CHAPTER-5

On Derivations in Rings and Banach Algebras

with Applications
Chapter 5

On derivations in rings and Banach algebras with applications

5.1 Introduction

By a derivation of \( R \) we mean an additive mapping \( d \) from \( R \) into itself which satisfies the rule \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). If \( R \) is an algebra we assume additionally that \( d \) is linear i.e., \( d(\alpha x) = \alpha d(x) \) for all \( x \in R \) and \( \alpha \) is in some field \( F \). The recent literature includes several papers on commutativity of rings and algebras with commutator constraints involving elements of the rings (or algebras) and images of elements under suitable mappings (see [51], [55], [122] and [123] for partial list of references). During the last few decades, there has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of \( R \). Recently, many authors viz., [59], [60], [62], [70], [176] and [197] have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. Bell ([51], [52]) studied the identity \([x^n, y] = [x, y^n]\) which is weaker than the identity \((x + y)^n = x^n + y^n\) for all \( x, y \in R \), where \( n > 1 \). As is usual for the study of commutativity involving derivations, the setting will be that of prime rings. In [57], Bell studied the commutativity of prime rings involving derivations satisfying above conditions. Precisely, he proved that a prime ring \( R \) with nonzero center, for which \( \text{char}(R) = 0 \) or \( \text{char}(R) > n \) where \( n > 1 \), must be commutative if it admits a nonzero derivation \( d \) such that \( d([x^n, y] - [x, y^n]) \in Z(R) \) for all \( x, y \in R \).

In Section 5.2, we generalize the result due to Bell [57] mentioned above. In fact, we prove the following: Let \( R \) be a 2-torsion free semiprime ring. If \( R \) admits a derivation
d such that $d([x^m, y^n]) = [x^m, y^n] \in Z(R)$ for all $x, y \in R$, then $R$ is commutative. Moreover, other interesting results are also obtained.

Following [68], an additive mapping $G : R \to R$ is called a generalized derivation of $R$ if there exists a derivation $d$ of $R$ such that $G(xy) = G(x)y + xd(y)$ for all $x, y \in R$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $G(x) = ax + xb$ for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example [134] and [152], where further references can be found). In [152], Lee extended the definition of generalized derivation as follows: by a generalized derivation he meant an additive mapping $G : J \to U$ such that $G(xy) = G(x)y + xd(y)$ for all $x, y \in J$, where $U$ is the left Utumi quotient ring of $R$, $J$ is a dense right ideal of $R$ and $d$ is a derivation from $J$ to $U$. He also showed that every generalized derivation can be uniquely extended to a generalized derivation of $U$. In fact, there exists $u \in U$ and a derivation $d$ of $U$ such that $G(x) = ax + d(x)$ for all $x \in U$ [152, Theorem 3]. A corresponding form of dense left ideals is as follows: let $I$ be a dense left ideal of $R$ and $U$ be the left Utumi quotient ring of $R$. An additive mapping $G : I \to U$ is called a generalized derivation if there exists a derivation $d : I \to U$ such that $G(xy) = G(x)y + xd(y)$ for all $x, y \in I$. Following the same methods in [152], one can extend $G$ uniquely to a generalized derivation of $U$, which we will also denote by $G$, and $G$ assumes to be of the form $G(x) = ax + d(x)$ for all $x \in U$ and some $a \in U$, where $d$ is a derivation of $U$. Notice that $G(x) = ax + (d - da)(x)$ for $x \in U$, where $da$ denotes the inner derivation induced by the element $a \in U$ i.e., $da(x) = [a, x]$. Setting $\delta = d - da$, we may always assume that a generalized derivation of a prime ring is one of the form $G(x) = ax + \delta(x)$ for all $x \in U$, where $a \in U$ and $\delta$ is a derivation of $U$.

Section 5.3 deals with the Engel type conditions involving generalized derivations on prime rings. In this section, we extend the result of Demir and Argaç [100]. In fact, we prove the following: Let $R$ be a noncommutative prime ring and $k, m, n, r$ be fixed positive integers. If there exists a generalized derivation $G$ of $R$ such that $[G(x^m)x^n + x^nG(x^m), x]^k = 0$ for all $x \in R$, then there exists an element $a \in C$ such that $G(x) = ax$ for all $x \in R$.

In Section 5.4, we study the result obtained by Yoog [214] and discuss the com-
mutativity of Banach algebras via linear derivations. Precisely, we prove the following result: Let \( A \) be a unital semisimple Banach algebra and \( G_1 \) and \( G_2 \) be open subsets of \( A \) such that for each \( x \in G_1 \), and \( y \in G_2 \) there is an integer \( m = m(x,y) > 1 \). If \( A \) admits a linear derivation \( d : A \rightarrow A \) such that either \( d((xy)^m) + d(x^m)d(y^n) \in Z(A) \) or \( d((xy)^m) - d(x^m)d(y^n) \in Z(A) \), then \( d(A) \subseteq Z(A) \).

The last section is based on the study of commutativity of prime and semiprime Banach algebras. The results of this section are applications of the results proved in Section 5.2 and in particular, we prove that if any one of the following expressions \( d([x^n, y^n]) + [x^m, y^n], d([x^n, y^n]) - [x^n, y^n], d(x^m \circ y^n), d([x^m, y^n]) \) is in the centre of a Banach algebra \( A \), where \( m > 1, n > 1 \) are positive integers depending on \( x \) and \( y \), then \( A \) is commutative.

### 5.2 Derivations in semiprime rings

In [95], Daif and Bell proved that a semiprime ring \( R \) must be commutative if it admits a derivation \( d \) such that either \( d([x, y]) - [x, y] = 0 \) for all \( x, y \in R \) or \( d([x, y]) + [x, y] = 0 \) for all \( x, y \in R \). Further, Hongan [132] extended this result as follows: Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a nonzero ideal of \( R \). If \( R \) admits a derivation \( d \) such that \( d([x, y]) - [x, y] \in Z(R) \) for all \( x, y \in I \) or \( d([x, y]) + [x, y] \in Z(R) \) for all \( x, y \in I \), then \( I \subseteq Z(R) \). Motivated by these results, Bell [57] investigated commutativity of prime rings satisfying certain differential identities. In fact, he proved the following result:

**Theorem 5.2.1.** Let \( n > 1 \) and let \( R \) be a prime ring with nonzero center, for which \( \text{char}(R) = 0 \) or \( \text{char}(R) > n \). If \( R \) admits a nonzero derivation \( d \) such that \( d([x^n, y^n]) - [x^n, y^n] \in Z(R) \) for all \( x, y \in R \), then \( R \) is commutative.

The goal of this section is to prove the next theorem which generalizes the results proved by Bell [57], Daif and Bell [95] and Hongan [132].

**Theorem 5.2.2.** Let \( R \) be a 2-torsion free semiprime ring and \( m, n \) be fixed positive integers. If \( R \) admits a derivation \( d \) such that \( d([x^m, y^n]) + [x^m, y^n] \in Z(R) \) for all \( x, y \in R \), or \( d([x^m, y^n]) - [x^m, y^n] \in Z(R) \) for all \( x, y \in R \), then \( R \) is commutative.

Before proving above theorem, we prove the following key lemma.
Lemma 5.2.1. Let $R$ be a prime ring such that $\text{char}(R) \neq 2$ and $U$ be a nonzero Lie ideal of $R$. If $R$ admits a derivation $d$ such that $d([u, v]) \pm [u, v] \in Z(R)$ for all $u, v \in U$, then $U \subseteq Z(R)$.

Proof. First we consider the case

$$d([u, v]) - [u, v] \in Z(R) \text{ for all } u, v \in U.$$  \hspace{1cm} (5.2.1)

If $d = 0$, then we get $[U, U] \subseteq Z(R)$. Hence, by Lemma 1.3.21, $U \subseteq Z(R)$. Now, we assume that $d \neq 0$ and from (5.2.1), we obtain

$$[d([u, v]) - [u, v], x] = 0 \text{ for all } u, v \in U \text{ and } x \in R.$$

This implies that

$$[[d(u), v], x] + [[u, d(v)], x] - [[u, v], x] = 0 \text{ for all } u, v \in U \text{ and } x \in R.$$ \hspace{1cm} (5.2.2)

Replacing $v$ by $[v, w]$ in (5.2.2), we get

$$[[d(u), [v, w]], x] + [[u, d([v, w])], x] - [[u, [v, w]], x] = 0 \text{ for all } u, v, w \in U \text{ and } x \in R.$$

This can be written as

$$[[d(u), [v, w]], x] + [[u, d([v, w]) - [v, w]], x] = 0 \text{ for all } u, v, w \in U \text{ and } x \in R.$$

Our hypothesis yields that

$$[[d(u), [v, w]], x] = 0 \text{ for all } u, v, w \in U \text{ and } x \in R.$$

That is,

$$[d(u), [v, w]] \in Z(R) \text{ for all } u, v, w \in U.$$

In view of Lemma 1.3.28, we have either $[v, w] \in Z(R)$ or $U \subseteq Z(R)$. If $U \subseteq Z(R)$, then we have done. Suppose $[v, w] \in Z(R)$ for all $v, w \in U$ i.e., $[U, U] \subseteq Z(R)$. Then, again by Lemma 1.3.21, we get $U \subseteq Z(R)$.

Similarly, we can prove the result for the case $d([u, v]) + [u, v] \in Z(R)$ for all
$u, v \in U$. This proves the lemma. \[\square\]

**Corollary 5.2.1.** Let $R$ be a prime ring such that $\text{char}(R) \neq 2$ and $U$ be a nonzero Lie ideal of $R$. If $R$ admits a derivation $d$ such that $d(u) + u \in Z(R)$ for all $u \in U$, then $U \subseteq Z(R)$.

**Proof.** Replacing $u$ by $[u, v]$ in the given condition, we obtain

$$d([u, v]) + [u, v] \in Z(R) \text{ for all } u, v \in U.$$ 

Hence, by Lemma 5.2.1, we get the required result. \[\square\]

**Proof of Theorem 5.2.2.** Firstly we assume that $R$ is a prime ring and consider

$$d([x^m, y^n]) - [x^m, y^n] \in Z(R) \text{ for all } x, y \in R.$$ 

Of course, in case $d = 0$, $[x^m, y^n] \in Z(R)$, for all $x, y \in R$. For $m, n = 1$, $R$ is obviously commutative. If $m > 1$ and $n = 1$ or $n > 1$ and $m = 1$, then by Lemma 1.3.34, $R$ is commutative and for $m, n > 1$ by Lemma 1.3.19, $R$ is commutative. Therefore in all that follows we assume $d \neq 0$.

Let $S_1$ be the additive subgroup generated by the subset $\{r^m \mid r \in R\}$ and $S_2$ the additive subgroup generated by the subset $\{r^n \mid r \in R\}$. It is easy to see that

$$d([x, y]) - [x, y] \in Z(R) \text{ for all } x \in S_1, y \in S_2.$$ 

In view of Lemma 1.3.14 and since $\text{char}(R) \neq 2$, we have that either $S_1$ contains a noncentral Lie ideal $L_1$, or $r^m \in Z(R)$ for all $r \in R$. It is well known that the later case forces $R$ to be commutative. Analogously we may assume there exists $L_2$ a noncentral Lie ideal of $R$, which is contained in $S_2$.

Moreover, by Lemma 1.3.20, there exist $I_1$ and $I_2$ nonzero two-sided ideals of $R$, such that $0 \neq [I_1, R] \subseteq L_1$ and $0 \neq [I_2, R] \subseteq L_2$. Hence we have that

$$d([x, y]) - [x, y] \in Z(R) \text{ for all } x \in [I_1, I], y \in [I_2, I].$$ 

Since $I_1$, $I_2$ and $R$ satisfy the same differential identities (see Lemma 1.3.30), then we
have
\[ d([x, y]) - [x, y] \in Z(R) \] for all \( x, y \in [R, R] \). \hfill (5.2.3)

Now for \( U = [R, R] \) applying Lemma 5.2.1, we get \( R \) is commutative.

Let now \( R \) be a semiprime ring. Since the derivation \( d \) can be uniquely extended in \( Q \), and \( R \) and \( Q \) satisfy the same differential identities (see again Lemma 1.3.30), then
\[ d([x^n, y^n]) - [x^n, y^n] \in C \] for all \( x, y \in Q \).

Let \( M \) be any maximal ideal of the complete Boolean algebra of idempotents of \( C \), denoted by \( B \). We know that \( MQ \) is a prime ideal of \( Q \). Let \( \bar{d} \) be the derivation induced by \( d \) in \( \bar{Q} = Q/MQ \). Therefore \( \bar{d} \) satisfy in \( \bar{Q} \) the same property of \( d \) on \( Q \). By the prime-case, for all \( M \) maximal ideal of \( B \), we have \([Q, Q] \subseteq MQ \), that is \([Q, Q] \subseteq \bigcap_M MQ = (0) \). Without loss of generality we have \([R, R] = (0) \) and we are done.

Similar conclusion holds for the case \( d([x^n, y^n]) + [x^n, y^n] \in Z(R) \) for all \( x, y \in R \). This completes the proof of the theorem. \( \square \)

In [57], Bell proved that if \( R \) is a prime ring such that \( char(R) \neq 2 \) and \( d \) is a nonzero derivation of \( R \) such that \( d([x, y]) \in Z(R) \) for all \( x, y \in R \), then \( R \) is commutative. The following theorem is a natural generalization of this result.

**Theorem 5.2.3.** Let \( R \) be a 2-torsion free semiprime ring and \( m, n \) be fixed positive integers. If \( R \) admits a nonzero derivation \( d \) such that \( d([x^n, y^n]) \in Z(R) \) for all \( x, y \in R \), then \( R \) is commutative.

**Proof.** Firstly we assume \( R \) is prime and
\[ d([x^n, y^n]) \in Z(R) \] for all \( x, y \in R \).

Now, using the same arguments as we have used in the proof of Theorem 5.2.2, we get
\[ d([x, y]) \in Z(R) \] for all \( x, y \in [R, R] \).

Therefore, for \( U = [R, R] \) by Lemma 1.3.28, we conclude that \( R \) is commutative.

Let now \( R \) be a semiprime ring. Since the derivation \( d \) can be uniquely extended
in $Q$, and $R$ and $Q$ satisfy the same differential identities by Lemma 1.3.30, then
\[ d([x^m, y^n]) \in C \text{ for all } x, y \in Q. \]

Let $M$ be any maximal ideal of the complete Boolean algebra of idempotents of $C$, denoted by $B$. We know that $MQ$ is a prime ideal of $Q$. Let $\tilde{d}$ be the derivation induced by $d$ in $\overline{Q} = Q/MQ$. Therefore $\tilde{d}$ satisfy in $\overline{Q}$ the same property of $d$ on $Q$. By the prime-case, for all $M$ maximal ideal of $B$, we have $[Q, \overline{Q}] \subseteq MQ$, that is $[Q, Q] \subseteq \bigcap_M MQ = (0)$. Without loss of generality, we have $[R, R] = (0)$ and we are done. Thus the proof of the theorem is completed. \( \square \)

**Corollary 5.2.2.** Let $R$ be a 2-torsion free semiprime ring. If $R$ admits a nonzero derivation $d$ such that $d([x, y]) \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.

If we replace the commutator by anti-commutator in Theorem 5.2.3, the corresponding result also holds.

**Theorem 5.2.4.** Let $R$ be a 2-torsion free semiprime ring and $m, n$ be fixed positive integers. If $R$ admits a nonzero derivation $d$ such that $d(x^m \circ y^n) \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.

**Proof.** First we consider the case when $R$ is prime and
\[ d(x^m \circ y^n) \in Z(R) \text{ for all } x, y \in R. \]

For $x = y$, we get $d(x^k) \in Z(R)$ for all $x \in R$, where $k = m + n$. Therefore, by Lemma 1.3.7, $R$ is commutative.

If we assume $R$ is semiprime, then using similar approach as we have used in the last paragraph of the proof of Theorem 5.2.2, we get the required result. \( \square \)

As an immediate consequence of Theorem 5.2.4 we get the following corollary.

**Corollary 5.2.3.** Let $R$ be a 2-torsion free semiprime ring. If $R$ admits a nonzero derivation $d$ such that $d(x \circ y) \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.
5.3 Generalized derivations with Engel condition in prime rings

If one defines \([x, y]_0 = x\) and \([x, y]_1 = [x, y] = xy - yx\), then an Engel condition is a polynomial \([x, y]_{n+1} = [[x, y]_n, y]\) in noncommutative indeterminates. In [177], Posner proved that if a prime ring \(R\) admits a nonzero derivation \(d\) such that \([d(x), x] \in Z(R)\) for all \(x \in R\), then \(R\) is commutative. The analogous result was obtained for automorphism in [172]. Many authors generalized Posner’s result in the setting of rings and algebras (see [57], [132], [197–199] and references therein). Considerable attention has been paid to commutativity theorems for rings and algebras (see for example [126, Chap. 3] and [85, Chap. 2]), where further references can also be seen.

In [145], Lanekpi proved that if \(R\) is a prime ring, \(d\) a nonzero derivation on \(R\), \(I\) a nonzero left ideal of \(R\) and \(k, n\) are positive integers such that \([d(x^n), x^n]\)_k = 0 for all \(x \in I\), then either \(d = 0\) or \(R\) is commutative. In [4], Albas et al. extended this result for generalized derivations. Further, Lee and Shiue [153] showed that if \(R\) is a noncommutative prime ring, \(I\) a nonzero left ideal of \(R\) and \(d\) a derivation of \(R\) such that \([d(x^m)x^n, x^n]_k = 0\) for all \(x \in I\), where \(k, m, n, r\) are fixed positive integers, then \(d = 0\) except when \(R \cong M_2(GF(2))\). Recently, Demir and Argaci [100] extended this result for generalized derivation \(G\) and they proved that if \(R\) is a prime ring and \(k, m, n, r\) are fixed positive integers such that \([G(x^m)x^n, x^{n}]_k = 0\) for all \(x \in R\), then there exists an element \(a \in U\) such that \(G(x) = xa\) for all \(x \in R\).

The goal of this section is to prove the following two theorems which are motivated by the above mentioned results.

**Theorem 5.3.1.** Let \(R\) be a noncommutative prime ring and \(k, m, n, r\) be fixed positive integers. If there exists a generalized derivation \(G\) of \(R\) such that \([G(x^m)x^n + x^nG(x^m), x^n]_k = 0\) for all \(x \in R\), then there exists an element \(a \in C\) such that \(G(x) = ax\) for all \(x \in R\).

**Theorem 5.3.2.** Let \(R\) be a noncommutative prime ring and \(k, m, n, r\) be fixed positive integers. If there exists a generalized derivation \(G\) of \(R\) and a derivation \(g\) of \(R\) such that \([G(x^m)x^n + x^n g(x^m), x^n]_k = 0\) for all \(x \in R\), then there exists an element \(a \in U\) such that \(G(x) = xa\) for all \(x \in R\).

In order to prove our main theorems we need to prove the following lemmas.
Lemma 5.3.1. Let $R = M_t(F)$, where $F$ is a field, $t \geq 2$ and $a, b \in R$. Suppose that
\[ [(ax^m + [b, x^m])x^n + x^n(ax^m + [b, x^m]), x^r]_k = 0 \]
for all $x \in R$, where $k, m, n, r$ are fixed positive integers. Then $a, b \in F$.

Proof. By the hypothesis, we have
\[ [(ax^m + [b, x^m])x^n + x^n(ax^m + [b, x^m]), x^r]_k = 0 \]  \hspace{1cm} (5.3.1)
for all $x \in R$. Put $x = e$ in (5.3.1), where $e$ is an idempotent of $R$ and multiplying right side by $(1 - e)$, we obtain $eb = ebe$. Again, put $x = e$ in (5.3.1) and multiplying from left side by $(1 - e)$, we obtain $(1 - e)(a + b)e = 0$. Since $eb = ebe$, so we get $ae = eae$. Hence, both $a$ and $b$ are diagonal matrices. Note that $ubu^{-1}$ must be diagonal for each invertible element $u \in R$, since
\[ [((uau^{-1})x^m + ([ubu^{-1}], x^m))x^n, x^r]_k = 0 \]  \hspace{1cm} (5.3.2)
for all $x \in R$. Take $b = \sum_{i=1}^{t} \alpha_i \epsilon_{ii}$, where $\alpha_i \in F$. Then for each $j > 1$ we see that the $(1, j)$-entry of $(1 + \epsilon_{jj})b(1 + \epsilon_{jj})^{-1}$ i.e., $\alpha_j - \alpha_i$ equals 0. That is, $\alpha_j = \alpha_i$ for $j > 1$ and hence $b \in F$. By symmetry, we conclude that $a \in F$. This proves the lemma. \(\Box\)

Corollary 5.3.1. Let $R = M_t(F)$, where $F$ is a field, $t \geq 2$ and $b \in R$. Suppose that
\[ [[b, x^m]x^n + x^n[b, x^m], x^r]_k = 0 \]
for all $x \in R$, where $k, m, n, r$ are fixed positive integers. Then $b \in F$.

The proof of next lemma is same as the proof of Lemma 5.3.1, so we skip its proof.

Lemma 5.3.2. Let $R = M_t(F)$, where $F$ is a field, $t \geq 2$ and $a, b, c \in R$. Suppose that
\[ [(ax^m + [b, x^m])x^n + x^n(ax^m + [b, x^m]), x^r]_k = 0 \]
for all $x \in R$, where $k, m, n, r$ are fixed positive integers. Then $a + b, c \in F$.

Lemma 5.3.3. Let $R$ be a noncommutative prime ring and $a, b \in R$ such that $[(ax^m + [b, x^m])x^n + x^n(ax^m + [b, x^m]), x^r]_k = 0$ for all $x \in R$, where $k, m, n, r$ are fixed positive integers. Then $a, b \in Z(R)$.

Proof. Suppose on the contrary that $a, b \notin Z(R)$. Then
\[ f(X) = [(aX^m + [b, X^m])X^n + X^n(ax^m + [b, X^m]), X^r]_k \]

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is a nontrivial generalized polynomial identity for \( R \). By Lemma 1.3.15, \( f(X) \) is also a GPI for \( Q \). Denote by \( F \) either the algebraic closure of \( C \) or \( C \), according to cases where \( C \) is either infinite or finite, respectively. Then, by a standard argument (see Lemma 1.3.29), \( f(X) \) is also a GPI for \( Q \otimes_C F \). Since \( Q \otimes_C F \) is centrally closed prime \( F \)-algebra (see Lemma 1.3.17), by replacing \( R, C \) with \( Q \otimes_C F, F \), respectively we may assume that \( R \) is centrally closed and \( C \) is either finite or algebraically closed.

In view of Lemma 1.3.32, \( R \) is a primitive ring having a nonzero socle \( H \) with \( C \) as its associative division ring.

Since \( a \not\in C \), we have \([a, h] \neq 0\) for some \( h \in H \). By Lemma 1.3.18, there exists an idempotent \( e \in H \) such that \( h, ah, ha \in eRe \). Note that \( ef(eXe)e \) is a GPI for \( R \). Therefore, \([(eae)X^m + [(ebe), X^m], X^n + X^n((eae)X^m + [(ebe), X^m]), X^r]_h \) is a GPI for \( eRe \). Since \( eRe \cong M_t(C) \) for some \( t \geq 1 \), so \( eae \) is central in view of Lemma 5.3.1. Then there exists \( c \in C \) such that \( ce = eae \). Hence \( ch = each = eah = ah \). Similarly \( hc = ha \). So \([a, h] = 0\), a contradiction. Hence \( a \in Z(R) \). Similarly we can show that \( b \in Z(R) \). This complete the proof.

**Corollary 5.3.2.** Let \( R \) be a noncommutative prime ring and \( a \in R \) be such that \([ax^m + x^nax^m, x^r]_h = 0\) for all \( x \in R \), where \( k, m, n, r \) are fixed positive integers. Then \( a \in Z(R) \).

Proceeding on the similar lines, we can prove the following lemma.

**Lemma 5.3.4.** Let \( R \) be a noncommutative prime ring and \( a, b, c \in R \) such that \([ax^m + [b, x^m]], x^n + x^n[c, x^m], x^r]_h = 0\) for all \( x \in R \), where \( k, m, n, r \) are fixed positive integers. Then \( a + b, c \in Z(R) \).

**Proposition 5.3.1.** Let \( R \) be a noncommutative prime ring and \( d, g \) be a derivations on \( R \) such that \([d(x^m)x^n + x^n g(x^m), x^r]_h = 0\) for all \( x \in R \), where \( k, m, n, r \) are fixed positive integers. Then \( d = g = 0 \).

**Proof.** By assumption, we have

\[
[d(x^m)x^n + x^n g(x^m), x^r]_h = 0
\] (5.3.3)

for all \( x \in R \). Replace \( \delta \) by \(-g\) in Lemma 1.3.31, we get \( d = g = 0 \). This completes the proof. \( \Box \)
Proof of Theorem 5.3.1. In view of the hypothesis, we have

\[ [G(x^m)x^n + x^nG(x^m), x^r]_k = 0 \]  \hspace{1cm} (5.3.4)

for all \( x \in R \). We know that every generalized derivation \( G \) on \( R \) can be uniquely extended to \( U \) and assumes the form \( G(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \). Then (5.3.4) can be written as

\[ [ax^{m+n} + x^nax^m + d(x^m)x^n + x^nd(x^m), x^r]_k = 0 \]  \hspace{1cm} (5.3.5)

for all \( x \in R \). If \( d = 0 \), then (5.3.5) reduces to \([ax^{m+n} + x^nax^m, x^r]_k = 0\) for all \( x \in R \).

By Lemma 1.3.15, \( U \) satisfy the above generalized polynomial identity. Moreover, since \( U \) remains prime by primeness of \( R \), replacing \( R \) with \( U \), we may assume that \( a \in R \) and \( C \) is just the center of \( R \). By Corollary 5.3.2, \( a \in Z(R) \). Thus \( G(x) = ax \) for all \( x \in R \). Now, we may assume \( d \neq 0 \). Suppose \( d \) is \( X \)-inner derivation induced by \( 0 \neq b \in U \) i.e., \( d(x) = [b, x] \) for all \( x \in U \). Then \( R \) satisfies the nontrivial generalized polynomial identity

\[ [(ax^m + [b, x^n])x^n + x^n(ax^m + [b, x^m]), x^r]_k = 0 \]  \hspace{1cm} (5.3.6)

for all \( x \in R \). By Lemma 1.3.15 \( U \) satisfy generalized polynomial identity (5.3.6). Moreover, since \( U \) remains prime by primeness of \( R \), we may assume that \( a, b \in R \) and \( C \) is just the center of \( R \). Therefore by Lemma 5.3.1, we have \( a, b \in Z(R) \), which is a contradiction as \( d \neq 0 \). Next, we suppose that \( d \) is outer derivation of \( U \). Now, we set \( h(Y, X) = \sum_{i=0}^{m-1} X^iYX^{m-1-i} \), a polynomial in two non-commuting variables \( X \) and \( Y \).

Note that \( d(x^m) = h(d(x), x) \) for all \( x \in R \). Then by (5.3.5) we have

\[ [ax^{m+n} + x^nax^m + h(d(x), x)x^n + x^nh(d(x), x), x^r]_k = 0 \]  \hspace{1cm} (5.3.7)

for all \( x \in R \). Apply Lemma 1.3.27 to relation (5.3.7), we get

\[ [ax^{m+n} + x^nax^m + h(y, x)x^n + x^nh(y, x), x^r]_k = 0 \]  \hspace{1cm} (5.3.8)

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for all $x, y \in R$. Taking $y = 0$ in (5.3.8), we get

$$[ax^{m+n} + x^n ax^m, x^r]_k = 0$$

for all $x \in R$. Thus, (5.3.5) reduces to

$$[d(x^m)x^n + x^n d(x^m), x^r]_k = 0$$

for all $x \in R$. Therefore, by Proposition 5.3.1, we must have $d = 0$, which is again a contradiction. This completes the proof of the theorem. 

Proof of Theorem 5.3.2. By the hypothesis, we have

$$[G(x^m)x^n + x^n g(x^m), x^r]_k = 0$$

(5.3.9)

for all $x \in R$. We know that every generalized derivation $G$ on $R$ can be uniquely extended to $U$ and assumes the form $G(x) = ax + d(x)$ for some $a \in U$ and a derivation $d$ on $U$. Then (5.3.9) can be written as

$$[ax^{m+n} + d(x^n)x^n + x^n g(x^m), x^r]_k = 0$$

(5.3.10)

for all $x \in R$. If $g = 0$, then the result follows from Lemma 1.3.16. Now, if $d = 0$, then from (5.3.10), we have

$$[ax^{m+n} + x^n g(x^m), x^r]_k = 0$$

(5.3.11)

for all $x \in R$. Here two cases arise, first that $g$ is $X$-inner derivation i.e., there exists an element $g \in U$ such that $g(x) = [g, x]$. Thus, (5.3.11) becomes

$$[ax^{m+n} + x^n [g, x^m], x^r]_k = 0$$

for all $x \in R$. Therefore, by Lemma 5.3.4, we get the result. Second case $g$ is outer derivation and set $h(Y, X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i}$. Then $g(x^m) = h(g(x), x)$ for all $x \in R$. Thus, (5.3.11) can be written as

$$[ax^{m+n} + x^n h(g(x), x), x^r]_k = 0.$$
Applying Lemma 1.3.27 in above, we get

\[ [ax^{m+n} + x^n h(y, x), x^r]_k = 0 \]

for all \( x, y \in R \). Taking \( y = 0 \), we get

\[ [ax^{m+n}, x^r]_k = 0. \]

In view of Lemma 5.3.4, we get the result. Thus in both cases either \( d = 0 \) or \( g = 0 \), we conclude that \( G(x) = xa \) for all \( x \in R \) and some \( a \in U \). Therefore in all that follows we assume both \( d \neq 0 \) and \( g \neq 0 \) and prove the result in the following cases:

**d and g are X-inner derivations.**

There exist \( 0 \neq b, c \in U \) such that \( d(x) = [b, x] \) and \( g(x) = [c, x] \). Then from (5.3.10), we have

\[ [ax^{m+n} + [b, x^m]x^n + x^n[c, x^m], x^r]_k = 0 \quad (5.3.12) \]

for all \( x \in R \). Proceeding on same lines as in the proof of Lemma 5.3.3 and making use of Lemma 5.3.4 and Lemma 5.3.2, we obtain that \( a + b, c \in Z(R) \), which is a contradiction as \( g \neq 0 \).

**d and g are C-independent modulo X-inner derivations.**

Set as above that \( h(Y, X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i} \). Then \( d(x^m) = h(d(x), x) \) and \( g(x^m) = h(g(x), x) \) for all \( x \in R \). Therefore, (5.3.10) can be written as

\[ [ax^{m+n} + h(d(x), x)x^n + x^n h(g(x), x), x^r]_k = 0 \]

for all \( x \in R \). Applying Lemma 1.3.27 to above expression, we obtain

\[ [ax^{m+n} + h(y, x)x^n + x^n h(z, x), x^r]_k = 0 \quad (5.3.13) \]

for all \( x, y, z \in R \). Taking \( y = [u, x] \) and \( z = [v, x] \) in (5.3.13) and using the fact that
\(h([u, x], x) = [u, x^m]\) and \(h([v, x], x) = [v, x^m]\), we arrive at

\[
[ax^{m+n} + [u, x^m]x^n + x^n[v, x^m], x^r]_k = 0
\]  

(5.3.14)

for all \(u, v, x \in R\). In view of Lemma 5.3.4, we have \(a + u, v \in Z(R)\) for all \(u, v \in R\), this forces that \(R\) is commutative. This leads to a contradiction.

\section*{d and \(g\) are \(C\)-dependent modulo \(X\)-inner derivations.}

Finally, we assume that \(d\) and \(g\) are not both inner derivations such that \(d\) and \(g\) are \(C\)-dependent modulo \(X\)-inner derivations. Thus there exist \(\alpha, \beta \in C\) and \(p \in U\) such that \(\alpha d(x) + \beta g(x) = [p, x]\). Assume first that \(\alpha = 0\), so that \(g(x) = [q, x]\) for all \(x \in R\), where \(q = \beta^{-1}p\) and \(d\) is not inner derivation. Thus (5.3.13) becomes

\[
[ax^{m+n} + h(d(x), x)x^n + x^n h([q, x], x), x^r]_k = 0
\]

for all \(x \in R\). By applying Lemma 1.3.27, we get

\[
[ax^{m+n} + h(y, x)x^n + x^n h([q, x], x), x^r]_k = 0
\]  

(5.3.15)

for all \(x, y \in R\). Taking \(y = 0\) in above expression, we get

\[
[ax^{m+n} + x^n h([q, x], x), x^r]_k = 0
\]  

(5.3.16)

for all \(x \in R\). By Lemma 5.3.4, we have \(a, q \in Z(R)\) and we reach at a contradiction \(g = 0\). Notice that the case \(\beta = 0\) is symmetric to the case \(\alpha = 0\).

At the end, we consider both \(\alpha \neq 0\) and \(\beta \neq 0\) and write \(g = \gamma d(x) + [q, x]\) for all \(x \in R\), where \(\gamma = -\beta^{-1}\alpha \neq 0\), \(q = \beta^{-1}p\) and \(d\) is not inner. Thus by (5.3.10), we have

\[
[ax^{m+n} + h(d(x), x)x^n + \gamma x^n h(d(x), x) + x^n [q, x^m], x^r]_k = 0
\]  

(5.3.17)

for all \(x \in R\). Again applying Lemma 1.3.27 to (5.3.17), we get

\[
[ax^{m+n} + h(y, x)x^n + \gamma x^n h(y, x) + x^n [q, x^m], x^r]_k = 0
\]  

(5.3.18)
for all \(x, y \in R\). Putting \(y = 0\) and proceeding as above, we have

\[
[h(y, x)x^n + \gamma x^n h(y, x), x^r]_k = 0
\]

for all \(x, y \in R\). Now, we take \(y = [u, x]\) and using the fact that \(h([u, x], x) = [u, x^m]\), we get

\[
[[u, x^m]x^n + x^n[u, x^m], x^r]_k = 0
\]

for all \(x, u \in R\). Application of Lemma 5.3.2 yields that \(u \in Z(R)\) for all \(u \in R\) and so \(R\) is commutative, this leads to a contradiction. This proves the theorem.

The following corollary is an immediate consequence of the above theorem.

**Corollary 5.3.3.** [100, Theorem 1] Let \(R\) be a noncommutative prime ring and \(k, m, n, r\) be fixed positive integers. If there exists a generalized derivation \(G\) of \(R\) such that \([G(x^n)x^r, x^r]_k = 0\) for all \(x \in R\), then there exists an element \(a \in U\) such that \(G(x) = xa\) for all \(x \in R\).

### 5.4 Derivations in semisimple Banach algebras

A number of theorems in ring theory, mostly due to Herstein (viz.; [117], [118], [120-123], [125], [127], [128] and [129], where further references can be found) are devoted to showing that certain rings must be commutative as consequences of conditions which are seemingly too weak to imply commutativity (see for example [124, Chapter 3]). In ([122], [123]), Herstein proved that a ring \(R\) is commutative if it has no nonzero nilpotent ideal and there is a fixed integer \(n > 1\) such that \((xy)^n = x^n y^n\) for all \(x, y \in R\) (see also [51]). In the special case of Banach algebra \(A\) with an identity, Yood [214] sharpened these results. More precisely, he proved the following theorem:

**Theorem 5.4.1.** Suppose that there are nonempty open subsets \(G_1\) and \(G_2\) of \(A\) such that for each \(x \in G_1\) and \(y \in G_2\) there is an integer \(n = n(x, y) > 1\) where either \((xy)^n - x^n y^n \in M\) or \((xy)^n - y^n x^n \in M\), where \(M\) is a closed linear subspace of \(A\). Then \([x, y] \in M\) for all \(x, y \in A\).

This result motivated us to prove following theorem by using linear derivations:
Theorem 5.4.2. Let $A$ be a unital semisimple Banach algebra and $G_1$ and $G_2$ be open subsets of $A$ such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If $A$ admits a linear derivation $d : A \to A$ such that either $d((xy)^m) + d(x^m)d(y^m) \in Z(A)$ or $d((xy)^m) - d(x^m)d(y^m) \in Z(A)$, then $d(A) \subseteq Z(A)$.

Proof. Fix $x \in G_1$ and for each $n$ we define the set $U_n = \{ y \in A \mid d((xy)^n) + d(x^n)d(y^n) \notin Z(A) \text{ and } y \in A \mid d((xy)^n) - d(x^n)d(y^n) \notin Z(A) \}$. We claim that $U_n$ is open. To show $U_n$ is open we have to show $U_n^c$, the complement of $U_n$, is closed. For this, we take a sequence $(z_k) \in U_n^c$ such that $z_k \to z$ as $k \to \infty$, we need to show $z \in U_n^c$. Since $z_k \in U_n^c$, then

$$d((xz_k)^n) + d(x^n)d(z_k^n) \in Z(A) \quad (5.4.1)$$

or

$$d((xz_k)^n) - d(x^n)d(z_k^n) \in Z(A). \quad (5.4.2)$$

Taking limit on $k$, we obtain

$$\lim_{k \to \infty} (d((xz_k)^n) + d(x^n)d(z_k^n)) \in Z(A) \quad (5.4.3)$$

or

$$\lim_{k \to \infty} (d((xz_k)^n) - d(x^n)d(z_k^n)) \in Z(A). \quad (5.4.4)$$

By Lemma 1.3.26, every linear derivation on semisimple Banach algebra is continuous. Then, the last expression yields that

$$d((x \lim_{k \to \infty} z_k)^n) + d(x^n)d(\lim_{k \to \infty} z_k^n) \in Z(A) \quad (5.4.5)$$

or

$$d((x \lim_{k \to \infty} z_k)^n) - d(x^n)d(\lim_{k \to \infty} z_k^n) \in Z(A). \quad (5.4.6)$$

Since $z_k \to z$ as $k \to \infty$, we are force to conclude that

$$d((xz)^n) + d(x^n)d(z^n) \in Z(A) \quad (5.4.7)$$

or

$$d((xz)^n) - d(x^n)d(z^n) \in Z(A). \quad (5.4.8)$$
This implies that $z \in U_n^c$. Hence, $U_n^c$ is closed i.e., each $U_n$ is open. Thus, by Baire's category theorem, if every $U_n$ is dense then their intersection is also dense, which contradict the existence of $G_2$. Hence there exists a positive integer $\tau$ such that $U_\tau$ is not dense. Therefore, there exists a nonempty open set $G_3$ in the complement of $U_\tau$ such that either $d((xy)\tau) + d(x^\tau)d(y^\tau) \in Z(A)$ or $d((xy)\tau) - d(x^\tau)d(y^\tau) \in Z(A)$ for all $y \in G_3$. Let $v_0 \in G_3$ and $w \in A$. Then $v_0 + tw \in G_3$ for sufficiently small real $t$. Then

$$d((x(v_0 + tw))\tau) + d(x^\tau)d((v_0 + tw)^\tau) \in Z(A) \quad (5.4.9)$$

or

$$d((x(v_0 + tw))\tau) - d(x^\tau)d((v_0 + tw)^\tau) \in Z(A) \quad (5.4.10)$$

for $v_0 \in G_3$ and $w \in A$. Thus at least one of (5.4.9) and (5.4.10) is valid for infinitely many $t$. Suppose (5.4.9) holds for these $t$. Then the expression $d((x(v_0 + tw))\tau) + d(x^\tau)d((v_0 + tw)^\tau)$ can be written as

$$d(A_{\tau,0}(x,v_0,w)) + d(x^\tau)B_{\tau,0}(v_0,w)$$

$$+ d(A_{\tau-1,1}(x,v_0,w)) + d(x^\tau)B_{\tau-1,1}(v_0,w)t$$

$$+ ...$$

$$+ d(A_{1,\tau-1}(x,v_0,w)) + d(x^\tau)B_{1,\tau-1}(v_0,w)t^{\tau-1}$$

$$+ d(A_{0,\tau}(x,v_0,w)) + d(x^\tau)B_{0,\tau}(v_0,w)t^{\tau},$$

where $A_{i,j}(x,y_0,w)$ denotes the sum of all terms in which $xv_0$ appears exactly $i$ times and $xw$ appears exactly $j$ times in the expansion of $d((x(v_0 + tw))\tau)$, where $i$ and $j$ are nonnegative integers such that $i + j = \tau$. Similarly, $B_{i,j}(v_0,w)$ is sum of all terms in which $v_0$ appears exactly $i$ times and $w$ appears exactly $j$ times in the expansion of $d((v_0 + tw)^\tau)$, where $i$ and $j$ are nonnegative integers such that $i + j = \tau$. The above expression is a polynomial in $t$ and the coefficient of $t^\tau$ in this polynomial is $d((xw)^\tau) + d(x^\tau)d(w^\tau)$. In view of Remark 1.2.14, we get $d((xw)^\tau) + d(x^\tau)d(w^\tau) \in Z(A)$. If (5.4.10) is holds for these $t$ then, we are force to conclude that $d((xw)^\tau) - d(x^\tau)d(w^\tau) \in Z(A)$. Thus, given $x \in G_1$ there is a positive integer $\tau$ depending on $w$ such that for each $w \in A$ either $d((xw)^\tau) + d(x^\tau)d(w^\tau) \in Z(A)$ or $d((xw)^\tau) - d(x^\tau)d(w^\tau) \in Z(A)$. Next, fix $y \in A$ and for each positive integer $k$, set $V_k = \{v \in A \mid d((vy)^k) + d(v^k)d(y^k) \not\in Z(A) \text{ and } d((vy)^k) - d(v^k)d(y^k) \not\in Z(A)\}$. Each $V_k$ is open (as we have shown above).
If each $V_k$ is dense, then by Baire’s category theorem so is intersection also but this contrary to what was shown earlier concerning the open set $G_1$. Hence, there is an integer $m > 1$ and a nonempty open subset $G_4$ in the complement of $V_m$. If $x_0 \in G_4$ and $y \in A$, then $x_0 + tu \in G_4$ for all sufficiently small $t$. Hence for positive integer $m > 1$ either

$$d(((x_0 + tu)y)^m) + d((x_0 + tu)^m)d(y^m) \in Z(A)$$

or

$$d(((x_0 + tu)y)^m) - d((x_0 + tu)^m)d(y^m) \in Z(A)$$

for each $u \in A$ and $x_0 \in G_4$. Arguing as above we see that either $d((uy)^m) + d(u^m)d(y^m) \in Z(A)$ or $d((uy)^m) - d(u^m)d(y^m) \in Z(A)$ for each $u \in A$.

Now $S_k$, $k > 1$, be the set of $y \in A$ such that for each $w \in A$ either $d((wy)^k) + d(w^k)d(y^k) \in Z(A)$ or $d((wy)^k) - d(w^k)d(y^k) \in Z(A)$, then the union of $S_k$ will be $A$. It can be easily prove that each $S_k$ is closed. Hence again by Baire’s category theorem some $S_l$ must have a nonempty open subset $G_5$. Let $y_0 \in G_5$, for all sufficiently small real $t$ and each $z \in A$ either

$$d((w(y_0 + tz))^t) + d(w^t)d((y_0 + tz)^t) \in Z(A)$$

or

$$d((w(y_0 + tz))^t) - d(w^t)d((y_0 + tz)^t) \in Z(A).$$

By earlier arguments, we have for each $w, z \in A$ either $d((wz)^t) + d(w^t)d(z^t) \in Z(A)$ or $d((wz)^t) - d(w^t)d(z^t) \in Z(A)$. Next, since $A$ is unital then, for all real $t$ either

$$d(((e + tx)y)^n) + d((e + tx)^n)d(y^n) \in Z(A)$$

or

$$d(((e + tx)y)^n) - d((e + tx)^n)d(y^n) \in Z(A)$$

for all $x, y \in A$. Hence taking coefficient of $t$ in the expansion of above equations, we get either

$$d(xy^n + \sum_{k=1}^{n-1} y^kxy^{n-k}) + nd(x)d(y^n) \in Z(A) \quad (5.4.11)$$
or
\[
d(x^n y + \sum_{k=1}^{n-1} y^k x y^{n-k}) - nd(x)d(y^n) \in Z(A)
\] (5.4.12)

for all \(x, y \in A\). Now, taking \(d((y(e + tx))y^n)\) instead of \(d((e + tx)y)y^n)\), we see that either
\[
d(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) + nd(y^n)d(x) \in Z(A)
\] (5.4.13)
or
\[
d(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - nd(y^n)d(x) \in Z(A)
\] (5.4.14)

for all \(x, y \in A\). Then at least one pair of equations \{((5.4.11), (5.4.13)), ((5.4.11), (5.4.14)), ((5.4.12), (5.4.13))\} and \{(5.4.12), (5.4.14)\} must hold. Then, we get
\[
d([x, y^n]) \pm n[d(x), d(y^n)] \in Z(A)
\]

for all \(x, y \in A\). Replace \(y\) by \(e + ty\) in above equation and using same arguments as we have used above, we get
\[
d([x, y]) \pm n[d(x), d(y)] \in Z(A)
\] (5.4.15)

for all \(x, y \in A\). First we assume that
\[
d([x, y]) + n[d(x), d(y)] \in Z(A)
\]

for all \(x, y \in A\). Replace \(y\) by \(yw\) in above expression, we obtain
\[
d([x, y])w + [x, y]d(w) + nd(y)[d(x), w] + n[d(x), d(y)]w + n[d(x), y]d(w) + ny[d(x), d(w)] \in Z(A)
\] (5.4.16)
for all $x, y, w \in A$. Taking $y = e + y$ in (5.4.16), we obtain

$$n[d(x), d(w)] \in Z(A)$$

(5.4.17)

for all $x, w \in A$. In particular, we have

$$[[d(x), d(w)], d(w)] = 0$$

(5.4.18)

for all $x, w \in A$. Then, by Lemma 1.3.13 each $z = [d(x), d(w)]$ is a generalized nilpotent element of $A$ i.e., $\rho(z) = \|z^n\|^{1/n} = 0$. Therefore, $[d(x), d(w)]$ is a generalized nilpotent element in $Z(A)$ and so is in radical of $Z(A)$. Since $A$ is semisimple, so $Z(A)$ will also be semisimple. Therefore, we conclude that

$$[d(x), d(w)] = 0$$

(5.4.19)

for all $x, w \in A$. Next, replace $w$ by $ya$ in (5.4.19), to get

$$d(y)[d(x), a] + [d(x), y]d(a) = 0$$

for all $a, x, y \in A$. Taking $a = x$ in above expression, we get

$$d(y)[d(x), x] + [d(x), y]d(x) = 0$$

(5.4.20)

for all $x, y \in A$. Again, replace $y$ by $xy$ in (5.4.20), to obtain

$$d(x)yd(x) + [d(x), x]yd(x) = 0$$

(5.4.21)

for all $x, y \in A$. By Lemma 1.3.12, we have

$$[d(x), x]yd(x) = 0$$

(5.4.22)

for all $x, y \in A$. In particular, it follows that $[d(x), x]yd(x) = 0$ for all $x \in A$. Since $A$ is semisimple, the last expression yields that $[d(x), x] = 0$ for all $x \in A$. Hence, $d(A) \subseteq Z(A)$.

In a similar manner, we can prove that the same conclusion holds for the case
\[ d([x, y]) - [d(x), d(y)] \in Z(A) \] for all \( x, y \in A \). Thereby the proof of the theorem is completed. \( \square \)

**Corollary 5.4.1.** Let \( A \) be a unital prime Banach algebra and \( G_1 \) and \( G_2 \) be open subsets of \( A \) such that for each \( x \in G_1 \), and \( y \in G_2 \) there is an integer \( m = m(x, y) > 1 \). If \( A \) admits a continuous linear derivation \( d : A \to A \) such that either \( d((xy)^m) + d(x^m)d(y^m) \in Z(A) \) or \( d((xy)^m) - d(x^m)d(y^m) \in Z(A) \), then \( A \) is commutative.

**Theorem 5.4.3.** Let \( A \) be a unital semisimple Banach algebra and \( G_1 \) and \( G_2 \) be open subsets of \( A \) such that for each \( x \in G_1 \), and \( y \in G_2 \) there is an integer \( m = m(x, y) > 1 \). If \( A \) admits a linear derivation \( d : A \to A \) such that either \( d((xy)^m) + x^my^m \in Z(A) \) or \( d((xy)^m) - x^my^m \in Z(A) \), then \( d(A) \subseteq Z(A) \).

**Proof.** Proceeding on same lines as in case of Theorem 5.4.2, we arrive at either
\[ d(((e+tx)y)^n) + (e+tx)^ny^n \in Z(A) \]

or
\[ d(((e+tx)y)^n) - (e+tx)^ny^n \in Z(A) \]

for all \( x, y \in A \). Hence taking coefficient of \( t \) in the expansion of above equations, we get either
\[ d(xy^n + \sum_{k=1}^{n-1} y^kxy^{n-k}) + nxy^n \in Z(A) \] (5.4.23)

or
\[ d(xy^n + \sum_{k=1}^{n-1} y^kxy^{n-k}) - nxy^n \in Z(A) \] (5.4.24)

for all \( x, y \in A \). Now, taking \( d(y(e+tx)^n) \) instead of \( d((e+tx)y)^n \), we see that either
\[ d(y^n x + \sum_{k=1}^{n-1} y^kxy^{n-k}) + ny^n x \in Z(A) \] (5.4.25)
or
\[ d(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n y^n x \in Z(A) \] (5.4.26)
for all \( x, y \in A \). Then at least one pair of equations \{(5.4.23), (5.4.25)\}, \{(5.4.23), (5.4.26)\}, \{(5.4.24), (5.4.25)\} and \{(5.4.24), (5.4.26)\} must hold. Then we obtain
\[ d([x, y^n]) \pm n[x, y^n] \in Z(A) \] (5.4.27)
for all \( x, y \in A \). Replace \( y \) by \( c + ty \) in above equation and using same arguments as we have used above, we get
\[ d([x, y]) \pm n[x, y] \in Z(A) \] (5.4.28)
for all \( x, y \in A \). Suppose
\[ d([x, y]) + n[x, y] \in Z(A) \] (5.4.29)
for all \( x, y \in A \). That is,
\[ [[d(x), y] + [x, d(y)] + n[x, y], x] = 0 \] (5.4.30)
for all \( x, y \in A \). Replace \( y \) by \( [y, z] \) in above expression, we get
\[ [d(x), [y, z]] + [x, d([y, z])] + n[x, [y, z]] \in Z(A) \] (5.4.31)
for all \( x, y \in A \). That is,
\[ [d(x), [y, z]] + [x, d([y, z])] + n[y, z] \in Z(A) \] (5.4.32)
for all \( x, y \in A \). Application of (5.4.29) yields that
\[ [d(x), [y, z]] \in Z(A) \] (5.4.33)
for all \( x, y \in A \). By Lemma 1.3.24, \( d(x) \in Z(A) \) for all \( x \in A \) i.e., \( d(A) \subseteq Z(A) \).
Similar conclusion holds for the case $d([x,y]) - [x,y] \in \mathcal{Z}(A)$ for all $x, y \in A$. This proves the theorem. 

**Corollary 5.4.2.** Let $A$ be a unital prime Banach algebra and $G_1$ and $G_2$ be open subsets of $A$ such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If $A$ admits a continuous linear derivation $d : A \to A$ such that either $d((xy)^m) + x^m y^m \in \mathcal{Z}(A)$ or $d((xy)^m) - x^m y^m \in \mathcal{Z}(A)$, then $A$ is commutative.

### 5.5 Applications

In this section we will use the previous algebraic results to study commutativity of Banach algebras involving linear derivations. Let us recall some elementary notions for the sake of completeness. A always denotes a Banach algebra which is a complex normed algebra and its underlying vector space is a Banach space. The Jacobson radical of $A$ is the intersection of all primitive ideals of $A$ and is denoted by $\text{rad}(A)$. We assume that all mappings on Banach algebra $A$ are linear mappings in the whole section, and that $m > 1$ and $n > 1$ are integers depending on $x$ and $y$.

In [138], Johnson and Sinclair proved that any linear derivation on semisimple Banach algebra is continuous. A result of Singer and Wermer [192] states that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Combining these two results one obtains that there are no nonzero linear derivations on a commutative semisimple Banach algebra. In 1988, Thomas [193] generalized the Singer-Wermer theorem by proving that any linear derivation on a commutative Banach algebra maps the algebra into its radical. Obviously, this result also implies that any linear derivation on a commutative semisimple Banach algebra is zero. First noncommutative extension of Singer-Wermer theorem has been proved by Yood [212], by showing that if for all pairs $x, y \in A$, where $A$ is noncommutative Banach algebra, the element $[d(x), y]$ lies in the $\text{rad}(A)$, then $d$ maps $A$ into $\text{rad}(A)$. In [81], Brešar and Vukman generalized the Yood's result by proving that in case $[d(x), x] \in \text{rad}(A)$ for all $x \in A$, then $d$ maps $A$ into $\text{rad}(A)$. A number of authors have proved the same kind of results, we refer the reader to [71] [78], [141], [176] and [213], where further references can be found.

We proceed our discussion with the following theorem.
Theorem 5.5.1. Let $A$ be a semiprime Banach algebra and $G_1$ and $G_2$ be nonempty open subsets of $A$. If $A$ admits a continuous derivation $d$ such that either $d([x^m, y^n]) - [x^m, y^n] \in \mathcal{Z}(A)$ or $d([x^m, y^n]) + [x^m, y^n] \in \mathcal{Z}(A)$ for all $x \in G_1$ and $y \in G_2$, then $A$ is commutative.

Proof. If $d = 0$, then $[x^m, y^n] \in \mathcal{Z}(A)$ for all $x \in G_1$ and $y \in G_2$, then by Lemma 1.3.38, $A$ is commutative. Now, we assume that $d \neq 0$ and for fix $x \in G_1$, we define the set $V_{m,n} = \{y \in A \mid d([x^m, y^n]) - [x^m, y^n] \notin \mathcal{Z}(A)$ and $d([x^m, y^n]) + [x^m, y^n] \notin \mathcal{Z}(A)\}$. We claim that each $V_{m,n}$ is open in $A$. To show $V_{m,n}$ is open we have to show $V_{m,n}^c$, the complement of $V_{m,n}$, is closed. For this, we take a sequence $(z_k) \in V_{m,n}^c$ such that $z_k \to z$ as $k \to \infty$, we need to show $z \in V_{m,n}^c$. Since $z_k \in V_{m,n}^c$, then

$$d([x^m, z_k^n]) - [x^m, z_k^n] \in \mathcal{Z}(A) \quad (5.5.1)$$

or

$$d([x^m, z_k^n]) + [x^m, z_k^n] \in \mathcal{Z}(A). \quad (5.5.2)$$

Taking limit on $k$, we obtain

$$\lim_{k \to \infty} (d([x^m, z_k^n]) - [x^m, z_k^n]) \in \mathcal{Z}(A) \quad (5.5.3)$$

or

$$\lim_{k \to \infty} (d([x^m, z_k^n]) + [x^m, z_k^n]) \in \mathcal{Z}(A). \quad (5.5.4)$$

Since $d$ is continuous, the last expression yields that

$$d([x^m, \lim_{k \to \infty} z_k^n]) - [x^m, \lim_{k \to \infty} z_k^n] \in \mathcal{Z}(A) \quad (5.5.5)$$

or

$$d([x^m, \lim_{k \to \infty} z_k^n]) + [x^m, \lim_{k \to \infty} z_k^n] \in \mathcal{Z}(A). \quad (5.5.6)$$

Since $z_k \to z$ as $k \to \infty$, we are force to conclude that

$$d([x^m, z^n]) - [x^m, z^n] \in \mathcal{Z}(A) \quad (5.5.7)$$

or

$$d([x^m, z^n]) + [x^m, z^n] \in \mathcal{Z}(A). \quad (5.5.8)$$
This implies that $z \in V_{m,n}^c$. Hence $V_{m,n}^c$ is closed i.e., each $V_{m,n}$ is open. By Baire’s category theorem, if every $V_{m,n}$ is dense then their intersection is also dense, which contradict the existence of $G_1$ and $G_2$. Hence there are positive integers $r, s$ such that $V_{r,s}$ is not dense in $A$. Therefore there exists a nonempty open subset $G_3$ in the complement of $V_{r,s}$ such that for each $y \in G_3$ either $d([x^r, y^s]) - [x^r, y^s] \in Z(A)$ or $d([x^r, y^s]) + [x^r, y^s] \in Z(A)$. Let $y_0 \in G_3$ and $w \in A$, then $y_0 + tw \in G_3$ for all sufficiently small real $t$. Therefore, for each $t$ either

$$d([x^r, (y_0 + tw)^s]) - [x^r, (y_0 + tw)^s] \in Z(A)$$  \hspace{1cm} (5.5.9)

or

$$d([x^r, (y_0 + tw)^s]) + [x^r, (y_0 + tw)^s] \in Z(A).$$  \hspace{1cm} (5.5.10)

Then at least one of (5.5.9) or (5.5.10) must be holds for infinitely many $t$. Suppose (5.5.9) holds for these $t$. Now, $d([x^r, (y_0 + tw)^s]) - [x^r, (y_0 + tw)^s]$ can be written as

$$d([x^r, A_{s,0}(y_0, w)]) - [x^r, A_{s,0}(y_0, w)]$$
$$+(d([x^r, A_{s-1,1}(y_0, w)]) - [x^r, A_{s-1,1}(y_0, w)])t$$
$$+...$$
$$+(d([x^r, A_{1,s-1}(y_0, w)]) - [x^r, A_{1,s-1}(y_0, w)])t^{s-1}$$
$$+(d([x^r, A_{0,s}(y_0, w)]) - [x^r, A_{0,s}(y_0, w)])t^s,$$

where $A_{i,j}(y_0, w)$ denotes the sum of all terms in which $y_0$ appears exactly $i$ times and $w$ appears exactly $j$ times in the expansion of $(y_0 + tw)^s$, where $i$ and $j$ are nonnegative integers such that $i + j = s$. The above expression is a polynomial in $t$ and the coefficient of $t^s$ in this polynomial is $d([x^r, w^s]) - [x^r, w^s]$. In view of Remark 1.2.14, we obtain $d([x^r, w^s]) - [x^r, w^s] \in Z(A)$. Similarly, if (5.5.10) holds for these $t$, then we are force to conclude that $d([x^r, w^s]) + [x^r, w^s] \in Z(A)$.

Thus, given $z \in G_1$ there are positive integers $r, s$ depending on $w$ so that for each $w \in A$ either $d([x^r, w^s]) - [x^r, w^s] \in Z(A)$ or $d([x^r, w^s]) + [x^r, w^s] \in Z(A)$. Let $F_1 = \{w \in A \mid d([x^r, w^s]) - [x^r, w^s] \in Z(A)\}$ and $F_2 = \{w \in A \mid d([x^r, w^s]) + [x^r, w^s] \in Z(A)\}$. Then $A$ must be union of $F_1$ and $F_2$, and each $F_k, (k = 1, 2)$ is closed (as we have shown above). Thus, by Baire’s category theorem, at least one of $F_1$ and $F_2$ must contain a nonempty open subset of $A$.

Suppose $F_1$ contains a nonempty open subset $G_4$ of $A$. Let $v_0 \in G_4$ and $z \in A$. Then
\( v_0 + tz \in G_4 \) for sufficiently small \( t \). For these \( t \), we have

\[
d([x^r, (v_0 + tz)^s]) - [x^r, (v_0 + tz)^s] \in Z(A).
\]

This can be written as a polynomial in \( t \) (as above) in which the coefficient of \( t^s \) is \( d([x^r, z^s]) - [x^r, z^s] \in Z(A) \) for all \( z \in A \). Likewise, if \( F_2 \) contains a nonempty open subset then \( d([x^r, z^s]) + [x^r, z^s] \in Z(A) \) for all \( z \in A \). Consequently, given \( x \in G_1 \) there are positive integers \( r \) and \( s \) so that either \( d([x^r, z^s]) - [x^r, z^s] \in Z(A) \) or \( d([x^r, z^s]) + [x^r, z^s] \in Z(A) \) for all \( z \in A \).

Now, we reverse the roles of \( G_1 \) and \( G_2 \) in the above settings. Proceeding as above we find that either \( d([x^r, z^s]) - [x^r, z^s] \in Z(A) \) or \( d([x^r, z^s]) + [x^r, z^s] \in Z(A) \) for all \( x, z \in A \). Then by Theorem 5.2.2, \( A \) is commutative. This proves the theorem completely. \( \square \)

The proof of the following theorem is same as the proof of Theorem 5.5.1 and the application of Theorems 5.2.2 and 5.2.4, so we only state the following result and do not provide its proof.

**Theorem 5.5.2.** Let \( A \) be a semiprime Banach algebra and \( G_1 \) and \( G_2 \) be nonempty open subsets of \( A \). If \( A \) admits a nonzero continuous derivation \( d \) such that either \( d([x^m, y^n]) \in Z(A) \) or \( d(x^m \circ y^n) \in Z(A) \) for all \( x \in G_1 \) and \( y \in G_2 \), then \( A \) is commutative.