Chapter 3

Critical Strange Nonchaotic Dynamics in Fibonacci Map

In addition to the Harper equation,

$$\psi_{n+1} + \psi_{n-1} + V_n \psi_n = E \psi_n,$$

(3.1)

where the the potential $V_n = 2 \cos(2\pi \omega)$, $\omega$ irrational, a variety of other quasiperiodic lattice potentials have been studied in the context of spectral theory, with particular focus on their localization properties [87]. Similar to the Harper map in Chapter 2 which obtains from the Harper equation, one can derive quasiperiodic mappings from other discrete Schrödinger equations, and in the present Chapter, I consider the Fibonacci lattice [78], which is one of the simplest cases.

The potential in the Fibonacci lattice takes two values ($\pm \alpha$, say) in a quasiperiodic manner; this gives a mapping where the driving is pulsed rather than continuous (see Eq. (3.3) below)). As a consequence, the attractors are piecewise linear and are geometrically simpler to study than the similar attractors of the Harper map (Fig. 2.1). Correspondences between the dynamical system and the eigenvalue problem have been explored in detail in the past few years [73]. When $V_n$ is quasiperiodic and when the parameter
$E$ in Eq. (3.3) is an eigenvalue of the corresponding Schrödinger equation Eq. (3.1), equivalence can be made between eigenfunctions of Eq. (3.1) and orbits of the dynamical system. The extensive work on the spectral theory of quasiperiodic Schrödinger operators [102] can offer fresh insights into the nature of the dynamical system, and in particular, it has been seen that localized eigenfunctions in the quantum problem necessarily correspond to nonchaotic attractors in the dynamical system [72, 73]. From the spectral theory it is known that the eigenvalue spectrum for the Fibonacci chain is singular–continuous and the Lyapunov exponent is exactly zero whenever $E$ is in the spectrum [78, 97].

The main result here for the Fibonacci map is that when all Lyapunov exponents are equal to zero, the orbits of the dynamical system lie on limit sets which appear to be fractal nonchaotic attractors. Such dynamical attractors are critical SNAs [116]. Although these SNAs are not robust [61], they should occur widely: there is a large class of aperiodic potentials which can be derived from substitutional sequences (Thue–Morse, Rudin–Shapiro etc.) all of which share a number of spectral properties with the Fibonacci chain. The transition to critical dynamics is studied in detail as a function of the parameter $E$, and it is seen that a novel mechanism operates for the formation of critical SNAs.

### 3.1 The Fibonacci Map

In the Fibonacci lattice, the potential takes values $\alpha$ and $-\alpha$ at different sites in the following manner. Consider the symbolic transformation $A \rightarrow B, B \rightarrow AB$, recursively applied to the starting sequence $A$. The first few applications of this transformation yields

$A, B, AB, BAB, ABBAB, BABABBAB, ABBABBABABBAB, \ldots.$
On a lattice labeled by such a symbolic sequence, assign the potential $V_n = \alpha$ if the $n$th symbol is $A$, and $V_n = -\alpha$ if the $n$th symbol is $B$; this defines the Fibonacci lattice.

A alternate algebraic prescription is possible as well. Defining a “phase” variable $\theta_n = \{n\omega\}$ where the notation $\{x\}$ denotes the fractional part of $x$, the Fibonacci model is given by

$$V_n(\theta_n) = \begin{cases} \alpha & 0 \leq \theta_n < \omega \\ -\alpha & \omega \leq \theta_n < 1 \end{cases}$$

(3.2)

for $\omega = (\sqrt{5} - 1)/2$, the inverse of the golden mean ratio.

Applying the transformation $\psi_{n-1}/\psi_n \rightarrow x_n$ to Eq. (3.1) with $V_n$ as above, and rewriting it as a two-dimensional skew-product mapping in the standard manner, one obtains the Fibonacci map

$$x_{n+1} = \frac{-1}{x_n - E + V_n(\theta_n)}$$

(3.3)

$$\theta_{n+1} = \{\theta_n + \omega\}$$

(3.4)

which is a (pulsed) modulated mapping on the infinite strip $(-\infty, \infty) \otimes [0, 1]$. Since the dynamics is invertible, there can be no chaos and the nontrivial Lyapunov exponent,

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \ln x_{j+1}^2,$$

(3.5)

is strictly nonpositive. As the $\theta$ dynamics is just a rigid rotation, the second Lyapunov exponent is zero.

In the limit $\alpha \rightarrow \infty$ or $E \rightarrow \infty$, the mapping takes a particularly simple form,

$$x_{n+1} \rightarrow -\frac{1}{V_n}, \text{ or}$$

$$x_{n+1} \rightarrow \frac{1}{E - V_n}.$$  

(3.6)

From Eq. (3.5) it is clear that the Lyapunov exponent must be negative, and as $V_n$ changes sign as a function of $n$, it is clear that the attractor of
the dynamics will have singularities. As $E$ or $\alpha$ decrease, these singularities proliferate, although it is not clear how the attractor develops as the Lyapunov exponent approaches zero. The variation of the nontrivial Lyapunov exponent $\lambda$ with parameter $E$ is shown in Fig. 3.1 for a fixed value of $\alpha$. The curve meets the $\lambda = 0$ line at a Cantor set of points, in a manner that is familiar from the case of the Harper system [6]. From the equivalences with the spectral theory, the points where the Lyapunov exponent is zero correspond to the spectrum of Eq. (3.1). Since the spectrum of the Fibonacci chain is known to be singular continuous for any value of $\alpha$, it is clear that any fixed value of $E$ can be in the spectrum of the Fibonacci chain for a number of different coupling strengths (possibly also a Cantor set). These can be simply determined from a graph of $\lambda$ as a function of $\alpha$ as shown for example in Fig. 3.2 for the value $E = 0$. 

Figure 3.1: The Lyapunov exponent versus energy at $\alpha = 1$ for the Fibonacci chain. The largest visible gaps are labeled $A$, $B$, $C$ and $D$ respectively. The dynamics for $\lambda = 0$ corresponds to SNAs.
Figure 3.2: The Lyapunov exponent versus parameter $\alpha$ at $E = 0$ for the Fibonacci chain.

3.2 Critical Attractors of the Fibonacci map

We now discuss the attractors of the Fibonacci map for $E=0$. Consider the system as a function of $\alpha$ with Lyapunov exponent as shown in Fig. 3.2. It is clear that the Lyapunov exponent vanishes for $E = 0$ for a number of values of $\alpha$, all of which are below $\alpha_c = 0.83267979 \ldots$. In the limit, $\alpha \to \infty$, the dynamical map for the Fibonacci system takes the simple form given by Eq. (3.6); the nontrivial Lyapunov exponent is negative, and the attractor of the dynamics consists of two branches at $x = \pm 1/\alpha$, with discontinuities at $\theta = \omega$ and $\{2\omega\}$. As $\alpha$ is reduced to $\alpha_c$ starting from the limit $\alpha \to \infty$, the attractor for the Fibonacci map for values of $\alpha > \alpha_c$ consists of piecewise constant segments with discontinuities. The number of segments is increasing as $\alpha$ decreases. As the critical value is approached the number of singularities of the attractor increases, initially appearing sequentially at $\theta = \{k\omega\}, k = 2, 3, \ldots$, leading to a dense set of singularities on the attractor.
Figure 3.3: The forward (thick line) and reverse (thin line) attractors for $E = 0$ approach each other as $\alpha \to \alpha_c$. 
at $\alpha = \alpha_c$. This increase in the number of singularities can be interpreted as a process of fractalization of the attractor similar to one mechanism of the emergence of SNA in the quasiperiodically forced logistic map \[94\].

Because the Fibonacci map is invertible, the repeller can also be constructed. This unstable set is obtained by iterating the mapping backwards. The attractor and the repeller are both shown in Fig. 3.3, and it can be seen that there is a distance of closest approach of these two sets as a function of $\theta$ which decreases progressively as a function of $\alpha$ and eventually vanishes at $\alpha_c$. The attractor and the repeller collide at $\alpha_c$, their vertical separation $d$ decreases as

$$d \sim (\alpha - \alpha_c)\nu$$

(3.7)

with $\nu \approx 0.635$; see Fig. 3.4.

The attractor for $\alpha = \alpha_c$ is shown in Fig. 3.5. Strictly speaking, since the nontrivial Lyapunov exponent is also exactly zero, this is properly a limit
Figure 3.5: The attractor at $\alpha = \alpha_c$. The forward and reverse attractor collides at this value and a critical SNA is born.

set. However, that the dynamics is confined to this set (with essentially no transients): regardless of initial conditions, this set is the unique global attractor of the dynamics. This is the critical SNA.

The formation of a fractal attractor from a nonfractal precursor has been called fractalization [94]. Another scenario for the creation of SNAs, which—in contrast to fractalization—is well understood, is the torus collision route [35]. In this case the stable and unstable tori collide in a dense set of points leading to an SNA. In the present case of the Fibonacci chain the formation of the critical attractor involves both, the torus collision and the fractalization process. We see here that critical SNAs in the Fibonacci map are indeed
gradually formed by a fractalization process, which ends in a collision between the stable and unstable set leading to strange nonchaotic behaviour. This appears to be a general mechanism for the creation of SNAs in the Fibonacci map.

3.3 Characterization of critical SNAs

The distinctive feature of SNAs is nondifferentiability. To distinguish between smooth and fractal attractors in quasiperiodically forced system, Feudel and Pikovsky [107] introduced the phase sensitivity exponent. The nondifferentiability of the attractor is measured by calculating the derivative $dx/d\theta$ along an orbit, and finding the maximal value for this quantity. In the Fibonacci map, however, the potential is a non-smooth step function of the phase $\theta$. Hence the calculation of phase sensitivity exponent will involve problems with the singularities induced by the step function.

A similar quantity which has been used to characterize SNAs is the parameter sensitivity $\Gamma$ [94],

$$\Gamma_N^{(\alpha)} = \min_{x_0, \theta_0} \sigma_N(x_0, \theta_0),$$

where $\sigma_N(x_0, \theta_0)$ is partial sum over an orbit of length $N$ with the initial point $(x_0, \theta_0)$ and is given by,

$$\sigma_N(x_0, \theta_0) = \max_{0 \leq n \leq N} \left| \frac{\partial x_n}{\partial \alpha} \right|.
$$

A fractal torus will be sensitive to the change of parameter $\alpha$ for some $\theta$; $\Gamma_N$ is an average property of the attractor since minimization is done over an ensemble of orbits corresponding to different initial conditions. For torus attractors $\Gamma$ saturates whereas for a fractal attractor this quantity grows without bound. It follows that a fractal attractor can not have finite derivative either with respect to variation of the external parameter or with respect
to the external phase. While a chaotic attractor is characterized by “sensitivity to initial conditions”, SNAs have parametric as well as phase sensitivity, the origins of which are related to the existence of positive local Lyapunov exponent for long time intervals on the SNA [107]. $\Gamma_N$ grows according to a power law on SNAs, namely $\Gamma_N = N^\mu$, $\mu$ being the parameter sensitivity exponent.

The process of fractalization of the torus leading to the formation of SNA can be characterized by the functional map approach introduced by Nishikawa and Kaneko [94]. The functional map can be used to obtain the piecewise invariant attractor. Furthermore a perturbation expansion of the functional equation can be applied, whereby the SNA can be characterized by a loss of convergence in the Fourier mode expansion of the derivative of the invariant curve. Consider SNAs in skew–product dynamical systems (as in the present instance). The iterative mapping in Eq. (3.3) generates an attractor, which can be expressed as a single–valued function $X(\theta)$ of the

Figure 3.6: The parameter sensitivity exponent $\Gamma_N^\alpha$ versus iteration. For $\alpha > \alpha_c$, $\Gamma_N^\alpha$ saturates for high $N$ whereas it is unbounded for $\alpha = \alpha_c$. 

\[ a = 0.833 \]
phase $\theta$, $V_n$ taking values $\alpha(-\alpha)$ when $\theta_{n+1}$ is $\geq \omega(<\omega)$. Since the attractor satisfies a functional equation we get, using Eq. (3.3)

$$X(\theta_{n+1}) = \frac{-1}{X(\theta_n) - E + V_n} \quad (3.10)$$

The functional mapping can be written as

$$X_{n+1}(\theta + \omega \mod 1) = \frac{-1}{X_n(\theta) - E + V_n} \quad (3.11)$$

The attractor of the map in Eq. (3.3) is obtained as a fixed point in the functional space by iteration of Eq. (3.11). Since $\theta$ is ergodic in the interval, the map given by Eq. (3.11) represents an infinite-dimensional dynamical system. To approximate $X(\theta)$, we approximate $\omega$ by the rational number $\omega_k = F_{k-1}/F_k$ where $F_k$ is the $k$th Fibonacci number. This approximation converts the functional map into an $F_k$-dimensional map which maps $F_k$ points $\theta_1, \ldots, \theta_{F_k}$ onto themselves through $X(\theta)$. The functional fixed point may be found by iterating Eq. (3.11) for rational $\omega_k$. In the limit $\lim_{k \to \infty} \omega_k = \omega$ the attractor is a fractal, $X_n(\theta)$ corresponds no longer to a finite set of points, and the Lyapunov exponent is exactly zero.

If we apply the Fourier mode analysis to the functional Eq. (3.3), by considering the Fourier expansion of $X(\theta)$, we get

$$X(\theta) = \sum_{k=-\infty}^{\infty} \tilde{X}(k) \exp(2\pi ik\theta) \quad (3.12)$$

$$\tilde{X}(k) = \int_{0}^{1} X(\theta) \exp(-2\pi ik\theta) d\theta \quad (3.13)$$

Substituting this in Eq. (3.11), we can numerically obtain the power spectrum $P(k) = |\tilde{X}(k)|^2$ for SNAs. On fractal tori $P(k)$ decays slower than $k^{-2}$ since there is a loss of convergence in the first derivative of the Fourier series given by Eq. (3.12); on smooth tori, the envelope of the spectra $P(k)$ decays faster than or equal to $k^{-2}$. Thus the spectral analysis can distinguish between smooth and strange nonchaotic dynamics.
Figure 3.7: The power spectrum $P(k)$, for an attractor near the critical value (top) and for the critical attractor (bottom). The superimposed line on the graph has a slope of $-2$. 
Defining the partial Fourier sum \[106\] for a process \(x_k\) as

\[
G(\Omega, T) = \sum_{k=1}^{T} x_k \exp(i2\pi k\Omega)
\] (3.14)

This defines a path on the plane \((\text{Re} G, \text{Im} G)\) where \(T\) is to be considered as time. For a completely random or Brownian path, \(|G(\Omega, T)|^2 \sim T\) and the spectrum is continuous. For a discrete spectrum with a frequency component \(\Omega\), the mean square displacement is then given by \(|G(\Omega, T)|^2 \sim T^2\). If SNAs have a singular continuous spectrum, then the scaling is \(|G(\Omega, T)|^2 \sim T^\mu\), where the value of \(\mu\) lies between 1 and 2. The path in the plane \((\text{Re} G, \text{Im} G)\) appears to be self-similar and is called a ‘fractal walk’ [10, 106].

Here we apply the standard diagnostics of strange nonchaotic dynamics to verify that the critical attractors of the Fibonacci map are SNAs. In order to calculate the parameter sensitivity \(\Gamma_N^{(a)}\), we randomly choose 100 initial conditions \((x_0, \theta_0)\) in the phase space. The quantity \(\Gamma_N^{(a)}\) is obtained as the minimum over this ensemble of 100 orbits. The parameter sensitivity for two
different attractors of the Fibonacci map is plotted in Fig. 3.6. Note that for an attractor near the critical point, at $\alpha = 0.833$, $\Gamma_{N}^{(\alpha)}$ saturates below $10^6$ iterations, while for the critical attractor at $\alpha = \alpha_c$, this quantity keeps growing (as a power-law).

The power spectrum is numerically obtained by a discrete FFT method [94] which uses the largest $T = 2^N$ points (we take $N=18$) out of $F_k$ points (in our case $F_k = 317811$) of the attractor of the functional map. It can be seen in Fig. 3.7. that the fractal torus possesses a power spectrum $P(k)$ which decays slower than $k^{-2}$ owing to a loss of convergence in the first derivative of the Fourier series given by Eq. (3.12). For the smooth torus, for example for the attractor at $\alpha = 0.833$, the maximum of the envelope of the spectrum $P(k)$ decays faster than or equal to $k^{-2}$ as seen in Fig. 3.7.

The partial Fourier transform for a typical trajectory on the critical SNA is shown in Fig. 3.8 for the parameter $E = 0$ and $\Omega = \omega/4$. The scaling followed by the mean square displacement can be found from the slope of the
curve; in the present case it is \( \approx 1.88 \). The self-similar 'fractal walk' in the 
\((\text{Re } G, \text{Im } G)\) plane is shown in Fig. 3.9.

## 3.4 Discussion

In this Chapter, I have investigated the dynamical transition in the Fibonacci 
map from a quasiperiodic attractor to a SNA as a function of the coupling 
strength. The process in which a critical SNA is formed appears to result both 
from the fractalization of a torus and the collision of the stable and unstable 
sets. Such a scenario where two different mechanisms for the creation of SNA 
occur simultaneously is a novel feature, which is particular to the transition 
to critical attractors of the Fibonacci map.

Another important situation where all orbits have all Lyapunov exponents 
equal to zero is in integrable (Hamiltonian) dynamical systems: this is a conse­quence of the existence of as many independent conserved dynamical quan­tities as freedoms [133]. There are no such symmetries in quasiperiodically 
driven maps, although there is an unusual symmetry in stretch exponents 
that causes the global Lyapunov exponent to vanish [120].

In the study of SNAs, the robustness of such dynamics is an issue. There 
are two senses in which the term is applicable, namely the persistence of 
the dynamics under variation of parameter and (more commonly) under the 
addition of noise. With additive noise, some studies have shown that SNAs 
persist, in the sense that the Lyapunov exponent remains negative, but the 
dynamics becomes more complex [75]. When the SNA occurs at a single 
point (or at a set of isolated points) in parameter space, the system is not 
robust with respect to parametric variation. For the Fibonacci map, the 
critical SNAs occur at isolated values of the parameter \( E \), and hence they 
are not robust. But nevertheless they occur at infinitely many points in the 
parameter space.

Although not robust, critical attractors are of interest and occur in a
number of quasiperiodically forced systems, as for example the Harper map [30, 117]. Other dichotomous potentials which have been studied in the context of critical localization are the Thue-Morse, period-doubling and Rudin-Shapiro sequences [87]. Therefore it is likely that quasiperiodic forcing with pulses patterned on such sequences will also lead to critical SNAs in this entire class of problems.