristic different from two admitting a nonzero generalized derivation $F : R \to R$ such that $[F(x), x^*] \in Z(R)$ for all $x \in R$, must be commutative. Moreover, Herstein [Canad. Math. Bull. 21(3)(1978), 369–370] proved that a prime ring $R$ of characteristic not two with a nonzero derivation $d$ satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. In the year 2014, Ali and Dar [Mosc. Math. J. (2014) (accepted)] extended Herstein’s result in the setting of prime rings with involution. In
Section 2.4, we shall continue the similar study for generalized derivations in prime rings with involution. Further, besides studying the Herstein’s result, we also explore the commutativity of prime rings with involution, which admits a nonzero generalized derivation $F$ satisfying any one of the following conditions:
(i) $F(x) F(x^*) = 0$, (ii) $F([x, x^*]) = 0$, (iii) $F(x \circ x^*) = 0$ for all $x \in R$.
Finally, in Section 2.6 suitable examples are also given to demonstrate that the restrictions imposed on the hypotheses of various results are not superfluous.

Chapter 3 is based on the study of certain Jordan $(\theta, \phi)^*$-derivations in rings with involution. Let $\theta$ and $\phi$ be endomorphisms of $R$. An additive mapping $d : R \to R$ is called a $(\theta, \phi)$-derivation (resp. Jordan $(\theta, \phi)$-derivation) if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$ (resp. $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a Jordan triple $(\theta, \phi)$-derivation if $d(xyx) = d(x)\theta(yx) + \phi(x)d(y)\theta(x) + \phi(xy)d(x)$ holds for all $x, y \in R$. Following [Comm. Algebra 32(8) (2004), 2977–2985], an additive mapping $F : R \to R$ is called a generalized $(\theta, \phi)$-derivation (resp. generalized Jordan $(\theta, \phi)$-derivation) if there exists a $(\theta, \phi)$-derivation $d$ (resp. Jordan $(\theta, \phi)$-derivation) of $R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ (resp. $F(x^2) = F(x)\theta(x) + \phi(x)d(x)$) holds for all $x, y \in R$. According to [Taiwanese J. Math. 11 (2007), 1397–1406], an additive mapping $F : R \to R$ is called a generalized Jordan triple $(\theta, \phi)$-derivation on $R$ if there exists a Jordan triple $(\theta, \phi)$-derivation $d$ of $R$ such that $F(xyx) = F(x)\theta(yx) + \phi(x)d(y)\theta(x) + \phi(xy)d(x)$ for all $x, y \in R$. Motivated by the notion of $(\theta, \phi)$-derivation (resp. Jordan $(\theta, \phi)$-derivation) in rings, Ali and Fošner [Int. J. Algebra 4 (2010), 99–108] introduced the concept of Jordan $(\theta, \phi)^*$-derivation in rings with involution as follows: an additive mapping $d : R \to R$ is called a $(\theta, \phi)^*$-derivation (resp. Jordan $(\theta, \phi)^*$-derivation) if $d(xy) = d(x)\theta(y^*) + \phi(x)d(y)$ (resp. $d(x^2) = d(x)\theta(x^*) + \phi(x)d(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a Jordan triple $(\theta, \phi)^*$-derivation if $d(xyx) = d(x)\theta(y^*x^*) + \phi(x)d(y)\theta(x^*) + \phi(xy)d(x)$ holds for all $x, y \in R$. Clearly, every $(\theta, \phi)^*$-derivation on a ring with involution is a Jordan triple $(\theta, \phi)^*$-derivation, but the converse is in general not true. Recently, Ali and Fošner [Int. J. Algebra 4 (2010), 99–108] proved that on a 6-torsion free semiprime *-ring $R$, every Jordan triple $(\theta, \phi)^*$-derivation is a Jordan $(\theta, \phi)^*$-derivation.
derivation. In [East-West J. Math. 13(2) (2011), 139-146], Ali improved this
result by removing 3-torsion free restriction. Motivated by the above mentioned
result, Ali and Dar [Ukrashen Math. J. (2014) (accepted)] proved the following
result: Let \( R \) be a 2-torsion free semiprime \( \ast \)-ring and let \( d : R \rightarrow R \) be an
additive mapping satisfying the relation \( d(xyz) = d(xy)x^\ast + xyd(x) \) for all \( x, y \in R \).
In this case, \( d \) is a \( \ast \)-derivation. In Section 3.2, we generalize above mentioned
result in the setting of \( (\theta, \phi) \ast \)-derivations. In fact, it is shown that any additive
mapping \( d : R \rightarrow R \) satisfying the relation \( d(xyz) = d(xy)\theta(x^\ast) + \phi(xy)d(x) \)
for all \( x, y \in R \) or \( d(xyz) = d(x)\theta(y^\ast x^\ast) + \phi(x)d(yx) \) for all \( x, y \in R \), is a
\( (\theta, \phi) \ast \)-derivation. A biadditive map \( B : R \times R \rightarrow R \) is called a symmetric if
\( B(x, y) = B(y, x) \) for all \( x, y \in R \). A symmetric biadditive map \( B : R \times R \rightarrow R \)
is called a symmetric biderivation (resp. symmetric Jordan biderivation) if
\( B(xy, z) = B(x, z)y + xB(y, z) \) (resp. \( B(x^2, z) = B(x, x)z + xB(x, z) \)) is fulfilled
for all \( x, y, z \in R \). Following [Kyungpook Math. J. 51(3) (2011), 301-309], a
symmetric biadditive map \( B : R \times R \rightarrow R \) is called a symmetric \( \ast \)-biderivation if
\( B(xy, z) = B(x, z)y^\ast + xB(y, z) \) holds for all \( x, y, z \in R \). In [Ukrashen Math. J.
(2014) (accepted)], Ali and Dar introduced the concept of symmetric Jordan \( \ast \)-
biderivation and symmetric Jordan triple \( \ast \)-biderivation as follows: a symmetric
biadditive map \( D : R \times R \rightarrow R \) is said to be a symmetric Jordan \( \ast \)-biderivation
if \( D(x^2, z) = D(x, z)x^\ast + xD(x, z) \) holds for all \( x, z \in R \). A symmetric biadditive
map \( D : R \times R \rightarrow R \) is said to be a symmetric Jordan triple \( \ast \)-biderivation if
\( D(xyz, z) = D(x, z)y^\ast x^\ast + xD(y, z)x^\ast + xyD(x, z) \) holds for all \( x, y, z \in R \). In
Section 3.3, we introduce the concept of symmetric Jordan \( (\theta, \phi) \ast \)-biderivation
and symmetric Jordan triple \( (\theta, \phi) \ast \)-biderivation as follows: a symmetric biaddi-
tive map \( D : R \times R \rightarrow R \) is said to be a symmetric Jordan \( (\theta, \phi) \ast \)-biderivation
if \( D(x^2, z) = D(x, z)\theta(x^\ast) + \phi(x)D(x, z) \) holds for all \( x, z \in R \). A symmetric
biadditive map \( D : R \times R \rightarrow R \) is called a symmetric Jordan triple \( (\theta, \phi) \ast \)
biderivation if \( D(xyz, z) = D(x, z)\theta(y^\ast x^\ast) + \phi(x)D(y, z)\theta(x^\ast) + \phi(xy)D(x, z) \)
holds for all \( x, y, z \in R \). Clearly, every symmetric \( (\theta, \phi) \ast \)-biderivation is a
Jordan triple \( (\theta, \phi) \ast \)-biderivation. However, the converse is not true in general.
In this section, we prove that under certain conditions on a prime ring with
involution, every symmetric Jordan triple \( (\theta, \phi) \ast \)-biderivation is a symmetric
Jordan \((\theta, \phi)^*\)-biderivation. In [Int. J. Algebra 1 (9-12) (2007), 551-555], Daif and El-Sayiad proved that if \(R\) is a 2-torsion free semiprime ring with involution and \(F: R \to R\) is an additive mapping with an associated derivation \(d: R \to R\) such that \(F(x^*) = F(x)x^* + xd(x^*)\) holds for all \(x \in R\), then \(F\) is a generalized Jordan \(*\)-derivation on \(R\). Later, Ashraf et al. [Math. Slovaca 62(3) (2012), 451-460] extended above mentioned result for generalized \((\theta, \phi)\)-derivation in semiprime rings with involution. In Section 3.4, we shall continue the similar study in the setting of \(*\)-closed Lie ideal of prime rings with involution.

Material of Chapter 4 concerns with the study of multiplicative (generalized)-derivations on prime and semiprime rings. In [Int. J. Math. Math. Sci. 14(3) (1991), 615-618], Daif introduced the notion of a multiplicative derivation as follows: a map \(d: R \to R\) is said to be multiplicative derivation if \(d(xy) = d(x)y + xd(y)\) for all \(x, y \in R\). Of course these maps are not assumed to be additive. The concept of multiplicative derivations appears for the first time in the work of Daif [Int. J. Math. Math. Sci. 14(3) (1991), 615-618], and it was motivated by the work of Martindale [Proc. Amer. Math. Soc. 21 (1969), 695-698]. Further, the complete description of those maps were given by Goldmann and Šemrl in [Monatsh. Math. 121(3) (1996), 189-197]. The notion of multiplicative derivation was extended to multiplicative (generalized)-derivation in [East-West J. Math. 9(1) (1997), 31-37] as follows: a map \(F: R \to R\) is called a multiplicative (generalized)-derivation if there exists a derivation \(d\) such that \(F(xy) = F(x)y + xd(y)\) for all \(x, y \in R\). In this definition, if we suppose \(d\) is any map (not necessarily additive) then it is called a multiplicative (generalized)-derivation. Following [Aequationes Math. 86(1-2) (2013), 65-79], a more precise definition of multiplicative (generalized)-derivation is as follows: a map \(F: R \to R\) (not necessarily additive) is said to be a multiplicative (generalized)-derivation if \(F(xy) = F(x)y + g(x)y\) holds for all \(x, y \in R\), where \(g\) is any map (not necessarily a derivation or an additive map). Hence, the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. Moreover, multiplicative (generalized)-derivation with \(g = 0\) covers the notion of multiplicative centralizers (not necessarily additive). In [Int. J. Math. Math. Sci. 15(1) (1992), 205-206], Daif and
Bell proved that if a semiprime ring $R$ admits a derivation $d$ such that either $d([x,y]) + [x,y] = 0$ for all $x, y \in I$ or $d([x,y]) - [x,y] = 0$ for all $x, y \in I$, where $I$ is a nonzero ideal of $R$, then $R$ is necessarily commutative. Later, Hongan [Int. J. Math. Math. Sci. 20(2) (1997), 413–415] generalized the above mentioned result, considering a semiprime ring $R$ satisfying the conditions $d([x,y]) + [x,y] \in Z(R)$ for all $x, y \in I$ and $d([x,y]) - [x,y] \in Z(R)$ for all $x, y \in I$. Later, Ali et al. [East-West J. Math. 7(1) (2005), 93–98] proved similar results in the setting of square closed Lie ideal of a prime ring $R$ with generalized derivations. In Section 4.2, we study similar results for multiplicative (generalized)-derivations on an appropriate subset of semiprime rings. In Section 4.3, we extend results proved by Ashraf [Southeast Asian Bull. Math. 31 (2007), 415–421] and Ali [East-West J. Math. 7(1) (2005), 93–98] on multiplicative (generalized)-derivations in the setting of square closed Lie ideal of prime rings.

An additive mapping $f: R \rightarrow R$ is called a homomorphism (resp. anti-homomorphism) of $R$ if $f(xy) = f(x)f(y)$ (resp. $f(xy) = f(y)f(x)$) for all $x, y \in R$. In the year 1989, Bell and Kappe [Acta. Math. Hungar. 53(3-4) (1989), 339–346] established the following result: Let $R$ be a prime ring and $d$ be a derivation on $R$. If $d$ acts as a endomorphism or as an anti-endomorphism on $R$, then $d = 0$.


The notion of homomorphism and anti-homomorphism can be extended to n-homomorphism and n-antihomomorphism as follows: an additive mapping $f: R \rightarrow R$ is called a $n$-homomorphism (resp. $n$-antihomomorphism) of $R$ if $f(\prod_{i=1}^{n} a_i) = \prod_{i=1}^{n} f(a_i)$ (resp. $f(\prod_{i=1}^{n} a_i) = \prod_{i=1}^{n} f(a_i)$) for all $a_1, a_2, ..., a_n \in R$. In particular, for $n = 3$, we call $f$ to be a 3-homomorphism (resp. 3-antihomomorphism) of $R$ if $f(xyz) = f(x)f(y)f(z)$ (resp. $f(xyz) = f(x)f(y)f(z)$) for all $x, y, z \in R$. The concept of $n$-homomorphism was studied for complex algebras by Hejazian, Mirzavaziri and Moslehian [Bull. Iranian Math. Soc. 31(1) (2005), 13–23] (see also [Trans. Amer. Math. Soc. 361(4) (2009), 1949–1961]), where some of their significant properties are investigated on Banach algebras. It can be easily seen that every homomorphism of $R$ is the $n$-homomorphism
of $R$ for all $n \geq 2$, but the converse is not true in general. For instance, let $f : R \to R$ be a nonzero homomorphism on $R$. Then $-f$ is a 3-homomorphism on $R$, but not the homomorphism of $R$. Thus it is reasonable to extend the study for homomorphism and anti-homomorphism to n-homomorphism and n-antihomomorphism respectively. In the last section, we describe the form of multiplicative (generalized)-derivations which acts as a 3-homomorphism and as a 3-antihomomorphism on a prime ring. More precisely, it is shown that if $F$ is a multiplicative (generalized)-derivation which acts as a 3-homomorphism or as a 3-antihomomorphism on a nonzero ideal $I$ of $R$, then $d = 0$ and $F = 0$ or $F(x) = \pm x$ for all $x \in I$.

The last chapter of the thesis is devoted to the study of commutativity of rings and Banach algebras involving derivations. The recent literature includes several papers on commutativity of rings and algebras with commutator constraints involving elements of the rings (or algebras) and images of elements under suitable mappings (see for instance [Arch. Math. (Basel) 21 (1970), 265-267], [Canad. Math. Bull. 21(4) (1978), 399-404], [Acta Math. Hungar. 66(4) (1995), 337-343], [Canad. Math. Bull. 30(1) (1987), 92-101], [Michigan Math. J. 8 (1961), 29-32], [Michigan Math. J. 10 (1963), 269-272], [Proc. Amer. Math. Soc. 109(1) (1990), 47-52] for partial list of references). In ([Arch. Math. (Basel) 21 (1970), 265-267], [Arch. Math. (Basel) 24 (1973), 34-38]), Bell studied the identity $[x^n, y] = [x, y^n]$ which is weaker than the identity $(x+y)^n = x^n + y^n$ for all $x, y \in R$, where $n > 1$. As is usual for the study of commutativity involving derivations, the setting will be that of prime rings. In [Quaest. Math. 22(3) (1999), 329-335], Bell studied the commutativity of prime rings involving derivations satisfying above conditions. Precisely, he proved that a prime ring $R$ with nonzero center, for which $\text{char}(R) = 0$ or $\text{char}(R) > n$ where $n > 1$, must be commutative if it admits a nonzero derivation $d$ such that $d([x^n, y] - [x, y^n]) \in Z(R)$ for all $x, y \in R$. In Section 5.2, we generalize the above mentioned result in the setting of semiprime rings and establish the following result: Let $R$ be a 2-torsion free semiprime ring and $m, n$ be fixed positive integers. If $R$ admits a derivation $d$ such that $d([x^m, y^n]) + [x^n, y^n] \in Z(R)$ for all $x, y \in R$ or $d([x^m, y^n]) - [x^n, y^n] \in Z(R)$ for all $x, y \in R$. Then $R$ is commutative. In Section 5.3, we study Engel type con-
ditions involving generalized derivations on prime rings and generalize the result of Demir and Argaç [J. Korean Math. Soc., 47(3) (2010), 483-494]. Precisely, we prove the following: Let $R$ be a noncommutative prime ring and $k, m, n, r$ be fixed positive integers. If there exists a generalized derivation $G$ of $R$ such that $[G(x^m)x^n + x^nG(x^m), x^r]_k = 0$ for all $x \in R$, then there exists an element $a \in C$ such that $G(x) = ax$ for all $x \in R$. In [Michigan Math. J. 8 (1961), 29-32], Herstein proved that a ring $R$ is commutative if it has no nonzero nilpotent ideal and there is a fixed integer $n > 1$ such that $(xy)^n = x^ny^n$ for all $x, y \in R$. In the special case of Banach algebra $A$ with an identity, Yood [Bull. London Math. Soc. 23(3) (1991), 278-280] sharpened these results as follows: Suppose that there are nonempty open subsets $G_1$ and $G_2$ of $A$ such that for each $x \in G_1$ and $y \in G_2$ there is an integer $n = n(x, y) > 1$ where either $(xy)^n - x^ny^n \in M$ or $(xy)^n - y^n x^n \in M$, where $M$ is a closed linear subspace of $A$. Then $[x, y] \in M$ for all $x, y \in A$. In Section 5.4, we study the above mentioned problem for semisimple Banach algebras involving linear derivations. In fact, we prove the following result: Let $A$ be a unital semisimple Banach algebra and $G_1$ and $G_2$ be open subsets of $A$ such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If $A$ admits a linear derivation $d : A \rightarrow A$ such that either $d((xy)^m) + d(x^m)d(y^m) \in Z(A)$ or $d((xy)^m) - d(x^m)d(y^m) \in Z(A)$, then $d(A) \subseteq Z(A)$. The final section of this chapter is devoted to study the applications of our results, obtained in Section 5.3 to semiprime Banach algebras. In fact, we prove that if any one of the following expressions $d([x^m, y^n]) + [x^m, y^n]$, $d([x^m, y^n]) - [x^m, y^n]$, $d(x^m \circ y^n)$, $d([x^m, y^n])$ is in the centre of a semiprime Banach algebra $A$, where $m > 1$ and $n > 1$ are positive integers depending on $x$ and $y$, then $A$ is commutative.