Chapter 2

The Nonlinear Schrodinger equation and sources

It has been well established that the nonlinear Schrodinger equation (NLSE) describes a wide class of physical phenomena e.g., modulational instability of water waves, propagation of heat pulses in anharmonic crystals, helical motion of a very thin vortex filament, nonlinear modulation of collisionless plasma waves, and self trapping of a light beam in a color dispersive system [1]. In optical fibers, the solitons of the NLSE provide a secure means to carry bits of information over many thousands of kilometers [2]. In many of these examples the equation appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating through nonlinear medium. When termed as Gross-Pitaevskii equation; the NLSE with an appropriate potential can be utilized to describe the dynamics of the Bose-Einstein condensate, both with the attractive and repulsive nonlinearities. It is our objective in this chapter to present the solutions of the NLSE with an external source and a gain or lossy term, and study the numerical stability of some of the solutions. Like KdV equation, the NLSE is a generic wave equation, arising in the study of unidirectional propagation of wave packets in a dispersive, energy conserving medium at the lowest order of nonlinearity.
2.1 The nonlinear Schrodinger equation

In certain dielectric materials, the refractive index increases in proportion to the square of the electric field, this property is known as Kerr effect. Then the refractive index can be written as

\[
n = n_s(\omega) + n_2|E|^2.
\]

Consider an electromagnetic wave (in a scalar form) represented by the function:

\[
\psi = a(z, t)e^{i(\omega_s t - k_s z)}, \quad (2.1)
\]

where \(a(z, t)\) is a dimensionless complex amplitude representing the slowly varying envelope of the wave and \(\omega_s\) is the wave central frequency \((k_s = n\omega_s/c)\). The wave intensity is given by 

\[
I = I_c \ |\psi(z, t)|^2,
\]

where \(I_c\) is a constant. The optical Kerr effect increases the refractive index by the quantity \(\delta n = n_2 I\), where \(n_2\) is the nonlinear refractive index coefficient. We assume that near the central frequency of the wave, the following dispersion relation holds:

\[
k(\omega) = k_s + (\omega - \omega_s)(\frac{\partial k}{\partial \omega})_s + (\omega - \omega_s)^2 \left( \frac{\partial^2 k}{\partial \omega^2} \right)_s + \frac{(\omega - \omega_s)^3}{3!} \left( \frac{\partial^3 k}{\partial \omega^3} \right)_s + \cdots + k_2 I + I(\omega - \omega_s) \left( \frac{\partial k_2}{\partial \omega} \right)_s + \cdots \quad (2.2)
\]

Equation (2.2) is a Taylor development of the wave vector near \(\omega_s\), with the addition of the effect of nonlinearity \(8k = k_2 I\) with \(k_2 = n_2\omega_s/c\). In this equation, if we replace the derivatives \(A, k''\) etc. through their relationship to the group velocity, \(v_g = \frac{\partial \omega}{\partial k}\), then we arrive at:

\[
\frac{1}{v_g} = \frac{\partial k}{\partial \omega} = (\frac{\partial k}{\partial \omega})_s + (\omega - \omega_s)\left( \frac{\partial^2 k}{\partial \omega^2} \right)_s + \frac{(\omega - \omega_s)^2}{2}\left( \frac{\partial^3 k}{\partial \omega^3} \right)_s + \cdots + I\left( \frac{\partial k_2}{\partial \omega} \right)_s + \cdots \\
= k' + (\omega - \omega_s)k'' + \frac{(\omega - \omega_s)^2}{2}k''' + \cdots \quad (2.3)
\]
At the frequencies of interest for solitons in single-mode fibers, the terms in $k''$ and $k_2 I$ in Eq. (2.2) have comparable magnitudes, while the higher order terms in $k'''$ and $I k'_2$ are like small perturbations. Thus Eq. (2.2) can be approximated by:

$$\Delta k \approx \Delta \omega k' + \frac{\Delta \omega^2}{2} k'' + k_2 I,$$

(2.4)

with $\Delta k = k - k_s$, and $\Delta \omega = \omega - \omega_s$. From Eq. (2.3), we find that,

$$k'' \approx \frac{\partial}{\partial \omega} \left( \frac{1}{v_g} \right) = -\frac{1}{v_g^2} \frac{\partial v_g}{\partial \omega},$$

(2.5)

which shows that $k''$ is the dispersion of the wave's group velocity.

We now consider the Fourier transform of the envelope function:

$$\hat{\psi}(\Delta k, \Delta \omega) = \int \psi(z,t) e^{i(\Delta \omega t - \Delta k z)} dt dz,$$

(2.6)

$$\psi(z,t) = \frac{1}{(2\pi)^2} \int \hat{\psi}(\Delta k, \Delta \omega) e^{-i(\Delta \omega t - \Delta k z)} d(\Delta \omega) d(\Delta k).$$

(2.7)

From Eqs. (2.6) and (2.7) we can show that the quantities $\partial \psi / \partial z = i \Delta k \psi$ and $\partial \psi / \partial t = -i \Delta \omega \psi$ are the Fourier transforms of $i \Delta k \hat{\psi}$ and $-i \Delta \omega \hat{\psi}$, respectively. Thus $\Delta k$, $\Delta \omega$ can be put into the form of operators $-i \partial / \partial z$, $i \partial / \partial t$. By substituting these operator forms into Eq. (2.4), we get:

$$-i \frac{\partial}{\partial z} \approx ik' \frac{\partial}{\partial t} - \frac{k''}{2} \frac{\partial^2}{\partial z^2} + k_2 I.$$

(2.8)

Applying Eq. (2.8) to the wave envelope $\psi(z,t)$ and using $I = I_c | \psi(z,t) |^2$, we obtain:

$$\frac{i}{k'} \frac{\partial \psi}{\partial z} + ik' \frac{\partial \psi}{\partial t} - \frac{k''}{2} \frac{\partial^2 \psi}{\partial t^2} + k_2 I_c | \psi(z,t) |^2 \psi \approx 0$$

(2.9)

Equation (2.9) can be transformed to correspond to a retarded time frame and be made dimensionless through the following substitutions: $\tau = (t - k' z) / t_c$, $\chi = z / z_c$, where $t_c$, $z_c$ are constants with dimensions of time and space, respectively. Hence, from Eq. (2.9), we get

$$-i \frac{t_c^2}{k'' z_c} \frac{\partial \psi}{\partial \chi} + \frac{1}{t_c^2} \frac{\partial^2 \psi}{\partial \tau^2} - \frac{t_c^2}{k'' k_2 I_c} | \psi |^2 \psi = 0.$$  

(2.10)
If the constants are defined as below, \( \frac{\partial^2}{\partial x^2} = -k'' \) and \( z_c = \frac{1}{k_2^2} \), then the above equation can be cast into the canonical form:

\[
\frac{i}{\partial x} \psi + \frac{1}{2} \frac{\partial^2 \psi}{\partial \tau^2} + |\psi|^2 \psi = 0.
\] (2.11)

This is the celebrated nonlinear Schrödinger equation. It is so called, because of its similarity in appearance with the Schrödinger equation in quantum theory. In the case where the medium has gain or loss, a term \(-i\Gamma\psi\) with \( \Gamma = z_c g/2 \) must be added to the left-hand side of Eq. (2.11), where \( g \) is the net power gain coefficient:

\[
\frac{i}{\partial x} \psi + \frac{1}{2} \frac{\partial^2 \psi}{\partial \tau^2} + |\psi|^2 \psi - i\Gamma \psi = 0.
\] (2.12)

It is our goal in this thesis, to present the solitary wave solutions of the NLSE in the presence of a source, and also with dispersion managed lossy and gain parameters. Before going into the details of NLSE interacting with external sources, we concentrate on free NLSE.

The NLSE is a second order nonlinear partial differential equation, which can contain localized solitons as solutions. Zakharov and Shabat solved the NLSE by the aid of inverse scattering method [3]. It is interesting to notice that, the solitons emerge, when the nonlinearity balances the dispersion. Solitons are stable, localized waves that propagate in a nonlinear medium without spreading. Solitons may be either bright or dark, depending on the details of the governing equation. A bright soliton is a peak in the amplitude; a dark soliton is a notch with a characteristic phase step across it. In addition to its solitons, NLSE supports periodic waves and exact N-soliton solutions [4].
2.2 The solutions of NLSE in terms of Jacobi elliptic functions

For the purpose of illustration we solve below, the NLSE in terms of the elliptic functions. The intended solutions are the traveling wave solutions, which can exhibit chirping. The NLSE is given by

\[ i\hbar \partial_t \psi + \frac{\hbar^2}{2m} \partial_x^2 \psi + g |\psi|^2 \psi = 0 \quad , \tag{2.13} \]

where \( g \) is real. In the scaled variables \( y = \sqrt{\frac{2m}{\hbar^2}} x \), and \( t = \frac{t'}{\hbar} \), Eq. \( (2.13) \) takes the form:

\[ i \partial_t \psi + \partial_y^2 \psi + g |\psi|^2 \psi = 0 \quad . \tag{2.14} \]

Using the following ansatz,

\[ \psi(y, t) = e^{i[\phi(\xi) - \omega t]} a(\xi) \quad , \tag{2.15} \]

where \( \xi = \alpha(y - vt) \), we can separate the real and the imaginary parts of the equation. The imaginary part

\[ -u \alpha' a' + \alpha^2 \psi'' a + 2 \alpha^2 \psi' a' = 0 \quad , \tag{2.16} \]

can be straightforwardly solved to give

\[ \psi' = \frac{u}{2\alpha} + \frac{P}{\alpha a^2} \quad , \tag{2.17} \]

where, \( P \) is the integration constant. For \( P \neq 0 \), the solutions exhibit chirping. For simplicity, we consider \( P = 0 \) in which case the real part is given by

\[ \epsilon a + \alpha^2 a'' + ga^3 = 0 \quad , \tag{2.18} \]

where \( \epsilon = \frac{v^2}{\alpha} - \omega \). The solutions of the above equation are the well-known elliptic functions. Below we tabulate a few cnoidal solutions of the above equation.
As is clear from the above solutions the signs of $y$ (attractive and repulsive) and $\epsilon$ play crucial roles in finding the solutions. This fact will become more explicit as we will use the solutions of the above equation repeatedly in our derivations.

### 2.3 The NLSE equation with a source

In this section, we present a wide class of rational solutions of the NLSE with a source, using a fractional transformation (FT). The solutions of NLSE, phase locked with a source, are exactly connected to the elliptic functions. These are necessarily of the rational type and are nonperturbative in nature. The numerical simulations revealed that some of these solitary waves are stable. We also present an elegant numerical technique to test the numerical stability of these exact solutions.

Much attention has been paid to the study of the externally driven NLSE after the seminal work of Kaup and Newell [5]. This equation features...
prominently in the problem of optical pulse propagation in asymmetric, twin-core optical fibers [6, 7, 8]; currently an area of active research. Of the several applications of an externally driven NLSE, perhaps the most important ones are to long Josephson junctions [9], charge density waves [10], and plasmas driven by rf fields [11]. The phenomenon of autoresonance [12], indicating a continuous phase locking between the solutions of NLSE and the driving field, has been found to be a key characteristic of this system. In the presence of damping, this dynamical system exhibits rich structure including bifurcation. This is evident from analyses around a constant background as well as numerical investigations [13, 14, 15]. Although NLSE is a well-studied integrable system [16], no exact solutions have so far been found for the NLSE with a source, to the best of the authors' knowledge. All the above inferences have been drawn through perturbations around solitons and numerical techniques.

In this chapter, we map exactly, the traveling wave solutions of the NLSE phase-locked with a source, to the elliptic functions, through the FT. It was found that the solutions are necessarily of the rational type, with both the numerator and denominator containing terms quadratic in elliptic functions, in addition to having constant terms. It is well-known that the solitary wave solutions of the NLSE [17, 8] are cnoidal waves, which contain the localized soliton solutions in the limit, when the modulus parameter equals one [19]. Hence, the solutions found here, for the NLSE with a source, are nonperturbative in nature. We find both bright and dark solitons as also singular ones. Solitons and solitary pulses show distinct behavior. In the case, when the source and the solutions are not phase matched, perturbation around these solutions may provide a better starting point.

For nonlinear equations, a number of transformations are well-known in the literature, which map the solutions of a given equation to the other [20, 21]. The familiar example being the Miura transformation [22], which
maps the solutions of the modified KdV to those of the KdV equation. To find static and propagating solutions, appropriate transformations have also been cleverly employed, to connect the nonlinear equations to the ones satisfied by the elliptic functions: \( f'' \pm a f' \pm A f^3 = 0 \). Here and henceforth, prime denotes derivative with respect to the argument of the function. Solitons and solitary wave solutions of KdV, NLSE, and sine-Gordon etc., can be easily obtained in terms of the elliptic functions in this manner.

Below, we consider the NLSE coupled to an external traveling wave field \( \eta \):

\[
\imath \hbar \partial_t q + \frac{\hbar^2}{2m} \partial_x^2 q + g |q|^2 q + \mu q - \eta = 0.
\]

where \( g \) and \( \mu \) are real. In the scaled variable; \( y = \sqrt{\frac{2m}{\hbar^2}} x \), and \( t = \frac{\imath t}{\hbar} \), the above equation takes the dimensionless form:

\[
i \partial_t q + \partial_x^2 q + g |q|^2 q + \mu q - \eta = 0.
\]

Using the following ansatz,

\[
q(y, t) = e^{i[\bar{\psi}(\xi) - \omega t]} a(\xi);
\]

we derive the moving solutions of Eq. (2.19). Here \( \xi = \alpha (y - vt) \), and choosing the source term as \( \eta(\xi) = k e^{i[\bar{\psi}(\xi) - \omega t]} \), we can separate the real and the imaginary parts of the equation as:

\[
\nu \alpha a \psi' + (\mu - \omega) a - \alpha^2 \psi'' a + \alpha^2 a'' + ga^3 - k = 0,
\]

and

\[
-\nu \alpha a' + \alpha^2 \psi'' a + \alpha^2 \psi' a' = 0,
\]

where the primes denote the derivatives with respect to \( \xi \) variable. Equation (2.21) can be straightforwardly solved to give

\[
\psi' = \frac{v}{2\alpha} + \frac{P}{\alpha a^2},
\]
where $P$ is the integration constant, which has been set to zero in the following. Thus the single nonlinear ordinary differential equation we have to solve is

$$\epsilon a + \alpha^2 a'' + g a^3 - k = 0,$$

with $\epsilon = \frac{v^2}{4} + \mu - \omega$. 

### 2.4 Solitary wave solutions of the NLSE with a source

#### 2.4.1 General solutions

We start with an ansatz solution of equation (2.23) of the form,

$$u(\xi) = \frac{A + B f^\delta(\xi, m)}{1 + D f^\delta(\xi, m)},$$

with $\delta$ taking integer values. After substitution, the coefficients of $f^n(\xi, m)$, for $n = 0, 2, 4, 6$ etc., can be set to zero, to reduce the problem to a set of algebraic equations. Since the goal is to map the solutions of Eq. (2.23) to elliptic functions, use was made of the following relations for various derivatives of $f$: $f'' = f - f^3$, and $f'^2 = f^2 - \frac{1}{2} f^4 + 2E_0$, where $E_0$ is the integration constant. It was found that $\delta$ takes the unique value 2 for consistency. The consistency conditions will be solved below for specific choices of $f$. However, it is worth noticing that several interesting special cases already emerge from preliminary analysis.

Case(i):

For $A = 0$ and $B \neq 0$, we find that the solution is given by

$$a(\xi) = \frac{k/4E_0f^2}{1 + (1/8E_0)f^2}.$$ 

Case(ii):

In another scenario, $B = 0$ and $A \neq 0$; the solution is

$$a(\xi) = \frac{2k}{1 - f^2}.$$
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which can be potentially singular. Although \( f \), in principle, can be any of the twelve Jacobi elliptic functions, we concentrate primarily on: \( cn(\xi, m) \), \( dn(\xi, m) \), and \( sn(\xi, m) \). The identities (\( cn^2 = 1 - sn^2 \), \( dn^2 = 1 - rasn^2 \)) satisfied by these cnoidal functions come handy in the algebraic analysis. For definiteness, we start with \( f = cn(\xi, m) \); then the coefficients of \( cn^n(\xi, m) \), for \( n = 0, 2, 4, 6 \) can be set to zero reducing the problem to a set of four algebraic equations as given below:

\[
Ae - 2\alpha^2(AD - B)(l - m) + gA^3 - k = 0, \quad (2.24)
\]

\[
2\epsilon AD + \epsilon B + 6\alpha^2(AD - B)D(l - m) - 4\alpha^2(AD - B)(2rn - 1) + 3gA^2B - 3kD = 0, \quad (2.25)
\]

\[
AeD^2 + 2\epsilon BD + 4\alpha^2(AD - B)D(2m......1) + 6\alpha^2(AD - B)m + 3gAB^2 - 3kD^2 = 0, \quad (2.26)
\]

\[
\epsilon BD^2 - 2\alpha^2(AD - B)Dm + gB^3 - kD^3 = 0. \quad (2.27)
\]

We notice that in Eqs. (2.24)-(2.27), the free parameters are \( A, B, D, \alpha, \) and the modulus parameter \( m \). In what follows, we demonstrate, under various limits, how the imbalance between the group velocity dispersion of the slowly varying envelope mounted on a weakly varying carrier wave with the cubic nonlinearity can lead to solitary wave solutions in the presence of an appropriate source. These solutions may find applications in long distance optical communications [16]. First we analyze the general cases and then move on to the special ones.

From the consistency conditions, it is clear that when the source is switched off, the solitary wave envelope is not of a rational type. This is because, for \( D = 0 \), Eq. (2.27) yields \( B = 0 \). This indicates that, excluding a constant solution, there exists no other solution for Eq. (2.23). For
\( AD = B \), it is observed that all the four equations are identical, a flat background solution is obtained. This shows that, \( a(\xi) = A + Bcn^2(\xi, m) \), type of solutions do not exist for Eq. (2.23). Instead of \( cn \), if one chooses \( dn \) or \( sn \) for \( f \), the same scenario emerges. Hence, \( a(\xi) \) is necessarily of the rational type. It should be noted that these are nonperturbative solutions, which can not be obtained from the elliptic function solutions of the NLSE given earlier, through perturbative means.

As mentioned earlier, the special cases lead to a number of interesting solutions. These contain both periodic and hyperbolic type solutions, some of which may be singular. We present them below, with the specifications for the regimes in which they exist.

### 2.4.2 Special solutions

#### Case(I): Trigonometric solution

For \( A = 0 \) and \( m = 0 \); we found that \( \alpha^2 = -\epsilon/4 \); hence, \( \epsilon \) has to be negative, which puts restriction on \( \nu, \mu, \) and \( \omega \). The parameters are also constrained as \( \epsilon = \left(-\frac{27}{2} g k^2\right)^{\frac{1}{3}} \). Then we arrive at the periodic solution, for the attractive regime \( \{g > 0\} \):

\[
a(\xi) = \left(-\frac{2k}{\epsilon}\right)\frac{\cos^2(\xi)}{1 - \frac{2}{3}\cos^2(\xi)}.
\] (2.28)

This periodic solution is found to be stable, as evidenced from the numerical simulations to be given later in the text. It should be noted that for \( B = 0 \), no rational periodic solutions are possible.

#### Case(II): Hyperbolic solution

In this case, we find that for \( A = 0 \), no rational solutions exist. For \( B = 0 \), and \( m = 1 \); we found that \( \alpha^2 = \epsilon/4 \). The parameters \( \nu, \mu, \) and \( \omega \) are related to the coupling strength as \( \epsilon = \left(-\frac{27}{2} g k^2\right)^{\frac{1}{3}} \). This yields, the hyperbolic solution, for the repulsive scenario:

\[
a(\xi) = \left(\frac{3k}{\epsilon}\right)\frac{1}{1 - \frac{3}{2}\text{sech}^2(\xi)}.
\] (2.29)
which is a singular one. The singularity here may correspond to extreme increase of the field amplitude due to self-focussing. We give below an example of a nonsingular solution. We take \( m = 1 \) and \( AD - B = 1 \) for simplicity. From Eq. (2.27), we determine the value of \( \alpha^2 \) as

\[
\alpha^2 = \frac{1}{2D} \left[ \epsilon BD^2 + gB^3 - kD^3 \right].
\]

For \( B = 0 \), we immediately arrive at a nonsingular solution

\[
a(\xi) = \left( \frac{3k}{\epsilon} \right) \frac{1}{1 + \left( \frac{\epsilon}{3k} \right) \text{sech}^2(\xi)}.
\]

subject to the following constraints: (i) for \( k \) positive, \( \epsilon > 0 \), hence \( \omega < \frac{\mu}{4} \), and (ii) for \( k \) negative, \( \epsilon < 0 \), hence \( \omega > \frac{\mu}{4} \).

Case(III): Pure cnoidal solutions

Below we give another periodic solution, where the modulus parameter takes a specific value. In this case, \( A = 0, D = 1, \) and \( m = 5/8 \); here \( \alpha^2 = \epsilon/2(3 - 2m) \). The parameters \( v, \mu, \) and \( UJ \) are related to the coupling strength as \( \epsilon = 7\left( -\frac{gk^2}{18} \right)^{1/3} \). This solution corresponds to the repulsive regime. It should be pointed out that in this case the solution is unique, as noticed.
above, various parameters are also related in this case. This gives rise to the cnoidal solution,

\[ a(\xi) = \left( \frac{k}{\epsilon} \right) \frac{3 - 2m}{1 - m} \frac{\cn^2(\xi, m)}{1 + \cn^2(\xi, m)}. \] (2.32)

For \( A = 0 \) and \( m = 1/2 \); it is found that \( \alpha^2 = \epsilon/2\sqrt{3} \) and \( \epsilon = (-27gk^2)^{1/3} \). This results in another cnoidal solution:

\[ a(\xi) = \left( 2 \frac{\sqrt{3}k}{\epsilon} \right) \frac{\cn^2(\xi, m)}{1 + \frac{1}{\sqrt{3}} \cn^2(\xi, m)}. \] (2.33)

For the sake of clarity, the above obtained solutions are tabulated below.

**Table I. Various limits for the exact solutions of NLSE with a source**

<table>
<thead>
<tr>
<th>Modulus parameter (m)</th>
<th>A</th>
<th>B</th>
<th>D</th>
<th>a(\xi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-2k/\epsilon</td>
<td>-2/3</td>
<td>\left( -\frac{2k}{\epsilon} \right) \frac{\cos^2(\xi)}{1 - \frac{2}{3} \cos^2(\xi)}</td>
</tr>
<tr>
<td>1</td>
<td>3k/\epsilon</td>
<td>0</td>
<td>-3/2</td>
<td>\left( \frac{3k}{\epsilon} \right) \frac{1}{1 \sech^2(\xi)}</td>
</tr>
<tr>
<td>5/8</td>
<td>0</td>
<td>14k/3\epsilon</td>
<td>1</td>
<td>\left( \frac{k}{\epsilon} \right) \frac{3 - 2m}{1 - m} \frac{\cn^2(\xi, m)}{1 + \cn^2(\xi, m)}</td>
</tr>
<tr>
<td>1/2</td>
<td>0</td>
<td>2\sqrt{3}k/\epsilon</td>
<td>1/\sqrt{3}</td>
<td>\left( \frac{2\sqrt{3}k}{\epsilon} \right) \frac{\cn^2(\xi, m)}{1 + \frac{1}{\sqrt{3}} \cn^2(\xi, m)}</td>
</tr>
</tbody>
</table>

We now give some general localized solutions. Taking \( m = 1 \) \( (\cn(\xi, 1) = \sech(\xi)) \), with the parameter values \( A = 1 \), \( B = -S \), and \( D = 1 \) \( S \) we obtain

\[ \rho(\xi) = \frac{1 + \delta \sech^2(\xi)}{1 + \Gamma \sech^2(\xi)}; \] (2.34)

here the amplitude, width, and velocity are related as,

\[ \delta = -\left(Q + \sqrt{Q^2 - 4PR}\right)/2Q, \]
with $Q = \epsilon - 2\alpha^2 - 3k$, $P = 2\epsilon - 3k$, $R = -k - 2\alpha^2$ and $\Gamma = \pm 8$. Hence, the width $rv$ is the only independent parameter. As the above form of the solution indicates, both nonsingular and singular solitons are possible solutions depending on the values of $\epsilon$, and the source strength $k$.

2.5 Numerical results

Since the localized solitons are usually robust, we have performed numerical simulations to check the stability of the solutions pertaining to Case(I), i.e., the trigonometric solution. It is worth pointing out that the numerical techniques based on the fast Fourier transform (FFT) are expensive as they require the FFT of the external source. Hence, we have used the Crank-Nicolson finite difference method to solve the NLSE with a source, which is quite handy, and unconditionally stable. Below a detailed description of the algorithm is given. We write $q = R + iI$, where $R$ and $I$ are real-valued functions. Then the NLSE with a source is equivalent to the following coupled system of equations:

\begin{equation}
\partial_t R = -\frac{1}{2} \partial_x^2 I - g(R^2 + I^2)I + k\sin(kx - \omega t),
\end{equation}

and

\begin{equation}
\partial_t I = \frac{1}{2} \partial_x^2 R + g(R^2 + I^2)R - k\cos(kx - \omega t).
\end{equation}

Euler algorithm:

The finite-difference scheme for the Eqs. (2.35) and (2.36) can be written as follows:

\begin{equation}
R_{i}^{n+1} = R_{i}^{n} - \frac{\Delta t}{2(\Delta x)^2} [I_{i+1}^{n} - 2I_{i}^{n} + I_{i-1}^{n}] - \\
g\Delta t [(R_{i}^{n})^2 + (I_{i}^{n})^2]I_{i}^{n} + k\Delta t \sin(kx - \omega t),
\end{equation}

and
Figure 2.2: Plot depicting the evolution of the trigonometric solution, for various times.

\[
I_{i+1}^n = I_i^n + \frac{\Delta t}{2(\Delta x)^2} [R_{i+1}^n - 2R_i^n + R_{i-1}^n] + \\
g\Delta t[(R_i^n)^2 + (I_i^n)^2]R_i^n - k\Delta t\cos(kx - \omega t).
\]  

(2.38)

Second-order algorithm:
We present here, the semi-implicit CNFD for the NLSE in the presence of an external source. The recurrence relations for the Eqs. (2.35) and (2.36)
are written as:

\[
R_i^{n+1} = R_i^n - \frac{\Delta t}{4(\Delta x)^2} [I_{i+1}^{n+1} - 2I_i^{n+1} + I_{i-1}^{n+1}] - \\
\frac{\Delta t}{4(\Delta x)^2} [I_{i+1}^n - 2I_i^n + I_{i-1}^n] - \frac{(g/2)\Delta t}{(R_i^n)^2} + \\
(I_i^n)^2 I_i^n - \frac{(g/2)\Delta t}{(R_i^n)(R_i^{n+1})} + \\
(I_i^{n+1})^2 I_i^{n+1} + k\Delta t \sin(kx - \omega t),
\]

and

\[
I_i^{n+1} = I_i^n + \frac{\Delta t}{4(\Delta x)^2} [I_{i+1}^{n+1} - 2I_i^{n+1} + I_{i-1}^{n+1}] + \\
\frac{\Delta t}{4(\Delta x)^2} [I_{i+1}^n - 2I_i^n + I_{i-1}^n] + \frac{(g/2)\Delta t}{(R_i^n)^2} + \\
(I_i^n)^2 I_i^n + \frac{(g/2)\Delta t}{(I_i^n)(I_i^{n+1})} + \\
(R_i^{n+1})^2 I_i^{n+1} - k\Delta t \cos(kx - \omega t).
\]  

Furthermore, \(R_i^n\) and \(I_i^n\) denote the approximation of the solution at \(t = n\Delta t\), and \(x_{\text{final}} = x_{\text{initial}} + ih\) with \(h = \frac{x_{\text{final}} - x_{\text{initial}}}{N}\), where \(N\) is the total number of grid points. The initial conditions chosen from the exact solution are knitted on a lattice with a grid size \(dx = 0.005\), and \(dt = 5.0 \times 10^{-6}\). The simulations carried out indicate clearly that the above-mentioned solution is stable (Fig. 2.2). The initial and the boundary conditions are:

\[
R_0^{n+1} = R_M^{n+1},
\]

and

\[
I_0^{n+1} = I_M^{n+1}.
\]

If \(i = N\), then

\[
R_N^{n+1} = R_2^{n+1},
\]

and for \(i = 1\)

\[
R_{i-1}^{n+1} = R_{N-1}^{n+1}.
\]

To summarize, we have used a fractional transformation to connect the solutions of the phase-locked NLSE with the elliptic functions, in an exact
The solutions are necessarily of the rational type that contain solitons, solitary waves, as also singular ones. Our procedure is applicable, both for the attractive and repulsive cases. Because of their exact nature, these will provide a better starting point for the treatment of general externally driven NLSE. Considering the utility of this equation in fiber optics and other branches of physics, these solutions may find practical applications.

2.6 NLSE in opaque medium with distributed coefficients and an external source

In this section we present a wide class of rational and periodic solutions of the nonlinear Schrodinger equation with a source, in an opaque medium with distributed coefficients. As we will see below certain relationships between the coefficients and a particular type of source will lead to exact solutions. It should be noted that space and time are interchanged in the following equation as compared to the previous section, as is appropriate for an optical fiber.

The damped nonlinear Schrodinger equation, coupled to an external space-time dependent source with distributed coefficients can be written as,

\[ i\psi_z - \frac{\beta(z)}{2} \psi_{\tau\tau} + \gamma(z) |\psi|^2 \psi = i\frac{g(z)}{2} \psi + \eta e^{i\Phi(\tau, z)}. \]  

(2.41)

It is assumed that the parameters \( \beta, \gamma, \delta, \) and \( g \) are all functions of the propagation distance \( z \). The explicit relationships between them will be given below.

The damped NLSE, for which the distributed terms are independent of the propagation distance appeared in a variety of contexts: breathers in charge-density-wave materials in the presence of an applied ac field [5], breathers in long Josephson junctions [24], in easy-axis ferromagnets in
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a rotating magnetic field [25, 26], and as solitons in the plasmas driven by rf fields [11, 27]. However, in recent years an important technology referred to as dispersion management (DM) has been developed by the researchers [28, 29]. DM means that optical fibers with sharply different dispersion characteristics, anomalous and normal, are combined together in subsections of the fiber and then this substructure is periodically repeated to cover the entire fiber length. DM technique is exploited profitably to enhance the power of the optical solitons, and reduce the effects of Gordon-Haus timing jitter [30]. Equation (2.41) describes the amplification or attenuation [for \( g(z) \) is negative] of pulses propagating nonlinearly in a single mode fiber, where \( \Psi(\tau, Z) \) is the complex envelope of the electric field in a comoving frame, \( \tau \) is the retarded time, \( \beta(z) \) is the group velocity dispersion (GVD) parameter, \( \gamma(z) \) is the nonlinearity parameter, and \( g(z) \) is the distributed gain function. In the absence of a source, numerically it was shown that, in the case where the gain due to the nonlinearity and the linear dispersion balance with each other; equilibrium solitons will be formed[31]. Recently, V. I. Kruglov et al have reported exact self-similar solutions characterized by a linear chirp [32, 33]. Solitary wave solutions of this type of NLSE helps in analyzing the compression problem of the laser pulse in a dispersion decreasing optical fiber. Motivated by this work and our results on solutions of NLSE with a source, we analyze below the effects of the distributed coefficients and damping on the exact rational solutions of Eq. (2.41). It is hoped that, these solutions may find experimental realization, particularly in the solitary wave based communication links [8, 28].

By writing the complex function \( \Psi(z, \tau) \) as

\[
\Psi(z, \tau) = P(z, \tau)e^{i\Phi(z, \tau)},
\]

(2.42)

where \( P \) and \( \Phi \) are real functions of \( z \) and \( T \); we look for the rational solutions of the NLSE assuming that the phase has the following quadratic
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form:

\[ P(z, \tau) = \frac{1}{\sqrt{1 - c_0 R(z)}} Q(\frac{\tau - \tau_c}{1 - c_0 R(z)}) \exp\left(\frac{1}{2} S(z)\right), \]  

(2.44)

Note that these solutions have a linear chirping. Now, Eqs. (2.42), (2.43) yield a self similar form of the amplitude:

\[ P(z, \tau) = \frac{1}{\sqrt{1 - c_0 R(z)}} Q(\frac{\tau - \tau_c}{1 - c_0 R(z)}) \exp\left(\frac{1}{2} S(z)\right), \]  

(2.44)

where \( \tau_c \) is the center of the pulse, and the functions \( a(z) \), \( c(z) \), \( R(z) \) and \( S(z) \) are given by

\[ a(z) = a_0 - \frac{\lambda}{2} \int_0^z \frac{\beta(z')dz'}{(1 - c_0 R(z'))^2}, \]  

(2.45)

\[ c(z) = \frac{c_0}{1 - c_0 R(z)}, \]  

(2.46)

\[ R(z) = 2 \int_0^z \beta(z')dz', \quad S(z) = \int_0^z g(z')dz', \]  

(2.47)

where \( a_0, A, \) and \( c_0 \) are the integration constants. Here, \( \eta = \frac{\beta(z)}{2(1-c_0D(z))^{3/2}}, \) and \( \varepsilon \) is the strength of the source. Furthermore, the function \( Q(T) \) which determines the amplitude \( P(z, \tau) \) in Eq. (2.44) can be found by solving the following nonlinear ODE

\[ Q'' - \lambda Q + 2 \kappa Q^3 - \varepsilon = 0, \]  

(2.48)

where the prime indicates the derivative with respect to \( T \). Here the scaling variables are given by

\[ T = \frac{\tau - \tau_c}{1 - c_0 R(z)}, \quad \kappa = -\frac{\gamma(0)}{\beta(0)}. \]  

(2.49)

2.7 The rational solutions

Our goal in this section is to present the rational solutions of Eq. (2.48), following the results of the previous section. In the same manner we start with a fractional transform (FT) [34]

\[ Q(T) = \frac{A + Bf^2(T, m)}{1 + Df^2(T, m)}, \]  

(2.50)
that connects the solutions of the damped NLSE with a source, to the
elliptic equation $f'' + af + \lambda f^3 = 0$. As has been done previously, the coef-
ficients of $f^n(T, m)$ for $n = 0, 2, 4, 6$ can be set to zero to reduce the problem
to an algebraic one, and obtain the solutions. In getting the algebraic
equations, use has been made of the following relations for various deriva-
tives of $f$: $f'' = f + f^3$, and $f^2 = \frac{1}{2} f^4 + 2E_0$, where $E_0$ is the integration
constant. Furthermore, it is assumed that $f$ can be taken as any of the
three Jacobi elliptic functions with an appropriate modulus parameter:
$c(T, m)$, $d(T, m)$, and $sn(T, m)$. The other nonsingular solutions can be
derived analogously. Various limiting conditions of cnoidal functions are:
$c^2(T, 0) = \cos^2(T)$, and $c^2(T, 1) = \sech^2(T)$; $d^2(T, 0) = 1$, $c^2(T, 1) = \sech^2(T)$;
$s^2(T, 0) = 0$, and $s^2(T, 1) = \tanh^2(T)$.

For definiteness, we start with the assumption $f = c(T, m)$; evidently,
the coefficients of $c^n(T, m)$, for $n = 0, 2, 4, 6$ can be set to zero, and thereby
yielding four algebraic equations. The identities satisfied by the cnoidal
functions make them amenable for finding the exact solutions of the non-
linear ODE of the form described by Eq. (2.48). In simplifying the second
derivative of $Q$, we used the following important identities satisfied by the
cnoidal functions.

$$ cn^2sn^2dn^2 = cn^2(1 - m) + (2m - 1)cn^4 - mcn^6 $$
$$ cn^4dn^2 - mcn^4sn^2 = cn^2(1 - 2m) + 2mcn^6 $$
$$ cn^2dn^2 - sn^2dn^2 - mcn^2sn^2 = 2cn^2(1 - 2m) + 3mcn^4 + m - 1. \quad (2.51) $$

The four consistency conditions are:

$$ -\lambda A - 2(AD - B)(1 - m) + 2\kappa A^3 - \varepsilon = 0, \quad (2.52) $$
$$ 2(AD - B)(3D - 3mD - 4m + 2) - 2X(AD + B) + 6\kappa A^2B - 3\varepsilon D = 0, \quad (2.53) $$
$$ 2(AD - B)(3m + 4mD - 2D) - XD(AD + 2B) + 6\kappa AB^2 - \varepsilon D^2 = 0, \quad (2.54) $$
$$ -\lambda BD^2 - 2m(AD - B)D + 2\kappa B^2 - \varepsilon D^3 = 0. \quad (2.55) $$

In general, the rational solutions are unstable and may blow up. A number
of rational solutions we have found are stable, as evidenced from the numerical stability. Thus, we shall present them here, with the specifications for the regimes in which they apply. Some of the periodic and hyperbolic solutions are presented below.

Case(I): Trigonometric solution

For $A = 0$; and $m = 0$; we find that

$$Q(T) = \frac{\cos^2(T)}{1 - (2/3)\cos^2(T)},$$

(2.56)

Case(II): Hyperbolic solution

For $B = 0$, and $m = 1$; we find that

$$Q(T) = \frac{(3/4)\varepsilon}{1 - (3/2)\text{sech}^2(T)}.$$

(2.57)

This is a singular solution. The singularity here corresponds to an extreme increase of the field amplitude due to self-focussing. If we consider the case, $AD = 1$, and $B = 0$; then we get a non-singular, hyperbolic solution

$$Q(T) = \frac{(\lambda - 8)}{6} \frac{1}{1 + D\text{sech}^2(T)}$$

(2.58)

where $D = \frac{6}{(\lambda - 8)}$, and $\lambda = \frac{16\pm\sqrt{256 - 4\alpha}}{2}$ with $\alpha = 18\varepsilon + 64$. To avoid the singularity, $\alpha$ should be always positive.

Case(III): Pure cnoidal solutions

Figure 2.3: Plot depicting the singular solitary solution.
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(i) For \( m = 5/8; \ A = 0 \); it is found that, for the specific value of \( A = 7/2 \) we obtain a periodic cnoidal solution

\[
Q(T) = (4/3)e^{\frac{\text{cn}^2(T, m)}{1 + \text{cn}^2(T, m)}}. \tag{2.59}
\]

(ii) For \( m = 5/8; \ A = 0 \); it is found that, for the specific value of \( A = \frac{7}{2} \), yields yet another pure cnoidal solution

\[
Q(T) = e^{\frac{\text{cn}^2(T, m)}{1 + \frac{2}{3}\text{cn}^2(T, m)}}. \tag{2.60}
\]

(iii) For \( m = 1/2; \ A = 0 \); it is found that another specific value of \( A = \pm 2\sqrt{3} \), yields yet another pure cnoidal solution

\[
Q(T) = e^{\frac{\text{cn}^2(T, m)}{1 + \frac{2}{3}\text{cn}^2(T, m)}}. \tag{2.61}
\]

For the sake of clarity, all these solutions are tabulated below.

**Table II. Various limits for the exact solutions of NLSE in an opaque medium with a source**

<table>
<thead>
<tr>
<th>Modulus parameter ( m )</th>
<th>( A )</th>
<th>( e )</th>
<th>( D )</th>
<th>Rational solution ( Q(T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( e/2 )</td>
<td>-2/3</td>
<td>( \frac{\left(\frac{5}{2}\right)^{\text{cn}^2(T)}}{1 - \frac{2}{3}\cos^2(T)} )</td>
</tr>
<tr>
<td>1</td>
<td>3( e/4 )</td>
<td>0</td>
<td>-3/2</td>
<td>( \frac{\left(\frac{3e}{4}\right)^{\text{sech}^2(T)}}{1 - \frac{1}{2}\text{sech}^2(T)} )</td>
</tr>
<tr>
<td>5/8</td>
<td>0</td>
<td>4( e/3 )</td>
<td>1</td>
<td>( \frac{\left(\frac{4e}{3}\right)^{\text{cn}^2(T, m)}}{1 + \text{cn}^2(T, m)} )</td>
</tr>
<tr>
<td>5/8</td>
<td>0</td>
<td>4( e/3 )</td>
<td>-5/9</td>
<td>( \frac{\left(\frac{4e}{3}\right)^{\text{cn}^2(T, m)}}{1 + \text{cn}^2(T, m)} )</td>
</tr>
<tr>
<td>1/2</td>
<td>0</td>
<td>( \pm (1/\sqrt{3}) )</td>
<td></td>
<td>( \frac{\text{cn}^2(T, m)}{1 + \frac{2}{3}\text{cn}^2(T, m)} )</td>
</tr>
</tbody>
</table>
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2.8 Numerical results

In this section we present the numerical corroborations of our analytical insights. We have solved Eq. (2.48) for various parameters values, using RK-4 method for a step size of $h = 10^{-5}$. We find oscillatory solutions, as was anticipated from the analytical result (Case(I)). After switching off the source, we also identify localized soliton solution for the same parameter values, in order to compare with the results reported in Ref.[33]. The same have been depicted in figures 2.4-2.6.

This technique may find applications in pulse compression. Since this area is rather new, one needs to explore the full potential of this possibility in more detail.
Figure 2.5: Plot depicting the oscillatory solution when the source is switched on. The parameter values are: $\varepsilon = 0.5$, $A = 1.0$, and $\kappa = 5.0$.

Figure 2.6: Plot depicting the oscillatory solution of Eq. (2.56) for same parameter values as above.
References


References


References


