Special Faithful Representation of a Po-universal Algebra into a Po-group

1. In Chapter I, we have proved that every universal algebra $A$, has a special faithful representation into a group $G$, that is $A$ is monomorphic to some specially derived algebra over $G$. In this paper, we have extended this result to po universal algebras.

2. A universal algebra $G$ with signature $\wedge$ is said to be a po universal algebra iff

(i) $G$ is a poset

(ii) $a_1 \leq b_1$, $i = 1, 2, \ldots, n$ where $a_i, b_i \in G$, $\omega \in \Omega_n$, $n \geq 2$

$$\Rightarrow a_1 a_2 \cdots a_n(\omega) \leq b_1 b_2 \cdots b_n(\omega).$$

Let $p$ be a po group and $\wedge$ be any signature. Let us
take a natural number $n$. The set of all $n$-ary operations belonging to $\mathcal{L}$ will be denoted by $\mathcal{L}_n$. If for each set of $n$-elements $a_1, a_2, \ldots, a_n \in p$, we get a uniquely defined element $(a_1 a_2 \ldots a_n) \omega \in p$, i.e. the $n$-ary operation $\omega$ is defined in $p$, for all $\omega \in \mathcal{L}_n$, $n \geq 1$, and further in the set $p$, we fix elements $0$ for all $\omega \in \mathcal{L}_0$, then the set $p$ becomes an $\mathcal{L}$ algebra $T$.

If further $a_1 \leq b_1, i = 1, 2, \ldots, n$ in $p$, $\omega \in \mathcal{L}_n, n \geq 2$

$\Rightarrow (a_1 a_2 \ldots a_n) \omega \leq (b_1 b_2 \ldots b_n) \omega$

then $T$ becomes a po algebra, which will be called a specially derived Po-algebra $T$ with signature $\mathcal{L}$ over the po group $p$.

We shall say that a po universal algebra $G$ with signature $\mathcal{L}$ has a special faithful representation in a po-group $p$,
iff $G$ is $0$-monomorphic to some specially derived po algebra with signature $\bigcap$ over the po group $P$.

3. We shall prove the following

Theorem: Every po universal algebra $G$ with signature $\bigcap$ has a special faithful representation in some po-group $P$.

Proof: Let $G$ be a universal algebra with signature $\bigcap$ and partially ordered with respect to the relation $\leq$.

The elements of the set $G$ are denoted by the symbols $x^+_{\alpha}, x^+_{\beta}$, etc.

Let us consider another set $G'$ of symbols $x^{-1}_{\alpha}, x^{-1}_{\beta}$, etc., which are in a bijective correspondence with the elements of the set $G$. In $G'$, define a binary relation $\preceq$ by $x^{-1}_{\alpha} \preceq x^{-1}_{\beta}$ iff $x^+_{\alpha} \preceq x^+_{\beta}$ in $G$,

where $x^+_{\alpha}, x^+_{\beta}$ are the elements in $G$ corresponding to
The relation "≤" in $G'$ as can be seen is a partial order.

In $H = G \cup G'$, define a binary relation "≤" by (i) if $u, v \in G$ that is $u = x_\alpha^{-1}$, $v = x_\beta^+$, then $x_\alpha \leq x_\beta$ is defined by the partial order "≤" in $G$.

(ii) if $u, v \in G'$, that is $u = x_\alpha^{-1}$, $v = x_\beta^{-1}$, then $x_\alpha^{-1} \leq x_\beta^{-1}$ is defined by the partial order "≤" in $G'$.

(iii) if $u \in G$ and $v \in G'$ say, then $u \parallel v$, i.e., $u$ is incomparable to $v$. In other words, no $x_\alpha^+$ is comparable to any $x_\beta^{-1}$.

The relation "≤" defined in $H$ is a partial order.

An expression

$$\Psi = \sum_{i=1}^{n} \epsilon_i x_{\alpha_i} x_{\beta_i} \cdots x_{\alpha_n} \epsilon_n \quad (\epsilon_i = \pm 1, i = 1, 2, \ldots, n) \quad (1)$$
that is an ordered system of a finite number of symbols of the form \( x^+ \) and \( x^- \) (where each symbol that enters into the expression (1) may occur several times) is called a word.

If in (1) no symbol \( x^+ \) stands next to its associated symbol \( x^- \), then \( \psi \) is called a reduced word. The number \( n \) is called the length of the word \( \psi \) and is denoted by \( \text{l}(\psi) \).

We also count as a word the empty word \( \phi \) which contains no symbol and we put \( \text{l}(\phi) = 0 \).

Let \( M \) be the set of all words over the alphabet \( H = G \cup G' \).

Let the empty word \( \phi \) belongs to \( M \).

Let \( a, b \in M \) \( \Rightarrow \) \( a = x_1 \cdot x_2 \cdot \ldots \cdot x_n \)

\( b = y_1 \cdot y_2 \cdot \ldots \cdot y_m \).

Define the product

\[ a \circ b = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m \]
The product thus defined is associative, which we have proved in Chapter I.

Thus $M$ is a semigroup with the empty word $\emptyset$ as the identity.

In $M$, define a binary relation "$\leq$" by $a \leq b$ iff $n = m$ and $x_i \leq y_i$, $i = 1, 2, \ldots, m$, in $H$.

The relation "$\leq$" defined in $M$ is a partial order.

Now in $M$, define a binary relation $R$ by the rule $aRb$ iff $b$ is obtained from $a$ by means of one or more or none of the following operations:

(A) if the word $a$ contains the word $a' = \frac{E_i}{i} \cdot \frac{-E_i}{i}$ as a sub word, then drop $a'$.

(B) insert the word $a' = \frac{E_i}{i} \cdot \frac{-E_i}{i}$ as a sub word in a word $a$.

Then we have proved in Chapter I, that $R$ is a congruence.

We consider the semigroup $M/R$. The binary operation of
this semigroup will be denoted by • "X".

In this semigroup, the class (Φ) containing the empty word is the identity. In fact, the class (Φ) contains, together with the empty word, the words of the form \( x_1^+ \cdot x_1^- \).

If \( a = x_1 \cdot x_2 \cdot \ldots \cdot x_n \), then we denote the element \( x_n \cdot x_{n-1} \cdot \ldots \cdot x_1 \) by \( a^{-1} \).

That is \( (x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{-1} = x_n \cdot x_{n-1} \cdot \ldots \cdot x_1 \).

The class \( (a^{-1}) \) is the inverse of \( (a) \) with respect to the identity (Φ) as follows from the product of classes and (A).

Consequently \( M/R \) is a group.

In \( M/R \), each class \( (a) \) consists of a unique reduced word since by applying (A) and (B) on a reduced word, we cannot obtain a different reduced word. This unique reduced word is called the normal word of \( (a) \) and will be denoted by \( n(a) \).
Let us now define a binary relation "$\prec" in $P = \mathcal{M}/R$ as follows:

Let $(a), (b)$ be any two elements of $P$ where

$$n(a) = \frac{\varepsilon_1}{\alpha_1} \cdot \frac{\varepsilon_2}{\alpha_2} \cdots \frac{\varepsilon_n}{\alpha_n}$$

and

$$n(b) = \frac{\delta_1}{\beta_1} \cdot \frac{\delta_2}{\beta_2} \cdots \frac{\delta_m}{\beta_m}$$

If $\{n(a)\} = \{n(b)\}$ define

$$(a) \prec (b) \iff \frac{\varepsilon_i}{\alpha_i} \ll \frac{\delta_i}{\beta_i} \quad i = 1, 2, \ldots, m \text{ in } H.$$ 

If $\{n(a)\} \neq \{n(b)\}$,

Let us consider the following operation:

(0.) Insert the word of the form $b.c^{-1}$ or drop the word of the form $c.b^{-1}$ anywhere in a word, where $b = \frac{\varepsilon_1}{\alpha_1} \cdot \frac{\varepsilon_2}{\alpha_2} \cdots \frac{\varepsilon_n}{\alpha_n}$,

$$c = \frac{\varepsilon'_1}{\gamma_1} \cdot \frac{\varepsilon'_2}{\gamma_2} \cdots \frac{\varepsilon'_n}{\gamma_n},$$

$b, c \in \mathcal{M}$ and $c \subseteq b$ in $\mathcal{M}$

that is $\frac{\varepsilon'_i}{\gamma_i} \ll \frac{\varepsilon_i}{\alpha_i} \quad i = 1, 2, \ldots, n \text{ in } H.$
If by applying \((0^\circ)^n\) one or more times to \(n(a)\) one gets a word
\[ a^1 = \frac{\varepsilon_{11}}{y_1} \cdot \frac{\varepsilon_{12}}{y_2} \cdots \frac{\varepsilon_{1m}}{y_m} \] such that
\[ \frac{\varepsilon_{1i}}{x_{y_1}} \ll \frac{\varepsilon_{i}}{x_{y_i}} \quad i = 1, 2, \ldots, m \text{ in } H, \]
then define \((a) \preceq (b)\).

Clearly "\(\preceq\)" is reflexive.

"\(\preceq\)" is transitive.

Let \((a), (b), (c)\) be three elements of \(P\)

where \(n(a) = \frac{\varepsilon_{11}}{x_1} \cdot \frac{\varepsilon_{12}}{x_2} \cdots \frac{\varepsilon_{1n}}{x_n}\)

\(n(b) = \frac{\varepsilon_{21}}{y_1} \cdot \frac{\varepsilon_{22}}{y_2} \cdots \frac{\varepsilon_{2m}}{y_m}\)

\(n(c) = \frac{\varepsilon_{31}}{z_1} \cdot \frac{\varepsilon_{32}}{z_2} \cdots \frac{\varepsilon_{3r}}{z_r}\).

Let \((a) \preceq (b)\) and \((b) \preceq (c)\).

The following cases will arise.

(i) \(|\{n(a)\}| = |\{n(b)\}| = |\{n(c)\}|\)
(ii) \( \mathcal{L}(b) \subseteq \mathcal{L}(c) \)

(iii) \( \mathcal{L}(a) \subseteq \mathcal{L}(b) = \mathcal{L}(c) \)

(iv) \( \mathcal{L}(a) \not\subseteq \mathcal{L}(b) \neq \mathcal{L}(c) \)

In case (i) it can be easily proved that "\( \subset \)" is transitive.

Let us prove the case (iv). Other cases will follow from this one.

Since (a) \( \subset (b) \) for \( n(b) \exists \) a word

\[
\begin{align*}
\mathcal{E}_1 & \quad \mathcal{E}_2 \\
1 & \quad 2
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_1(c_1) & \quad \mathcal{E}_1(c_2) \\
j_1 & \quad j_2
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_1(c_3) & \quad \mathcal{E}_1(c_4) \\
j_3 & \quad j_4
\end{align*}
\]
\[ x_{11} x_{12} \cdots x_{1m} \]

obtained from \( x_1, x_2, \ldots, x_n \) by inserting the words \( b_i^{-1} c_i \) \( i = 1, 2, \ldots, \ell + r \) where \( b_i^{-1}, c_i \in M \)

and \( c_i \subseteq b_i \) in \( M \) and dropping the words:

\[ \varepsilon_1(d_{i-t+1}) \quad \varepsilon_1(d_{i-t+2}) \quad \ldots \quad \varepsilon_1(d_{i+t}) \quad \varepsilon_1(d_{i+t+1}) \]

\[ \varepsilon_1(d_{i+t+2}) \quad \ldots \quad \varepsilon_1(d_{i+T}) \]

\[ \varepsilon_1(d_{i+t+T}) \] 

\( i = 1, 2, \ldots, \ell \) in \( n(a) \) where

\[ \varepsilon_1(d_{i-t+1}) \quad \varepsilon_1(d_{i-t+2}) \quad \ldots \quad \varepsilon_1(d_{i+t}) \quad \varepsilon_1(d_{i+t+1}) \]

\[ \varepsilon_1(d_{i+t+2}) \quad \ldots \quad \varepsilon_1(d_{i+T}) \]

\[ \varepsilon_1(d_{i+t+T}) \]

\[ \varepsilon_1(d_{i+t+T+1}) \]

\[ \varepsilon_1(d_{i+t+T+2}) \]

\[ \ldots \]

\[ \varepsilon_1(d_{i+T+1}) \]

\[ \varepsilon_1(d_{i+T+2}) \]

\[ \ldots \]

\[ \varepsilon_1(d_{i+T+T}) \]
\[ \varepsilon_i(x^{d_i - t_i + 1}) \subset \varepsilon_i(x^{d_i + t_i}) \subset \varepsilon_i(x^{d_i + t_i + 1}) \]

in \( M \) (we conveniently insert and drop words of the form \( b \cdot c^{-1} \)) in \( n(a) \) whenever it is necessary.) such that

\[ \varepsilon_i \preccurlyeq \varepsilon_j, \quad i = 1, 2, \ldots, m. \]

Similarly, since \( (b) \preccurlyeq (c) \) for \( n(c) \) there is a word

\[ \varepsilon_1 y_1, \quad \varepsilon_2 y_2, \quad \ldots, \quad \varepsilon_{2g_1} y_1, \quad \varepsilon_{2g_2} y_2, \quad \ldots, \quad \varepsilon_{2g_1} y_1, \quad \varepsilon_{2g_2} y_2, \quad \ldots, \quad \varepsilon_{2g_m} y_m \]

obtained from \( y_1^{s_1} g_1 \cdot y_2^{s_2} g_2 \cdot \ldots \cdot y_{2g_m}^{s_{2g_m}} g_{2g_m} \).
by applying \((0^1)\) several times such that

\[
\begin{align*}
\mathcal{E}_i &\ll \mathcal{E}_n^i, & i = 1, 2, \ldots, r
\end{align*}
\]

Thus for \(z = z^1, z^2, \ldots, z^r\) we get

\[
\begin{align*}
\mathcal{E}_1 \cdot \mathcal{E}_2 \\
\mathcal{E}_3 \\
\mathcal{E}_4 \\
\mathcal{E}_5
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}(h^{-r}) &\ll \mathcal{E}_n^{(h^{-r} + 1)} \\
\mathcal{E}_n^{(h^{-r} + 1)} &\ll \mathcal{E}_n^{(h^{-r} + 2)} \\
\mathcal{E}_n^{(h^{-r} + 2)} &\ll \mathcal{E}_n^{(h^{-r} + 3)} \\
\mathcal{E}_n^{(h^{-r} + 3)} &\ll \mathcal{E}_n^{(h^{-r} + 4)}
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_1 &\ll \mathcal{E}_2 \\
\mathcal{E}_3 &\ll \mathcal{E}_4 \\
\mathcal{E}_5 &\ll \mathcal{E}_6
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}(h^{-r}) &\ll \mathcal{E}_n^{(h^{-r} + 1)} \\
\mathcal{E}_n^{(h^{-r} + 1)} &\ll \mathcal{E}_n^{(h^{-r} + 2)} \\
\mathcal{E}_n^{(h^{-r} + 2)} &\ll \mathcal{E}_n^{(h^{-r} + 3)} \\
\mathcal{E}_n^{(h^{-r} + 3)} &\ll \mathcal{E}_n^{(h^{-r} + 4)}
\end{align*}
\]

obtained from \(x \cdot x \cdot x \cdot x \cdot x\) by applying \((0^1)\)

\[
\begin{align*}
\mathcal{E}_1 &\ll \mathcal{E}_2 \\
\mathcal{E}_3 &\ll \mathcal{E}_4 \\
\mathcal{E}_5 &\ll \mathcal{E}_6
\end{align*}
\]

such that \(x \cdot x \cdot x \cdot x \cdot x\), \(i = 1, 2, \ldots, r\).
(a) \not\leq (c)

:: "\not\leq" is transitive.

In P, let us define a relation R* by (a)R*(b) iff (a) \not\leq (b) and (b) \not\leq (a).

Then R* is an equivalence.

Also R* is a congruence.

Let (a), (b), (c), (d) \in P.

Then \( n(a) = \frac{\varepsilon_{11}}{1} \cdot \frac{\varepsilon_{12}}{2} \cdot \frac{\varepsilon_{1n}}{n} \)

\( n(b) = \frac{\varepsilon_{21}}{y} \cdot \frac{\varepsilon_{22}}{y} \cdot \frac{\varepsilon_{2m}}{m} \)

\( n(c) = \frac{\varepsilon_{31}}{p} \cdot \frac{\varepsilon_{32}}{p} \cdot \frac{\varepsilon_{3r}}{r} \)

\( n(d) = \frac{\varepsilon_{41}}{q} \cdot \frac{\varepsilon_{42}}{q} \cdot \frac{\varepsilon_{4s}}{s} \)

Let (a) R* (b) and (c) R* (d).
Now \( n(a) \circ n(c) = x_1 x_2 \quad \cdots \quad x_n p_1 \quad \cdots \quad p_p \)

\[ n(b) \circ n(d) = \frac{E_{21}}{y_1} \quad \frac{E_{22}}{y_2} \quad \cdots \quad \frac{E_{m}}{q_m} \quad \frac{E_{41}}{q_1} \quad \cdots \quad \frac{E_{4_s}}{q_s} \]

Suppose that \( x_{n-i+1}^* = p_i \) for all \( i, 1 \leq i \leq k \)

where \( k \) is subject to the condition \( 0 \leq k \leq \min(n,r) \)

but that \( x_{n-k} \neq p_{k+1} \), then

\[ n(a \circ c) = \frac{E_{11}}{x_1} \quad \frac{E_{12}}{x_2} \quad \cdots \quad \frac{E_{1(n-k)}}{x_{n-k}} \quad \frac{E_{3(k+1)}}{p_{k+1}} \quad \cdots \quad \frac{E_{3r}}{p_r} \]

Similarly let,

\[ n(b \circ d) = \frac{E_{21}}{y_1} \quad \frac{E_{22}}{y_2} \quad \cdots \quad \frac{E_{2(n-1)}}{q_{m-1}} \quad \frac{E_{4(l+1)}}{q_{l+1}} \quad \cdots \quad \frac{E_{4_s}}{q_s} \]

Since \( (a) \pitchfork (b) \) and \( (c) \pitchfork (d) \), we have

\( (a) \ll (b) \) and \( (b) \ll (a) \),

\( (c) \ll (d) \) and \( (d) \ll (c) \).
Let us prove our result for the case $|{n(a)}| \neq |{n(b)}| \neq |{n(c)}| \neq |{n(d)}|$. Other cases will follow from this.

From (a) $\not< (b)$ it follows that for $n(b) = \varepsilon_2^1 \varepsilon_2^2 \cdots \varepsilon_2^m$,

$$
\exists \text{ a word } x_1^1 \cdots x_1^{11}, x_1^{12}, \cdots, x_1^{1n} \text{ obtained from } x_1^{11} \cdots x_1^{1m}
$$

by applying the operation $(0_1)$ one or more times, such that $x_1^i \ll y_1^i$, $i = 1, 2, \ldots, m$ in $H$.

Similarly from (c) $\not< (d)$, it follows that for $n(d) = \varepsilon_3^1 \varepsilon_3^2 \cdots \varepsilon_3^s$,

$$
= q_1^1 \cdot q_1^2 \cdots \cdot q_1^s, \quad \exists \text{ a word } p_1^1 \cdot p_1^2 \cdots \cdot p_1^s
$$

obtained from $\varepsilon_3^1_1 \varepsilon_3^1_2 \cdots \cdot \varepsilon_3^1_r$ by applying the

operation $(0_1)$ several times such that $p_1^i \ll q_1^i$. \hfill
Thus for $n(bod) = \varepsilon_2 y \cdot y_2 \cdot \varepsilon_2 (m-1) \cdot q_{l+1}$,

\[
\frac{\varepsilon_4 S}{S} \supset \text{a word} \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_{m-1} \cdot q_{l+1} \quad \text{obtained from } n(aoc) \text{ by applying the operation } (0,1) \text{ such that}
\]

\[
\frac{x_i}{y_1} \ll \frac{\varepsilon_2 i}{y_i}, \quad i = 1, 2, \ldots, m-1
\]

and \[
\frac{\delta_j}{1_1} \ll \frac{\varepsilon_4 j}{q_{l+1}}, \quad j = l+1, \ldots, s.
\]

\[
\therefore (aoc) \ll (bod).
\]

Similarly from (b) \ll (a) and (d) \ll (c), it follows that

\[
(bod) \ll (aoc).
\]

\[
\therefore (aoc) R^* (bod).
\]
$R^*$ is a congruence.

Thus $P/R^*$ is a group.

The elements of $P/R^*$ will be denoted by $[(a)]$.

The binary operation in $P/R^*$ is defined by $[(a)] \times [(b)] = [(a \cdot b)]$.

$[(\emptyset)]$ is the identity of $P/R^*$.

Further the inverse of $[(a)]$ is $[(a^{-1})]$, where $(a^{-1})$ has the usual meaning in $N_R = P$.

In $P/R^*$, we define "\subseteq" by $[(a)] \subseteq [(b)]$ iff $\forall (a) \in [(a)]$ and $\forall (b) \in [(b)]$ we have $(a) \not< (b)$ in $P$.

Then with respect to "\subseteq", $P/R^*$ is a poset.

Further $[(a)] \subseteq [(b)]$ and $[(c)] \subseteq [(d)]$.

$\Rightarrow [(a)] \times [(c)] = [(a) \times (c)] = [(a \circ c)] \subseteq [(b \circ d)]$.
\[ \left[ (b) \times (d) \right] = \left[ (b) \right] \otimes \left[ (d) \right] \quad (\text{Since } R^* \text{ is a congruence}) \]

Consequently, \( P/R^* \) is a po group. We put \( P/R^* = Q \).

Before proving the remaining part of the theorem, let us prove the following lemma:

**Lemma 1**: If \( n(a) \) be an element of \( G \), then \( \left[ (a) \right] \) consists of \( (a) \) only.

**Proof**: We have \( (b) \in \left[ (a) \right] \)

\[ \iff (a) \preceq (b) \text{ and } (b) \preceq (a). \]

Since \( n(a) \in G \), \( n(a) \) is a reduced word of length 1, that is

\[ \ell \left\{ n(a) \right\} = 1 \text{ and let } n(a) = a_1. \]

Since \( (a) \preceq (b) \), \( n(b) \) must be a word of odd length.

First suppose \( \ell \left\{ n(b) \right\} = \ell \left\{ n(a) \right\} = 1 \) and \( n(b) = b_1. \)

\( (a) \preceq (b) \) and \( (b) \preceq (a) \iff a_1 \preceq b_1 \) and \( b_1 \preceq a_1. \)
in \( H \) \( \iff \) \( a_1 = b_1 \) (Since "\( \ll \)" is antisymmetric)

\( \iff \) \( (a) = (b) \).

Next let, \( \{ n(b) \} = 3 \),

and \( n(b) = y_1 \cdot y_2 \cdot y_3 \).

\( (a) \ll (b) \Rightarrow \exists \text{ a word } x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot x_3^{\varepsilon_3} \) obtained from \( a_1 \)

by applying \((0_1)\) such that

\[
\begin{align*}
\varepsilon_i' &\ll y_1^{\varepsilon_i}, & i &= 1, 2, 3 \text{ in } H.
\end{align*}
\]

Now \( x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot x_3^{\varepsilon_3} \) can be obtained from \( a_1 \) by the following way—

(1) Let \( a_1 \) be in the relation "\( \ll \)" with one of the component

of \( y_1^{\varepsilon_1} \cdot y_2^{\varepsilon_2} \cdot y_3^{\varepsilon_3} \), say \( a \ll y_1^{\varepsilon_1} \) in \( H \)

We might also

take \( a \ll y_3^{\varepsilon_3} \). The case \( a \ll y_2^{\varepsilon_2} \) will not arise. Let \( a \ll y_2^{\varepsilon_2} \).

Here first inserting the words \( x_1^{\varepsilon_1} \) and \( a \cdot x_3^{\varepsilon_3} \) where
Now let \( a_i \) is not "\( \ll \)" related with \( y_i \) (or any other component). We shall show that this case does not arise.

Because first inserting \( x_i \) \( x_1 \) and \( x_2 \) \( x_3 \) in \( a_i \), where
Now suppose (b) $\subset$ (a). Then proceeding as in the previous case, we get a word $y$ from $y_1 \cdot y_2 \cdot y_3$ such that $y \ll a$.

$y$ can be obtained from $y_1 \cdot y_2 \cdot y_3$ by the way given below —

(ii) Let $y_3 \ll a$, in this case putting $y_3 = y$ and dropping $y_1 \cdot y_2$ in $y_1 \cdot y_2 \cdot y_3$ (Here $y_1 \ll y_2$) we get $y$. ——— (3)

(In case of our starting with $a \ll y_3$ in (i), here we were to start with $y_1 \ll a$, because $y_1 \cdot y_2$ can not be dropped and...
Let $y_3^3$ is not "" related with $a$. As before we can show that this case does not arise.

Now from (2) and (3) it follows that

$$a \ll y_1^1 \ll y_2^2 \ll y_3^3 \ll a \quad \text{in } \mathbb{H} \Rightarrow y_1^1 = y_2^2 = y_3^3 = a$$

$$\Rightarrow y_1^1 \cdot y_2^2 \cdot y_3^3 = a \cdot a \cdot a \Rightarrow n(b) \text{ is not a reduced word.}$$

Contradiction.

Next let us assume that $n(b)$ is not a reduced word when

$$\ell \{n(b)\} = 2r + 1.$$ 

Next let $\ell \{n(b)\} = 2r + 3$

and $n(b) = y_1^1 \cdot y_2^2 \cdot \ldots \cdot y_{2r+3}^{2r+3}$

We want to show that $n(b)$ is not a reduced word.

Since (a) $\ll$ (b), for $y_1^1 \cdot y_2^2 \cdot \ldots \cdot y_{2r+3}^{2r+3}$
A word $a' = x_{11}^\epsilon x_{12}^\epsilon x_1 r_1 x_1 (r_1 + 1) \cdots x_1 (2r_1) x_2^\epsilon x_2 (r_2 + 1) \cdots x_2 (2r_2)$

Obtained from $a_1$ by applying $o_i$ in $n(a)$ where $l(a') = 2r + 3$

$-E_i(n_i - j) + E_i(n_i + j)$

and $x_i(r_i - j) \leq x_i(r_i + j + 1)$, $0 \leq j \leq r_i - 1$, $i = 1, 2$.

$\cdots (4)$

such that $E_{11} \leq y_1^o, E_{12} \leq y_2^o, \cdots a_i \leq y_i^o$.

$E(1+\epsilon)(2r_i+1) \leq x^{(l+\epsilon)}(2r_i+1) \\
x^{(l+\epsilon)}(2r_i+1) \leq y^{2r+3}$. \hspace{1cm} (5)

Hence $n(b)$ can be written in the form

$E_{11} E_{12} E_1(2r) E_1 E_1^H E_{1+2}^l E(1+\epsilon)(2r_i+1) \\
y_{11} y_{12} y_1(2r_1) y_1(r_1+1) y^{2r+1}$. \hspace{1cm} (6)
From (4) and (5) it follows that

\[ \frac{-E_\ell(r_\ell-j)}{y_\ell(r_\ell-j)} \ll \frac{+E_\ell(r_\ell+j+1)}{y_\ell(r_\ell+j+1)} \]

\[ 0 \ll j \ll r_\ell-1, \quad i = 1, 2, \ldots, \ell, \ell+2, \ldots, \ell+g. \] (6)

Again \((b) \ll (a) \implies \exists\) a word say \(y^r\) obtained from \(n(b)\) by

applying \((0_1)\) in \(n(b)\) such that \(y^r \ll a_i\).

Let the word \(b' = E_{ll} \cdot E_{l2} \cdot \ldots \cdot E_{(d-1)(2r_\ell-1)} \cdot E_{l2} \cdot \ldots \cdot E_{(2r_\ell-1)} \cdot E_{l1} \cdot \ldots \cdot E_{(2r_\ell+1)} \cdot y_{(l+2)1} \cdot \ldots \cdot y_{(l+g)(2r_\ell+1)}\)

where \(l(b) = 2r+1\) and \(b'\) is obtained from \(n(b)\) by omitting

the terms \(y_{l1}\) and \(y_{l(2r_\ell)}\) which do not coincide with \(y^r\).

Then for \(b' \exists\) a word \(a''\) obtained from \(a'\) by omitting the
corresponding terms \( \mathcal{E}_{l_1} \) and \( \mathcal{E}_{l_2} \) that is obtained from

\( a \) by applying \( 0 \) in \( n(a) \) such that \( a \subset b \) in \( M \).

Thus \( (a) \preceq (b) \) where \( n(b) = b \).

Again to obtain \( \mathcal{E}_{r_1} \) from \( n(b) \) by applying \( (0,1) \), suppose along

with other operations we have to drop \( \mathcal{E}_{l_1} \) and \( \mathcal{E}_{l_2} \),

where \( \mathcal{E}_{l_1} \ll b^{-1} \) and \( c \ll \mathcal{E}_{l_2} \).

From (6) it follows that \( \mathcal{E}_{l_2} \ll \mathcal{E}_{l_1} \).

Hence \( c \ll \mathcal{E}_{l_2} \ll \mathcal{E}_{l_1} \ll b^{-1} \Rightarrow c \ll b^{-1} \).

Thus \( \mathcal{E}_{r_1} \) can be obtained from \( b' \) by applying the same operations as above and dropping \( b.c \) (\( b.c \) can be dropped since \( c \ll b^{-1} \) in \( H \)).

\( \Rightarrow (b) \preceq (a) \Rightarrow (b) \in [a] \).

Since \( \ell(b') = 2r+1 \), it follows from our assumption that \( b' \) is
not a reduced word.

Hence \( n(b) = \frac{\ell(1)}{\ell(2n)} \) is also not a reduced word.

Hence by mathematical induction it follows that

\[
\ell_n(b) = 1 \text{ and in that case } (a) = (b).
\]

Hence \( n(a) \in G \Rightarrow [a] \) consists of \( (a) \) only.

Theorem contd.

Now we construct a specially derived po algebra with signature

\( \Omega \) over the po group \( Q \).

If \( \omega \in \Omega \) determining \( Q(\omega) \) in \( G \), then in \( Q \) we fix \( [Q(\omega)] \).

If \( \omega \in \Omega_n, n \geq 1 \), then

\[
\begin{bmatrix} (a_1) \end{bmatrix}, \begin{bmatrix} (a_2) \end{bmatrix}, \ldots, \begin{bmatrix} (a_n) \end{bmatrix}
\]

corresponds to

\[
\begin{bmatrix} (a_1) \end{bmatrix} \times \begin{bmatrix} (a_2) \end{bmatrix} \times \ldots \times \begin{bmatrix} (a_n) \end{bmatrix}
\]

that is

\[
\left\{ \begin{bmatrix} (a_1) \end{bmatrix}, \begin{bmatrix} (a_2) \end{bmatrix}, \ldots, \begin{bmatrix} (a_n) \end{bmatrix} \right\} \omega = \begin{bmatrix} (a_1) \end{bmatrix} \times \begin{bmatrix} (a_2) \end{bmatrix} \times \ldots \times \begin{bmatrix} (a_n) \end{bmatrix}
\]
where \( n(a_1), n(a_2), \ldots, n(a_n) \) are not all elements of \( G \) and

\[
\{ [a_1] \times [a_2] \times \cdots [a_n] \} \omega = \{ n(a_1)n(a_2) \cdots n(a_n) \omega \}
\]

where \( n(a_1), n(a_2), \ldots, n(a_n) \) are all elements of \( G \).

The operation thus defined is independent of the choice of representatives.

Further let

\[
[a_1] \sqsubseteq [b_1] \quad i = 1, 2, \ldots, n \text{ in } G,
\]

\( \omega \in \bigcap_n \, n \geq 2 \) then evidently

\[
[a_1] \times [a_2] \times \cdots [a_n] \sqsubseteq [b_1] \times [b_2] \times \cdots \times [b_n]
\]

\( \cdots \times [b_n] \) when \( n(a_1), n(a_2), \ldots, n(a_n), n(b_1) \)

\( \cdots \times n(b_n) \) are not all elements of \( G \) (Since \( R^+ \) is a

congruence), and
when \( n(a_1), n(a_2), \ldots, n(a_n), n(b_1), \ldots, n(b_n) \) are all elements of \( G \).

Hence

\[
\left\{ \left[ (a_1) \right], \left[ (a_2) \right], \ldots, \left[ (a_n) \right] \right\} \subseteq \left\{ \left[ (b_1) \right], \left[ (b_2) \right], \ldots, \left[ (b_n) \right] \right\}
\]

Thus we get a specially derived po algebra \( \mathcal{A} \) with signature \( \Omega \) over the po-group \( Q \).

Let us define a mapping \( f : G \rightarrow \mathcal{T} \) by the rule:

\[
a f = \left[ (a) \right], \quad \forall \quad a \in G.
\]

\( f \) is isotone.

\( a, b \in G \Rightarrow \) they are reduced words of length 1.

\[
\Rightarrow \quad n(a) = a \quad \text{and} \quad n(b) = b.
\]

\[
a \triangleleft b \iff n(a) \subseteq n(b) \iff (a) \triangleleft (b) \iff \left[ (a) \right] \subseteq \left[ (b) \right] \quad \text{(by lemma 1)}.
\]
Further $f$ is injective.

$$a \neq b \iff n(a) \neq n(b) \iff (a) \neq (b) \iff [(a)] \neq [(b)].$$

Now let $a_1, a_2, \ldots, a_n \in G$, $\omega \in \bigcap_{n \in \mathbb{N}} n$. Then

$$(a_1 a_2 \cdots a_n \omega)f = \left[(a_1 a_2 \cdots a_n \omega)\right]$$

$$= \left[(n(a_1) n(a_2) \cdots n(a_n) \omega)\right]$$

$$= \left\{\left[(a_1)\right] \left[(a_2)\right] \cdots \left[(a_n)\right]\right\}(\omega)$$

$$= (a_1 f a_2 f \cdots a_n f) \omega.$$ 

If $\omega_0 \in \bigcap_{n \in \mathbb{N}}$ then $0_{\omega_0} f = \left[(0_{\omega_0})\right]$.

Consequently $f$ is an $O$-monomorphism.