Chapter-2

On limit theorems for degenerate U- and V-statistics in the *-mixing processes
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On Limit Theorems for Degenerate $U$- and $V$-Statistics in the $\ast$-Mixing Processes

2.1. Introduction

Much effort has been devoted to study the asymptotic distribution of non-degenerate $U$- and $V$-statistics for i.i.d. random variables. Nauhaus (1977) showed that degenerate $V$-statistic $n(V_n^* - \theta(F))$ converges in distribution to a linear combination of squares of Wiener processes. Ronzhin (1986) showed that a degenerate $U$-statistic $n^{k_n}(U_n^* - \theta(F))$ converges weakly to a linear combination of Hermite polynomials of degree $k$ in independent Wiener processes. Using the idea from the theory of Poisson point process and Gaussian random measure, Dynkin and Mandelbaum (1983) obtained asymptotic distribution of degenerate symmetric statistics in terms of multiple Wiener integrals. In a subsequent paper Mandelbaum and Taqqu (1984) extended the above result to derive its invariance principles. Denker, Grillenberger and Keller (1985) first reduced their problem to the case of observations from uniform distribution on $[0,1]$ and then expressed the limiting distribution for $U$- and $V$-statistics as a $C[0,1]$-valued integral with respect to Kiefer processes.

Sen (1972) considered the case of strictly stationary $\ast$-mixing process and showed that the nondegenerate $U$-statistic converges weakly to standard Brownian motion. Limiting behaviour of nondegenerate $U$-statistic under different mixing conditions are obtained by Denker and Keller (1983), Dehling and Taqqu (1987), Yoshihara (1992) and Denker et al. (1985). For degenerate $U$-statistic under the $\ast$-mixing setup, Carlstein (1988) strove to derive the asymptotic distribution of degenerate $U$-statistic of order of degeneracy one under a very strong condition.
In this chapter we have derived the limiting distribution of the functional version of degenerate $U$- and $V$-statistics under a $\ast$-mixing setup. The limiting distribution is obtained in the form of a multiple stochastic integral with respect to Kiefer processes. A special feature of the $\ast$-mixing setup is that, unlike the independent case, the Kiefer process involved in the stochastic integral incorporates a tail part. This is in itself of independent interest in the theory of stochastic integrals.

The Vinogradov symbol $'<<'$ is used to indicate an inequality containing some unspecified positive constant factor.

2.2. Main Results

Let $\{X_i\}$ be a strictly stationary $\ast$-mixing stochastic process defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $E(\cdot)$ denote expectation with respect to the probability measure $P$ and $\mathcal{M}_t^t$ and $\mathcal{M}_{t+n}^t$ denote the $\sigma$-algebras generated by $\{X_i : i \leq t\}$ and $\{X_i : i \geq t+n\}$ respectively.

Then $\ast$-mixing coefficient $\Psi(n) \to 0$ as $n \to \infty$, where

$$\Psi(n) = \sup_{A \in \mathcal{M}_t^t, B \in \mathcal{M}_{t+n}^t} \left| P(AB) - P(A)P(B) \right| / P(A)P(B)$$

Also for random variables $X \in \mathcal{M}_t^t$ and $Y \in \mathcal{M}_{t+n}^t$ we have

$$|E(XY) - E(X)E(Y)| \leq \psi(n)E\left|X\right|\left|Y\right|.$$  \(2.2.1\)

Let $h(x_1,\ldots,x_k)$ be a symmetric measurable function with $k$ arguments and define

$$\theta(F) = \int \ldots \int h(x_1,\ldots,x_k)dF(x_1)\ldots dF(x_k),$$

where $F(x) = P(X_1 < x)$.

This chapter deals with two types of estimators of $\theta(F)$. One is the $U$-statistic due to Hoeffding given by

$$U_n(h) = \binom{n}{k} \sum_{i_1 < \ldots < i_k} h(X_{i_1},\ldots,X_{i_k})$$

and the other one is Von Mises $V$-statistic

$$V_n(h) = \int \ldots \int h(x_1,\ldots,x_k) \prod_{i=1}^k dF_n(x_i),$$

where $F_n$ is the empirical distribution function based on a sample $\{X_1,\ldots,X_n\}$.

Since for any measurable function $g$ the process $\{g(X_i)\}$ is also strictly stationary $\ast$-mixing, we can assume, without loss of generality, that the marginal distribution of each $X_i$ is uniform over $[0,1]$. 

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Now define the following norm of $h$:

$$
\|h\|_p = \left( \int_0^1 \cdots \int_0^1 |h(x_1, \ldots, x_k)|^p \prod_{i=1}^k dx_i \right)^{1/p}
$$

and

$$
L_p[0,1]^k = \{ h : \|h\|_p < \infty \}.
$$

Also define

$$
A_p(\psi) = \sum_{m=1}^{\infty} (m+1)^p \psi(m)
$$

and

$$
\tilde{h}(x_1, \ldots, x_k) = h(x_1, \ldots, x_k) - \theta(F).
$$

Throughout the chapter we assume that $h$ is a degenerate kernel i.e.

$$
\int_0^1 \tilde{h}(x_1, \ldots, x_k) dx_i = 0, \quad i = 1, \ldots, k.
$$

Let

$$
\|h\|_2 < \infty
$$

and

$$
A_{k*}(\psi) = \sum_{m=1}^{k*} (m+1)^{k*} \psi(m) < \infty,
$$

where $k* = \max(2, k-1)$.

Define

$$
\tilde{U}_n(h) = n^{k*2}U_n(\tilde{h}).
$$

Then the following theorem holds.

**Theorem 1**: Under conditions (2.2.3) and (2.2.4), $\tilde{U}_n(h)$ converges in distribution to $J_k(\tilde{h})$, where

$$
J_k(\tilde{h}) = \int_{[0,1]^k} \tilde{h}(x_1, \ldots, x_k) \prod_{i=1}^k \beta(dx_i),
$$

$\beta(x)$ is Gaussian with covariance function

$$
\text{cov}(\beta(x), \beta(x')) = \Gamma(x, x')
$$

$$
= (x \wedge x' - xx') + \sum_{j=2}^{k*} \left\{ P(X_1 \leq x, X_j \leq x') - xx' \right\}
$$

$$
+ \sum_{j=2}^{k*} \left\{ P(X_j \leq x, X_i \leq x') - xx' \right\}
$$

and the Wiener-Itô integral $\int_{[0,1]^k}$ is a multiple stochastic integral which is not extended over the hyperdiagonals of $[0,1]^k$. 

The following is a functional version of the above theorem. 
Define

\[ U'_{\alpha}(\tilde{h}) = \begin{cases} \binom{[nt]}{k} \binom{n}{k} U_{[\omega]}(\tilde{h}) & \text{if } 0 < t \leq 1 \\ 0 & t = 0 \end{cases} \]

and

\[ \tilde{U}'_{\alpha}(\tilde{h}) = n^{k/2}U'_{\alpha}(\tilde{h}). \]

We assume that

\[ h \in L_{\alpha}[0,1]^k \]

i.e.

\[ \int_0^1 \cdots \int_0^1 h^k(x_1, \ldots, x_k) \prod_{i=1}^k dx_i < \infty \] \hspace{1cm} \text{(2.2.5)}

and

\[ A_{\alpha}(\psi) = \sum_{m=1}^\infty m^{k^*} \psi(m) < \infty, \]

where

\[ k^* = \max(2, 2k - 1) \]

Now we state the following theorem:

**Theorem 2**: Under conditions (2.2.5) and (2.2.6)

\[ \tilde{U}'_{\alpha}(\tilde{h}) \longrightarrow_{D} J'_{\alpha}(\tilde{h}) \text{ as } n \to \infty, \]

where

\[ J'_{\alpha}(\tilde{h}) = \int_{[0,1]^k} \tilde{h}(x_1, \ldots, x_k) \prod_{i=1}^k K(dx_i, t), \]

\( K(x, t) \) is a Kiefer process with covariance function \( \text{Cov}(K(x, t), K(x', t')) = (t \wedge t') \Gamma(x, x') \) and as before \( \int_{[0,1]^k} \) is a Wiener-Itô stochastic integral.

**Remark**: If \( h \) is a degenerate kernel with \( k=2 \) arguments then under conditions (2.2.3) and (2.2.4) Theorem 1 of Carlstein (1988) follows while its functional version can be obtained under the assumptions (2.2.5) and (2.2.6).

Now we study the asymptotic distribution of degenerate \( V \)-statistics. For a partition \( Q = (Q_1, \ldots, Q_q) \) of \( \{1,2, \ldots, k\} \) into \( q \) sets define \( h_q(y_1, \ldots, y_q) = h(x_1, \ldots, x_k) \), where \( y_j = x_i \) if and only if \( i \in Q_j, i = 1, \ldots, k \) and \( j = 1, \ldots, q \). Let \( \mathcal{P} \) be the set of all
partitions of \( \{1,2,\ldots,k\} \) and define
\[
\|h\|_p = \sum_{\omega \in \Omega} |h(\omega)|_p
\]
and
\[
\hat{L}_p[0,1]^k = \left\{ h : \|h\|_p < \infty \right\}.
\]
Now assume that
\[
\|h\|_2 < \infty.
\]
Define
\[
V_n(h) = nk \cdot \hat{V}_n(h).
\]
Then the following holds.

**Theorem 3**: Under conditions (2.2.4) and (2.2.7), \( \hat{V}_n(h) \) converges in distribution to \( \hat{H}_k(h) \), where
\[
\hat{H}_k(h) = \int_{[0,1]^k} \tilde{h}(x_1,\ldots,x_k) \prod_{i=1}^k \beta(dx_i)
\]
and \( \beta(x) \) is a separable Gaussian process defined in Theorem 1 and the stochastic integral is meant to be the "full integral" extended also over the hyperdiagonals of \([0,1]^k\).

The following is a functional version of Theorem 3. Define
\[
V'_{n}(\tilde{h}) = \begin{cases} 
\left[ \frac{m^k}{n^k} \right] V[\omega](\tilde{h}) & \text{if } 0 < k \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
and assume that
\[
\|h\|_4 < \infty.
\]
Let
\[
V'_{n}(\tilde{h}) = n^{k/2} V_{n}(\tilde{h}).
\]
Then the following holds.

**Theorem 4**: Under conditions (2.2.6) and (2.2.8)
\[
\hat{V}_n(h) \overset{a.s.}{\longrightarrow} \hat{H}_k(h) \quad \text{as } n \to \infty,
\]
where
\[
\hat{H}_k(h) = \int_{[0,1]^k} \tilde{h}(x_1,\ldots,x_k) \prod_{i=1}^k K(dx_i,t)
\]
and \( K(x,t) \) is a Kiefer process defined in Theorem 2.
2.3. Some Auxiliary Results

Lemma 1: For $t_1 < t_2 < ... < t_k$ let $F', G$ and $H$ be the multivariate distribution functions of $(X_{i_1}, ..., X_{i_l})$, $(X_{t_1}, ..., X_{t_j})$ and $(X_{t_{j+1}}, ..., X_{t_k})$ respectively. Then

$$\left| \int_0^{t_1} \cdots \int_0^{t_l} h(x_{i_1}, ..., x_{i_l}) d\left[F'(x_{i_1}, ..., x_{i_l}) - G(x_{i_1}, ..., x_{i_l})H(x_{t_{j+1}}, ..., x_{t_k}) \right] \right|$$

$$\ll \psi(t_{j+1} - t_j) \int_0^{t_1} \cdots \int_0^{t_l} |h(x_{i_1}, ..., x_{i_l})| \prod_{i=1}^{k-l} dx_i.$$

Lemma 2: Under the assumptions of Theorem 1

$$E(\hat{U}_n(h))^2 \ll \|h\|^2_2.$$

Lemma 3: Under the assumptions of Theorem 2

$$E(\hat{U}_n(h))^4 \ll \|h\|^4_4.$$

Lemma 4: Define

$$U'_n(g) = \sum_{i_1 < i_2 < ... < i_l} g(X_{i_1}, ..., X_{i_l}), \quad l \geq k,$$

where $g(x_{1}, ..., x_{k})$ is a symmetric measurable degenerate kernel function satisfying (2.2.5) with $g$ in lieu of $h$. Also assume that the $^*$-mixing coefficient satisfies condition (2.2.6).

Then

$$\lim_{n \to \infty} E\left( \sup_{1 \leq n \leq N} \frac{U'_n(g)}{n^{1/2}} \right)^2 \ll \int_0^{t_1} \cdots \int_0^{t_l} g^2(x_{i_1}, ..., x_{i_l}) \prod_{i=1}^{k-l} dx_i.$$

2.4. Proof of Main Results

Proof of Theorem 1: Since $\widehat{h} \in L^2[0,1]^k$, following Lee (1990) pp. 88-90 we get

$$\xi(x_{1}, ..., x_{k}) = \sum_{j=1}^{p} \lambda_j \Phi_j(x_{1}) \cdots \Phi_j(x_{k}) \quad \cdots (2.4.1)$$

satisfying $\|h - \xi\|_2 < \varepsilon$ for some arbitrary $\varepsilon > 0$ and large $n$, where $\Phi_j$'s are square integrable functions of one variable with $E\Phi_j(X_{t_j}) = 0$, $j = 1, 2, ..., p$.

Write

$$\hat{U}_n(\widehat{h}) - J_k(\widehat{h})$$

$$= (\hat{U}_n(\widehat{h}) - \hat{U}_n(\xi)) + (\hat{U}_n(\xi) - J_k(\eta)) + (J_k(\xi) - J_k(\widehat{h})). \quad \cdots (2.4.2)$$

By Lemma 2 we get
\[ E(\hat{U}_n(\vec{h}) - \hat{U}_n(\xi))^2 \]
\[ = E(\hat{U}_n(\vec{h} - \xi))^2 \]  ...(2.4.3)
\[ \ll \| \vec{h} - \xi \|_2^2 \]
\[ \ll e^2 \]

Write
\[ \hat{U}_n(\xi) = n^{k/2} \left( \frac{n}{k} \right)^{-1} \sum_{j=1}^k \lambda_j \sum_{1 \leq i < \cdots < k \leq n} \phi_j(X_i) \cdots \phi_j(X_k) \]  ...(2.4.4)

Since by (2.2.4),
\[ \sum_{i=1}^\infty \psi^{1/2}(m) = \sum_{i=1}^\infty m^{-1/2} \psi^{1/2}(m) \leq \left( \sum_{i=1}^\infty \frac{1}{m^2} \right) \left( \sum_{i=1}^\infty m^2 \psi(m) \right) < \infty, \]
by the multivariate analogue of Theorem 20.1 of Billingsley (1968), A2, Ergodic theorem and (3) of Ronzhin (1986) pp. 808 we get from (2.4.4),
\[ \hat{U}_n(\xi) \xrightarrow{d} \sum_{j=1}^p \lambda_j H_j \left( \int_0^1 \phi_j(x) d\beta(x) \right) \text{ as } n \to \infty. \]  ...(2.4.5)

Now by A3 we have
\[ \sum_{j=1}^p \lambda_j H_j \left( \int_0^1 \phi_j(x) d\beta(x) \right) = \sum_{j=1}^p \lambda_j \int_0^1 \phi_j(x) d\beta(x) \]
\[ = J_k(\xi) \]
and hence by (2.4.5),
\[ \hat{U}_n(\xi) \xrightarrow{d} J_k(\xi) \text{ as } n \to \infty. \]  ...(2.4.6)

By A4 we have
\[ E(J_k(\xi) - J_k(\vec{h}))^2 \]
\[ = E(J_k(\xi - \vec{h}))^2 \]  ...(2.4.7)
\[ \ll \| \xi - \vec{h} \|_2^2 \]
\[ \ll e^2 \]
Thus the theorem follows from (2.4.2), (2.4.3), (2.4.6) and (2.4.7).

**Proof of Theorem 2**: Since \( \vec{h} \in L_4[0,1]^k \), by Lemma 4.1 of Dehling (1989) we have
\[ \eta(x_1, \ldots, x_k) = \sum_{j=1}^k \mu_j \alpha_j(x_1) \cdots \alpha_j(x_k) \]  ...(2.4.8)
satisfying \( \| \vec{h} - \eta \|_4 < \epsilon \) for some arbitrary \( \epsilon > 0 \), where \( \alpha_j \)'s are functions of one
variable taking finitely many values only and $\int_0^1 \alpha_j(x) \, dx = 0$, $j = 1, \ldots, q$.

Now note that

$$L(\hat{U}_n^*(\tilde{h}), J_n^*(\tilde{h}))$$

$$\leq L(\hat{U}_n^*(\tilde{h}), \hat{U}_n^*(\eta)) + L(\hat{U}_n^*(\eta), J_n^*(\eta)) + L(J_n^*(\eta), J_n^*(\tilde{h})) \quad \vdots \quad (2.4.9)$$

$$= I + II + III,$$

where $L(Z_1, Z_2)$ denotes the Levy-Prohorov distance between the $D[0,1]$ valued random variables $Z_1$ and $Z_2$. Since $\|\tilde{h} - \eta\|_4 < \varepsilon$, putting $g = \tilde{h} - \eta$ in Lemma 4 we get

$$\lim E \left( \sup_{s \in [0,n]} \frac{U_n^*(\tilde{h} - \eta)}{n^{1/2}} \right)^2 \ll \|\tilde{h} - \eta\|_2^2 \ll \varepsilon^2$$

and hence

$$\lim L(\hat{U}_n^*(\tilde{h}), \hat{U}_n^*(\eta)) \ll \varepsilon. \quad \vdots \quad (2.4.10)$$

Now we consider II of (2.4.9). Write

$$\hat{U}_n^*(\eta) = n^{1/2} \left( \begin{array} { c } { n } \\ { k } \end{array} \right) \sum_{j=1}^k \mu_j \sum_{1 \leq \xi_j \leq \xi_k \leq m} \alpha_j(X_\xi) \ldots \alpha_j(X_m). \quad \vdots \quad (2.4.11)$$

Define

$$S^{(j)}_m(t) = n^{-m/2} \sum_{i=1}^{[m]} \alpha^m_j(X_i), \ j = 1, 2, \ldots, q, \ m = 1, 2, \ldots, k.$$ 

Now putting $g(x) = \alpha^2_j(x) - E\alpha^2_j(X_j)$ in Lemma 4 we get

$$\begin{align*}
P \left( \sup_{0 \leq s \leq t} \left[ \frac{\sum_{i=1}^{[m]} \alpha^2_j(X_i) - E\alpha^2_j(X_i)}{n} \right] > \varepsilon \right) \\
\leq \frac{1}{nE^2} E \left( \sup_{0 \leq s \leq t} \left[ \frac{\sum_{i=1}^{[m]} \left( \alpha^2_j(X_i) - E\alpha^2_j(X_i) \right)}{\sqrt{n}} \right]^2 \right) \\
\ll \frac{1}{nE^2} \int_0^1 \alpha^2_j(x) \, dx \to 0 \text{ as } n \to \infty.
\end{align*}$$

Thus we get

$$\lim P \left[ \sup_{0 \leq s \leq t} |S^{(j)}_m(t) - tE\alpha^2_j(X_j)| > \varepsilon \right] = 0. \quad \vdots \quad (2.4.12)$$

For $m \geq 3$ note that

$$|S^{(j)}_m(t)| \leq n^{-(m/2-1)} n^{-3} \sum_{i=1}^t |\alpha^m_j(X_i)|$$

and hence by Ergodic theorem

$$\lim P \left[ \sup_{0 \leq s \leq t} |S^{(j)}_m(t)| > \varepsilon \right] = 0. \quad \vdots \quad (2.4.13)$$
Thus by multivariate analogue of Theorem 20.1 of Billingsley (1968), (2.4.13) and (4) of Ronzhin (1986) pp. 809 and A3 we get
\[ \lim_{n \to \infty} L \left( \hat{U}_n^i(\eta), J_i^j(\eta) \right) = 0. \tag{2.4.14} \]
Since \( \| \hat{h} - \eta \|_4 < \varepsilon \), putting \( \phi = \hat{h} - \eta \) in A5 we get
\[ E \left( \sup_{t \in \mathbb{R}_+} J_i^j(\hat{h} - \eta) \right)^2 << \| \hat{h} - \eta \|_4^2 << \varepsilon^2 \]
and hence
\[ L \left( J_i^j(\hat{h}), J_i^j(\eta) \right) << \varepsilon. \tag{2.4.15} \]
Thus by (2.4.9), (2.4.10), (2.4.14) and (2.4.15) we have
\[ \lim_{n \to \infty} L \left( \hat{U}_n^i(\hat{h}), J_i^j(\hat{h}) \right)^2 << \varepsilon. \tag{2.4.16} \]
\( \varepsilon \) being arbitrary (2.4.16) implies that
\[ \lim_{n \to \infty} L \left( \hat{U}_n^i(\hat{h}), J_i^j(\hat{h}) \right) = 0 \]
and hence the theorem follows.

**Proof of Theorem 3 and Theorem 4:** The proofs follow in the same line as those of Theorem 1 and Theorem 2 with A6 used instead of A4 and and A7 instead of A5 respectively.

### 2.5. Proof of Auxiliary Results

**Proof of Lemma 1:** For simple function \( h \) by (2.2.2) We have
\[ \left| \int_0^1 \cdots \int_0^1 h(x_i, \ldots, x_k) \prod_{i=1}^{k} [ F'(x_i, \ldots, x_k) - G(x_i, \ldots, x_k)] H(x_{j_1}, \ldots, x_{j_k}) \right| \]
\[ \ll \psi(t_{j_1} - t_j) \int_0^1 \cdots \int_0^1 \left| h(x_i, \ldots, x_k) \prod_{i=1}^{k} [ G(x_i, \ldots, x_k) \right| \right| dG(x_i, \ldots, x_k) \right| dH(x_{j_1}, \ldots, x_{j_k}) \]
and the lemma follows by repeated application of the above result. For general function \( h \) taking limits over simple functions the lemma follows at once.

**Proof of Lemma 2:** For \( k=1 \) under \( A_0(\psi) = \sum_{m=1}^{\infty} \psi(m) < \infty \) we have by Lemma 1
\[ E(n U_n(\hat{h}))^2 = E( \sum_{1 \leq i < j \leq n} \hat{h}(X_i) \hat{h}(X_j) ) \]
\[ = \sum_{1 \leq i < j \leq n} \{ E[\hat{h}(X_i) \hat{h}(X_j)] - E[\hat{h}(X_i)] E[\hat{h}(X_j)] \} \]
\[ \ll n \| \hat{h} \| _2^2. \]
Hence

\[ E(n^{1/2} \tilde{U}_n(\tilde{h}))^2 \ll \| h \|^2 . \]

For \( k=2 \) when \( A_1(\psi) = \sum_{m=1}^{\infty} (m+1)\psi(m) < \infty \), following the proof of Lemma 2 of Yoshihara (1976) and applying Lemma 1 we have

\[ E\left( \binom{n}{2} U_n(\tilde{h}) \right)^2 \ll n^2 \| h \|^2 \]

i.e.

\[ E(\tilde{U}_n(\tilde{h}))^2 \ll \| h \|^2 . \]

For general \( k \) the lemma can be proved similarly once we note that

\[ \left( \sum_{m=1}^{n} (m+1)^{2k-1}\psi(m) \right) \ll n^{k-1} \left( \sum_{m=1}^{\infty} (m+1)^{k-1}\psi(m) \right) \ll n^{k-1} . \]

**Proof of Lemma 3**: First we prove the lemma for \( k=1 \). Write

\[ E(nU_n(\tilde{h}))^4 = \sum_{i_1, i_2, j_1, j_2, \ldots, \in \mathbb{N}} E(\tilde{h}(X_{i_1})\tilde{h}(X_{j_1})\tilde{h}(X_{i_2})\tilde{h}(X_{j_2})) . \]

Now since \( A_1(\psi) = \sum_{m=1}^{\infty} (m+1)\psi(m) < \infty \), applying Lemma 1 and following the proof of Lemma 2 of Yoshihara (1976) we get

\[ E(nU_n(\tilde{h}))^4 \ll n^2 \| h \|^4 \]

i.e.

\[ E(\tilde{U}_n(\tilde{h}))^4 \ll \| h \|^4 . \]

For \( k=2 \) under \( A_1(\psi) = \sum_{m=1}^{\infty} (m+1)^2\psi(m) < \infty \) applying Lemma 1 and following the proof of Lemma 3 of Yoshihara (1976) we get

\[ E\left( \binom{n}{2} U_n(\tilde{h}) \right)^4 \ll n^4 \| h \|^4 \]

i.e.

\[ E(\tilde{U}_n(\tilde{h}))^4 \ll \| h \|^4 . \]

For general \( k \) the lemma can be proved similarly once we note that

\[ \left( \sum_{m=1}^{n} (m+1)^{2k-1}\psi(m) \right) \ll n^{2k-1} \left( \sum_{m=1}^{\infty} (m+1)^{2k-1}\psi(m) \right) \ll n^{2k-1} . \]

**Proof of Lemma 4**: For \( k=1 \) the result can be proved following Philipp and Stout (1975). We prove the result for \( k=2 \) only. The proof for general \( k \) follows in
the same way. At first we define blocks of integers inductively as given in Philipp and Stout (1975) by requiring that $I_j$ contains $\lceil j^{3/2} \rceil$ consecutive integers and that there are no gaps between consecutive blocks.

Let

$$h_j = \min \{ s : s \in I_j \}.$$  

Then

$$h_j = \sum_{s=1}^{j-1} \lfloor s^{3/2} \rfloor + 1$$

and thus

$$j^{3/2} \ll h_j \ll j^{3/2}.$$  

...(2.5.1)

Define $I'_j$ to be the set of lattice points $(s, l)$ such that $h_j \leq l < h_{j+1}$, $1 \leq s < h_{j+1}$ and $s \leq l$ (i.e. lattice points in the shaded region of Fig. 2.1) and the number of elements in $I'_j$ is bounded by $(h_{j+1} - h_j) h_j + \frac{1}{2} (h_{j+1} - h_j)^2 = O(j^2)$.

![Fig. 2.1](image-url)

Define

$$\tilde{g}_s(x_s, x_i) = g(x_s, x_i) - Eg(X_s, X_i),$$

$$u_j = \sum_{(s,i) \in I'_j} g(X_s, X_i)$$

and

$$\tilde{u}_j = \sum_{(s,i) \in I'_j} \tilde{g}_s(X_s, X_i).$$

For $j \geq 1$ set

$$\mathcal{F}_j = \{ \sigma < X_s : s < h_{j+1} \} >$$

$$\mathcal{F}_0 = \{ \phi, \Omega \}.$$
\[ \bar{v}_j = \sum_{s=0}^{\infty} E(\bar{u}_{j+s+1}|\mathcal{F}_{j-1}) \]

and

\[ \tilde{v}_j = \sum_{s=0}^{\infty} \left[ E(\bar{u}_{j+s+1}|\mathcal{F}_j) - E(\bar{u}_{j+s+1}|\mathcal{F}_{j-1}) \right] \]

\[ = \sum_{s=0}^{\infty} \left[ E(u_{j+s+1}|\mathcal{F}_j) - E(u_{j+s+1}|\mathcal{F}_{j-1}) \right]. \]

Clearly for \( j \geq 1 \), \( \{\tilde{v}_j, \mathcal{F}_j\} \) is a martingale difference sequence and \( \bar{u}_j \) is \( \mathcal{F}_j \) measurable. Then

\[ \bar{u}_j = \tilde{v}_j + \bar{v}_j - \bar{v}_{j+1}. \]

Write

\[ u_j = \sum_{1 \leq r < \ell \leq j+1} g(X_r, X_\ell) \]

\[ = \sum_{1 \leq r < \ell \leq j+1} g(X_r, X_\ell) + \sum_{h_j \leq \ell < h_{j+1}} g(X_r, X_\ell) \]

\[ = \sum_{1 \leq r < \ell \leq j+1} g(X_r, X_\ell) + \sum_{h_j \leq \ell < h_{j+1}} g(X_r, X_\ell). \]...

(2.5.2)

Then

\[ Eu_j^4 \leq 2^4 \left( E\left( \sum_{1 \leq r < \ell \leq j+1} g(X_r, X_\ell)^4 \right) + E\left( \sum_{h_j \leq \ell < h_{j+1}} g(X_r, X_\ell)^4 \right) \right) \]

\[ \leq 2^4 (I + II), \text{ say.} \]...

(2.5.3)

Since (2.2.5) and (2.2.6) holds, applying Lemma 3 and (2.5.1) it follows that

\[ II \ll (h_{j+1} - h_j)^4 \ll j^2. \]...

(2.5.4)

Now write

\[ I = c_1 \sum_{1 \leq r < \ell \leq j+1} Eg^4(X_r, X_\ell) + c_2 \sum_{1 \leq r < \ell \leq j+1} E\{g^2(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)\} \]

\[ + c_3 \sum_{1 \leq r < \ell \leq j+1} E\{g^2(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)\} \]

\[ + c_4 \sum_{1 \leq r < \ell \leq j+1} E\{g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)g(X_r, X_\ell)\} \]

\[ = A + B + C + D, \text{ say.} \]...

(2.5.5)

where \( c_1, c_2, c_3 \) and \( c_4 \) are some constants.

By (2.5.1) we have

\[ A \ll h_j(h_{j+1} - h_j) \ll j^2 \]

and

\[ B \ll h_j^2(h_{j+1} - h_j)^2 \ll j^4. \]

Since (2.2.5) and (2.2.6) holds, applying Lemma 1 and (2.5.1) and following the proof of Lemma 3 we have

\[ C \ll j^4 \text{ and } D \ll j^4. \]
Thus by (2.5.5) we get
\[ I \ll j^4. \]  
...(2.5.6)
Hence combining (2.5.4) and (2.5.6) with (2.5.3) we get
\[ \mu_j \ll j^4. \]  
...(2.5.7)
Now for \( \gamma > 0 \) it follows from (2.5.7) that
\[ P(\mu_j > j^{1+\gamma}) \ll \frac{E(\mu_j^4)}{j^{4+\gamma}} \ll j^{-\gamma}. \]

Hence for \( \gamma > 1/4 \) we get by Borel-Cantelli lemma
\[ u_j \ll j^{1+\gamma} \text{ a.s.} \]  
...(2.5.8)
Let \( M_n \) be the index of \( I_j \) containing the coordinates \((s,l)\) with \( h_{M_s} \leq s \leq n \) and \( n \leq l < h_{M_{s+1}} \). Then
\[ u_{M_n} \ll n^{2/3+2/3\gamma} \text{ a.s.} \]  
...(2.5.9)
Applying Lemma 1 once again
\[ \mu_{M_n} \ll n^{1/3}. \]  
...(2.5.10)
i.e.
\[ \mu_{M_n} \ll n^{1/3}. \]  
...(2.5.11)
Thus by (2.5.9) and (2.5.11) and choosing \( \gamma \) such that \( 1/4 < \gamma \) and \( 2/3 + 2\gamma /3 < 1 \) we get
\[ \mu_n \ll n^{2/3+2/3\gamma} \ll n^{1-\gamma} \text{ a.s.}, \]  
...(2.5.12)
where \( \gamma' > 0 \).

For \( p \geq 1 \) and \( \delta > 0 \) write
\[ E \left| \frac{E(\mu_{j+p}|F_{j-1})}{2^{+\delta}} \right|^{2+\delta} \]
\[ = E \left[ E(\mu_{j+p}|F_{j-1})E(\mu_{j+p}|F_{j-1})E(\mu_{j+p}|F_{j-1}) \right]^{\delta} \]
\[ = E \left[ \mu_{j+p}E(\mu_{j+p}|F_{j-1})E(\mu_{j+p}|F_{j-1}) \right]^{\delta} \]
\[ = \left( \sum_{(1)} + \sum_{(2)} + \sum_{(3)} + \sum_{(4)} \right) E \left[ \mathcal{G}_d(X_s, X_l)E(\mu_{j+p}|F_{j-1})E(\mu_{j+p}|F_{j-1}) \right]^{\delta} \]  
...(2.5.13)
where the summations \( \sum_{(1)}, \sum_{(2)}, \sum_{(3)} \) and \( \sum_{(4)} \) extend respectively over all
\[ h_{j+p} \leq s < h_{j+p+1}, \quad h_j \leq s < \frac{h_j + h_{j+p}}{2}, \quad \frac{h_j + h_{j+p}}{2} \leq s < h_{j+p}, \quad 1 \leq s < h_j \quad \text{and} \quad h_{j+p} \leq l < h_{j+p+1}. \]
for all the cases, where \( s < l \).
Write
\[
\sum_{(3)} \mathbb{E} \left[ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{6}
\]

\[
= \sum_{(3)} \left\{ \mathbb{E} \left[ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right] \mathbb{E} \left[ \tilde{g}_u(X_s, X_t) \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6} \quad \text{(2.5.14)}
\]

Applying Lemma 1, then using Holder's inequality, (2.2.5) with \(g\) in lieu of \(h\), (2.2.6) and (2.5.1), we get from (2.5.14)

\[
\sum_{(3)} \mathbb{E} \left\{ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6}
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \sum_{(3)} \psi(s - h_j)
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \sum_{(3)} \frac{(h_{j+p+1} - h_{j+p})^2}{(h_{j+p} - h_j)^2}
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \frac{(j + p)}{(j + p)^{3/2} - j^{3/2}}^4
\]

Similarly we get

\[
\sum_{(3)} \mathbb{E} \left\{ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6}
\]

\[
= \sum_{(3)} \left\{ \mathbb{E} \left[ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right] \mathbb{E} \left[ \tilde{g}_u(X_s, X_t) \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6} \quad \text{(2.5.15)}
\]

\[
\sum_{(3)} \mathbb{E} \left\{ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6}
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \sum_{(3)} \psi(l - s)
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \sum_{(3)} \frac{(h_{j+p+1} - h_{j+p})^2}{(h_{j+p} - h_j)^2}
\]

\[
\ll \left[ \mathbb{E} \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right]^{2+5} \frac{(j + p)^{1/2}}{(j + p)^{3/2} - j^{3/2}}^3
\]

\[
\sum_{(3)} \left\{ \tilde{g}_u(X_s, X_t) \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \mathbb{E} \left[ \tilde{u}_{j+p} | \mathcal{F}_{j-1} \right] \right\}^{6}
\]

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\[
\sum_{(3)} \left\{ E[g_u(X_s, X_i)E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})^5] \right. \\
- E[g_u(X_s, X_i)]E[E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})^5] \} \] 
\leq \sum_{(3)} \psi(s-h_j) \quad \text{...(2.5.17)}
\]

\[
\sum_{(4)} \left\{ E[g_u(X_s, X_i)E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})^5] \right. \\
- E[E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})^5] \} \sum_{(4)} E[g(X_s, X_i)] \] 
\leq \sum_{(4)} \psi(l-h_j) \quad \text{...(2.5.18)}
\]

Now note that
\[
(j + p)^{3/2} - j^{3/2} \gg \begin{cases} 
 j^{3/2} & \text{for } 1 \leq p \leq j \\
 p^{3/2} & \text{for } p > j. 
\end{cases} \quad \text{...(2.5.19)}
\]

Thus by (2.5.15), (2.5.16), (2.5.17), (2.5.18) and (2.5.19) the bound in (2.5.13) can be obtained as
\[
E[E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|^{2+\delta} \ll \frac{1}{p^{\delta}} [E[E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})|^{2+\delta}]^{(1+\delta)/(2+\delta)}] 
\]
i.e.
\[
\|E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})\|_{2+\delta} \ll \frac{1}{p^{\delta}}, \text{ for } p \geq 1. \quad \text{...(2.5.20)}
\]
Now
\[
E[E(\tilde{u}_{j+\rho}|\mathcal{F}_{j-1})]^{4} \\
= E[\tilde{u}_{j}E(\tilde{u}_{j}|\mathcal{F}_{j-1})|E(\tilde{u}_{j}|\mathcal{F}_{j-1})^{2}] 
\]
\[
\leq \left[ E(\tilde{u}_j)^4 \right]^{3/4} \left[ E\left(E(\tilde{u}_j | \tau_{j-1})\right)^4 \right]^{3/4}
\]

and hence by (2.5.7) and (2.5.10) we get

\[
\left\| E(\tilde{u}_j | \tau_{j-1}) \right\|_4 \ll j.
\]

Combining the above inequality with (2.5.20) we get

\[
\left\| \bar{v}_j \right\|_{2+\varepsilon} \ll j \quad \text{(2.5.21)}
\]

Since

\[
\tilde{u}_j = \tilde{w}_j + v_j - v_{j+1}, \quad j \geq 1,
\]

for \( m > 1 \) we get by (2.5.21)

\[
P\left[ \sum_{j \leq m} (\tilde{u}_j - \tilde{w}_j) > m^{3(2+\varepsilon)} + 55/(4(2+\varepsilon)) \right] = P\left[ \tilde{v}_1 = \tilde{v}_{m+1} > m^{3(2+\varepsilon)} + 55/(4(2+\varepsilon)) \right]
\]

\[
\ll \frac{E(\tilde{v}_j)^{2+\varepsilon} + E(\tilde{v}_{m+1})^{2+\varepsilon}}{m^{3+55/4}}
\]

\[
\ll \frac{m^{2+\varepsilon}}{m^{3+55/4}}
\]

\[
\ll m^{-(1+55/4)}.
\]

Since \( n^{2/3} \ll M_n \ll n^{2/3} \), applying Borel-Cantelli lemma it follows from the above that

\[
\sum_{j \leq m} (\tilde{u}_j - \tilde{w}_j) \leq m^{3(2+\varepsilon)} + 55/(4(2+\varepsilon)) \quad \text{a.s.}
\]

\[
\ll n^{-\varepsilon/6(2+\varepsilon)} \quad \text{a.s.}
\]

\[
\ll n^{-\beta} \quad \text{a.s. for some } 0 < \beta < 1. \quad \text{(2.5.22)}
\]

Define

\[
U_t'(g) = \sum_{j \leq m} \tilde{w}_j, \quad t \geq 2.
\]

Since

\[
U_t'(g) = \sum_{1 \leq r \leq t} \tilde{g}_u(X_s, X_r),
\]

we have

\[
U_t'(g) - EU_t'(g) - U_t'(g)
\]

\[
= \sum_{j \leq m, \tau_{j-1} \leq \chi_{m+1}, \chi_{m+1} \leq \chi_{m+1}} (\tilde{u}_j - \tilde{w}_j) - \sum_{1 \leq r \leq t} \tilde{g}_u(X_s, X_r).
\]

Thus by (2.5.12) and (2.5.22) we get

\[
\left| U_t'(g) - EU_t'(g) - U_t'(g) \right| \ll \max(t^{-\beta}, t^{-\gamma}) \quad \text{a.s., } t \geq 2
\]

and hence
Since \( \{U'_i(g), \mathcal{F}_{M_0}\}_{i \geq 1} \) is a martingale, it follows from Lemma 2 and (2.5.23) that

\[
E \left[ \sup \frac{U_i'(g)}{n} \right]^2 \leq E \left[ \sup \frac{U'_i(g)}{n} \right]^2 + \sup \left[ \frac{EU'_i(g)}{n} \right]^2 + n^{-2\alpha}
\]

\[
\leq E \left[ \frac{U'_i(g)}{n} \right]^2 + \sup \left[ \frac{EU'_i(g)}{n} \right]^2 + n^{-2\alpha}
\]

\[
\leq E \left[ \frac{U'_i(g)}{n} \right]^2 + \left( \frac{EU'_i(g)}{n} \right)^2 + \sup \left[ \frac{EU'_i(g)}{n} \right]^2 + n^{-2\alpha}
\]

\[
\leq \|g\|_2^2 + n^{-2\alpha}.
\]

Thus

\[
\lim_{n \to \infty} E \left[ \sup \frac{U_i'(g)}{n} \right]^2 \leq \|g\|_2^2
\]

and hence the lemma is proved.

### 2.6. Examples

The solution of a stable linear difference equation induced by a sequence of independent bounded random variables furnishes an example of a random sequence satisfying \(*\)-mixing condition. Numerous applications of degenerate \(U\)-statistics of order of degeneracy one can be found in Serfling (1980), Carlstein (1988) and Lee (1990). For kernel of higher order degeneracy our result can be applied to estimate higher order cumulant spectral density for all frequencies under \(*\)-mixing condition (see Lee (1990) and Rosenblatt (1990) for details).

Let \( \{X_i\} \) be a strictly stationary \(*\)-mixing process with zero mean and finite variance. Consider the product kernel

\[ h(x_1, \ldots, x_k) = x_1 \cdots x_k, \quad k \geq 2. \]

Then

\[ U_n(h) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} X_{i_1} \cdots X_{i_k} \binom{n}{k} \]

is a degenerate \(U\)-statistic. Since for i.i.d. observations \( EU_n = 0 \), the statistic \( U_n(h) \) can be used to test serial dependence. Skaug and Tjøstheim (1993) considered pairwise independence of the elements in \( (X_{t+1}, \ldots, X_t) \), but not simultaneous
independence of the whole set. The above degenerate $U$-statistic may be utilized to test independence of the whole set of random variables.

### 2.7. Multiple Stochastic Integral

**A1.** Let $\{K(x,t)\}$ be a Kiefer process defined on $([0,1], B([0,1]), P) \times ([0,1], B([0,1]), \lambda)$, where $\lambda$ is the Lebesgue measure on $[0,1]$, such that $E K(x,t) = 0$ and the covariance function is given by

$$ Cov\{K(x,t), K(x',t')\} = (t \wedge t') \Gamma(x, x'), $$

where

$$\Gamma(x, x') = (x \wedge x' - xx') + \sum_{j=2}^{m} \{P(X_j \leq x, X_j \leq x') - xx'\} + \sum_{j=2}^{m} \{P(X_j \leq x, X_j \leq x') - xx'\}. $$

Now for partition

$$0 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_j < x_{j+1} < \ldots < x_m = 1$$

define

$$\Delta K(x_i,t) = K(x_{i+1},t) - K(x_i,t).$$

Then for fixed $t$, $\Delta K(x_1,t), \ldots, \Delta K(x_m,t)$ are Gaussian with covariance function $\widetilde{\Lambda}(t) = ((u'_{ij}))$, where

$$u'_{ij} = t[-\Delta x_i \Delta x_j + \sum_{s=2}^{m} \{P(x_i < X_s \leq x_{i+1}, x_j < X_s \leq x_{j+1}) - \Delta x_i \Delta x_j\}]$$

$$+ \sum_{s=2}^{m} \{P(x_i < X_s \leq x_{i+1}, x_j < X_s \leq x_{j+1}) - \Delta x_i \Delta x_j\}]$$

and

$$u''_{ij} = t[\Delta x_i - \Delta^2 x_j + 2 \sum_{s=2}^{m} \{P(x_i < X_s \leq x_{i+1}, x_j < X_s \leq x_{j+1}) - \Delta x_i \Delta x_j\}]$$

Calculating the joint characteristic functions of the Gaussian increments we get

$$E \prod_{i=1}^{2s} \Delta K(x_i,t) = \sum u'_{i_1i_2} u'_{i_3i_4} \ldots u'_{i_{2s-1}i_{2s}}, \quad \ldots \quad (2.7.1)$$

where $i_j = 1, 2, \ldots, 2s$, $I_j = 1, 2, \ldots, 2s$, $j = 1, 2, \ldots, 2s$ and the summation is extended over all possible combinations of the indices $1, 2, \ldots, 2s$ into pairs. (See Filippova (1961) pp. 34).

Now using (2.2.1) the bounds of $u'_{ij}$ and $u''_{ij}$ can be obtained as

$$u'_{ij} \leq t[\Delta x_i \Delta x_j + 2A_0(\psi)\Delta^2 x_i], \quad \ldots \quad (2.7.2)$$

and

$$u''_{ij} \leq t[-\Delta x_i \Delta x_j + 2A_0(\psi)\Delta x_i \Delta x_j], \quad \ldots \quad (2.7.3)$$
where
\[ A_0(\psi) = \sum_{m=1}^\infty \psi(m). \]

**A2.** Let \( \phi(\cdot) \) be a simple function such that
\[
\phi(u) = \begin{cases} 
\phi(x_i) & \text{if } u \in (x_i, x_{i+1}] \\
0 & \text{otherwise,}
\end{cases}
\]
where
\[ 0 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_j < x_{j+1} < \ldots < x_m = 1. \]
Now define
\[
J'_i(\phi) = \int_0^1 \phi(u) K(du, t)
= \sum_{i=1}^m \phi(x_i) \Delta K(x_i, t).
\]
Then
\[ J'_i : L_2[0,1] \to C([0,1], C), \]
where \( C[0,1] \) is the space of all real valued continuous functions on \([0,1]\) equipped with the uniform topology and \( C \) is the \( \sigma \)-algebra defined on \( C[0,1] \).

Then
\[ EJ'_i(\phi) = 0 \text{ and } E(J'_i(\phi))^2 = \Lambda_1(\phi), \]
where
\[ \Lambda_1(\phi) = i[E\phi^2(X_i) - [E\phi(X_i)]^2] + 2 \sum_{j=1}^n \{E[\phi(X_i)\phi(X_j)] - [E\phi(X_i)]^2\}. \]
By (2.2.2) we get
\[ \Lambda_1(\phi) < \|\phi\|_2^2 \]
and hence the operator \( J'_i \) can be extended to the whole of \( L_2[0,1] \) satisfying (2.7.5).

**A3.** For square integrable functions of single variable \( \phi_1, \ldots, \phi_p \) and \( \alpha \) define
\[ \phi(x_1, \ldots, x_p) = \phi_1(x_1) \ldots \phi_p(x_p) \]
and
\[ (\phi_\alpha) = \alpha(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p) = \int_0^1 \phi(x_1, \ldots, x_p) \alpha(x_k) dx_k \]
and assume that
\[ A_0(\psi) = \sum_{m=1}^\infty \psi(m) < \infty \]
and
\[ \int_0^1 \phi^2(x) \alpha^2(x) dx < \infty. \]
Then
\[ J'_{p-1}(\phi(\alpha)) = J'_{p}(\phi)(J'_{p}(\alpha) - \sum_{k=1}^{p} J'_{p-1}(\phi(\alpha) \times \alpha) \text{ a.s.} \quad 0 \leq t \leq 1. \]

Proof: We prove the assertion for \( p = 1 \) only. The result, for general \( p \), can be derived in the same way following the proof of Theorem 3.1 of Itô (1951).

Define
\[ \mathcal{F}_s = \sigma \{ K(x, s) : 0 \leq x \leq 1, 0 \leq s \leq t \} \]
and
\[ \mathcal{F}_0 = \{ \phi, \Omega \} \]
Then \( \{K(x, t), \mathcal{F}_s\} \) is a martingale for \( 0 \leq t \leq 1 \).

Let \( \phi \) be a simple function defined by (2.7.4) and \( \alpha \) also a simple function defined similarly.

Define
\[ Y'_m = J'_1(\phi(\alpha)) - J'_1(\phi)(J'_1(\alpha) - \sum_{i=1}^{m} \phi(x_i)\alpha(x_i)) \]
\[ = \sum_{i=1}^{m} \phi(x_i)\alpha(x_i)[\Delta^2 K(x_i, t) - \Delta x_i - \Delta^2 x_i + 2 \sum_{s=2}^{\infty} \{P(x_i < X_s < x_{i+1}, x_i < X_s < x_{i+1}) - \Delta x_i\}] \]
\[ = \sum_{i=1}^{m} \phi(x_i)\alpha(x_i)[\Delta^2 K(x_i, t) - \Delta x_i - \Delta^2 x_i + 2 \sum_{s=2}^{\infty} \{P(x_i < X_s < x_{i+1}, x_i < X_s < x_{i+1}) - \Delta x_i\}] \]
\[ \text{...(2.7.6)} \]

For \( 0 \leq s < t \leq 1 \) write
\[ \sum_{i=1}^{m} \phi(x_i)\alpha(x_i)[\Delta^2 K(x_i, t) - \Delta x_i - \Delta^2 x_i + 2 \sum_{s=2}^{\infty} \{P(x_i < X_s < x_{i+1}, x_i < X_s < x_{i+1}) - \Delta x_i\}] \]
\[ = \sum_{i=1}^{m} \phi(x_i)\alpha(x_i)[\Delta K(x_i, t) - \Delta K(x_i, s) + 2 \Delta K(x_i, s)(\Delta K(x_i, t) - \Delta K(x_i, s))] \]
Thus by (2.7.6) we have
\[ \mathbb{E}(Y_m^t | \mathcal{F}_t) = Y_m^t \]
implying that \( \{Y_m^t, \mathcal{F}_t\} \) is a martingale for \( 0 \leq t \leq 1 \). By maximal inequality, (2.7.1), (2.7.2) and (2.7.3) and following Itô (1951) we get
\[ \mathbb{E}(\sup_{0 \leq s \leq t} Y_m^s)^2 \leq \mathbb{E}(Y_m^1)^2 < \varepsilon. \] (2.7.7)
This implies that
\[ Y_m^t \xrightarrow{p} 0, \quad \text{uniformly in } t, \quad 0 \leq t \leq 1 \quad \text{as} \quad m \to \infty \quad \text{and} \quad \max_{1 \leq s \leq t} |x_{is} - x_i| \to 0. \]
Again by (2.2.1) and since \( \max_{1 \leq s \leq t} |x_{is} - x_i| \to 0 \) as \( m \to \infty \) we have
\[ \sum_{i=1}^{m} \phi(x_i) \alpha(x_i) \Delta x_i - 2 \sum_{i=1}^{m} \{P(x_i < X_i \leq x_{is}, x_i < X_s \leq x_{is}) - \Delta^2 x_i\} \to \int_0^1 \phi(x) \alpha(x) dx \quad \text{as} \quad m \to \infty. \] (2.7.8)
Thus taking limit on (2.7.6) as \( m \to \infty \) it follows from (2.7.7) and (2.7.8) that
\[ J_2^t(\alpha) = J_1^t(\phi)J'_1(\alpha) - \int_0^1 \phi(x) \alpha(x) dx \quad \text{a.s.}. \]

A4. For symmetric measurable degenerate kernel function \( \phi(x_1, \ldots, x_k) \in L_2[0,1]^k \) with \( \int_0^1 \phi(x_1, \ldots, x_k) dx_k = 0 \) we have
\[ \mathbb{E}(J_2(\phi))^2 < \|\phi\|_2^2, \]
where the \(*\)-mixing coefficient satisfies (2.2.4).

**Proof**: The assertion for \( k=1 \) has been already proved in A2 (see (2.7.5)). Since for general \( k \) the proof is tedious we prove it for \( k=2 \) only. The proof for general \( k \) follows similarly.

For partition
\[ 0 = x_0 < x_1 < \ldots < x_i < x_{is1} < \ldots < x_j < x_{js1} < \ldots < x_m = 1 \]
define simple function
\[ \phi(u, v) = \begin{cases} \phi(x_i, x_j) & \text{if } (u, v) \in (x_i, x_{is1}) \times (x_j, x_{js1}) \\ 0 & \text{otherwise}. \end{cases} \] (2.7.9)
Then by (2.7.1), (2.7.2) and (2.7.3) we have
\[ \mathbb{E}(J_2(\phi))^2 < \|\phi\|_2^2. \] (2.7.10)

The space \( L_2[0,1]^2 \) being complete the above inequality is true for general square integrable function also.
A5. For symmetric measurable degenerate kernel function \( \phi(x_1, \ldots, x_k) \in L_2[0,1]^k \) with \( \int_0^1 \phi(x_1, \ldots, x_k) dx_1 = 0 \) under the assumption that \( * \)-mixing coefficient satisfies (2.2.6) we have

\[
E(\sup_{\sigma \in \mathcal{S}_1} J_k'(\phi))^2 \ll \| \phi \|_2^2.
\]

**Proof**: The result for \( k = 1 \) has already been proved in (2.7.5) of A2. For general \( k \) let \( \eta(x_1, \ldots, x_k) \) be defined as in (2.4.8) such that \( \| \phi - \eta \|_d < \varepsilon \) for some arbitrary \( \varepsilon > 0 \).

Then by (2.4.14) we have

\[
\hat{U}_n^j(\eta) \xrightarrow{p} J_k'(\eta) \quad \text{as} \quad n \to \infty.
\]

Hence by Lemma 4 (putting \( \eta \) in place of \( g \)) and Theorem 5.3 of Billingsley (1968) we have

\[
E(\sup_{\sigma \in \mathcal{S}_1} J_k'(\eta))^2 \leq \lim_{n \to \infty} E(\sup_{\sigma \in \mathcal{S}_1} \hat{U}_n^j(\eta))^2
\]

\[
\leq \lim_{n \to \infty} E(\sup_{\sigma \in \mathcal{S}_1} \hat{U}_n^j(\eta))^{k_2^2}
\]

\[
\ll \| \eta \|_2^2.
\]

\( C[0,1] \) being complete, it follows from (2.7.12) that \( J_k'(\phi) \) is well defined and

\[
E(\sup_{\sigma \in \mathcal{S}_1} J_k'(\phi))^2 \ll \| \phi \|_2^2.
\]

A6. For symmetric measurable degenerate kernel function \( \phi(x_1, \ldots, x_k) \in \hat{L}_2[0,1]^k \) with \( \int_0^1 \phi(x_1, \ldots, x_k) dx_1 = 0 \) under the assumption that the \( * \)-mixing coefficient satisfies (2.2.4) we have

\[
E(\hat{H}_k(\phi))^2 \ll f,
\]

where \( f \) is a polynomial in \( \| \phi \|_2 \), \( Q \) varies over all partitions of \( \{1,2,\ldots,k\} \), and \( \phi \) is defined similarly as \( h_2 \).

**Proof**: Since for general \( k \) the proof is tedious we prove the result for \( k = 2 \) only. The proof for general \( k \) will be the same. For simple function \( \phi(x,y) \) defined in (2.7.9) write

\[
E(\hat{H}_2(\phi))^2
\]

\[
= \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{m=0}^1 \sum_{n=0}^1 \sum_{\Delta K(x_1,\tau)\Delta K(x_2,\tau)\Delta K(x_1,\tau)\Delta K(x_1,\tau)} E(\phi(x_1,x_2)\phi(x_1,x_1)) \Delta K(x_1,\tau) \Delta K(x_2,\tau) \Delta K(x_1,\tau) \Delta K(x_1,\tau)), \quad (2.7.13)
\]

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where
\[
\sum_i = \sum_{i=j,k=1}^l + \sum_{i=k,j=1}^l \\
\sum_2 = \sum_{i=j,k=1}^l + \sum_{j=l, j=k}^l \\
\sum_3 = \sum_{j=k, j=1}^l + \sum_{i=l, j=k}^l \\
\sum_4 = \sum_{i=j, k=1}^l \\
\sum_5 = \sum_{i=k, j=1}^l + \sum_{i=l, j=k}^l \\
\sum_6 = \sum_{i=j, k=1}^l 
\]
and
\[
\sum_6 = \sum_{i=j, k=1}^l 
\]

By (2.7.1), (2.7.2) and (2.7.3) we have
\[
\sum_p E\{\phi(x_i, x_j)\phi(x_k, x_l)\Delta K(x_i, t)\Delta K(x_j, t)\Delta K(x_k, t)\Delta K(x_l, t)\} \\
\ll \|\phi\|_2^2 (\int_0^1 \phi^2(x, x)dx)^{1/2} \quad \text{for } p = 1 \\
\ll \|\phi\|_2^2 \quad \text{for } p = 2, 3, 5 \text{ and } 6 \\
\int_0^1 \phi^2(x, x)dx \quad \text{for } p = 4. 
\]
\[
\text{(2.7.14)} 
\]
Thus by (2.7.13) by (2.7.14) we have
\[
E(\hat{H}_2(\phi))^2 \ll f. \quad \text{(2.7.15)}
\]
The space \( \hat{L}_2[0,1]^2 \) being complete it follows that (2.7.15) is true also for general function \( \phi \in \hat{L}_2[0,1]^2 \).

\textbf{A7.} For symmetric measurable degenerate kernel function \( \phi \in \hat{L}_2[0,1]^2 \) with \( \int_0^1 \phi(x_1, \ldots, x_d)dx = 0 \) under the assumption that the \(*\)-mixing coefficient satisfies (2.2.6) we have
\[
E(\sup_{0 \leq t \leq 1} \hat{H}_4(\phi))^2 \ll f, 
\]
where \( f \) is a polynomial as defined in A6.

\textbf{Proof:} The proof is in the same line as that of A5.