CHAPTER 4

MULTIVARIATE OPTIMUM ALLOCATION WITH PROBABILISTIC NON-LINEAR COST
CHAPTER-4
MULTIVARIATE OPTIMUM ALLOCATION WITH
PROBABILISTIC NON-LINEAR COST CONSTRAINT

4.1 INTRODUCTION

In multivariate surveys when the characteristics are highly correlated, the individual optimum allocations for different characteristics may differ relatively little. For such situations Cochran (1977) suggested the use of the character wise average of the individual optimum allocations as a usable compromise allocation.

From the time of Neyman (1934) many researchers have conjectured various criteria for obtaining a usable compromise allocation. A comprehensive but not exhaustive list of these researcher is provided in section 2.1 of Chapter 2 of this thesis. Generally, the optimum allocation in stratified sampling is worked out using either of the following two criteria depending on the situation. The cost of the survey is minimized for a desired precision of the estimate or the precision is maximized for a given budget of the survey. Kokan and Khan (1967) formulated the problem of determining the compromise allocation by minimizing the cost of the survey for desired precisions of the estimates of the population means as the following convex programming problem (CPP).
Minimize \[ C = \sum_{h=1}^{L} c_h n_h \]  
\[ \text{Subject to } \sum_{h=1}^{L} \frac{a_{jh}}{n_h} - \sum_{h=1}^{L} \frac{a_{jh}}{N_h} \leq k_j \]  
and \[ 2 \leq n_h \leq N_h, \; h = 1, \ldots, L \] \tag{4.1}

where \( a_{jh} = W_h^2 \sigma_{jh}^2 \); \( h = 1, 2, \ldots, L; \; j = 1, 2, \ldots, p \).

If the budget of the survey \( C \), is fixed in advance, then the individual allocation problem to minimize the variance for a fixed cost may be expressed as the following \( p \)-CPDs.

Minimize \[ V = \sum_{h=1}^{L} \frac{W_h^2 \sigma_{jh}^2}{n_h} - \sum_{h=1}^{L} \frac{W_h^2 \sigma_{jh}^2}{N_h} \]  
\[ \text{Subject to } \sum_{h=1}^{L} c_h n_h + c_0 \leq C \]  
and \[ 2 \leq n_h \leq N_h; \; h = 1, \ldots, L \] \tag{4.2}

Further, in a survey the costs \( c_h; \; h = 1, 2, \ldots, L \) of enumerating a characteristic in various strata random variables. In such cases the formulated allocation problem becomes a stochastic programming problem. Stochastic programming problem was first formulated by Dantzig (1955), who suggested a two stage programming technique for its solutions. Later, Charnes & Cooper (1959) developed the chance constrained programming
technique in which the chance constraints are converted into equivalent deterministic constraints.

When the constants \( c_h \) and \( a_{jh}, (h = 1, \ldots, L, j = 1, \ldots, p) \) are known, the solution to problem (4.1) is given by Kokan and Khan (1967) using results of \( n \) - dimensional geometry. Prekopa (1995) also developed a method from stochastic point of view. When the strata variances \( s_{jh}^2 \) are unknown and are estimated from a sample as \( s_{jh}^2 \), they also become random variables. Diaz-Garcia et al. (2007) discussed this situation. Javaid et al. (2009) considered the case of random costs in (4.1) and used modified E- model for solving it. Bakhshi et al. (2010) also computed the optimum allocation in Multivariate Stratified Sampling with a probabilistic cost constraint.

In this chapter a non-linear cost function with random coefficients is considered. The equivalent deterministic model for the problem is obtained by applying the chance constrained programming technique. The problem of finding the optimum compromise allocation is formulated as a Stochastic Convex Programming Problem (SCPP) and a method is developed to work out the compromise allocation in a multivariate stratified survey using the compromise criterion "Minimizing the sum of sampling variances of the estimators of the population means of various characteristics". A numerical
example is also worked out to illustrate the computational details of the method.


4.2 PROBLEM FORMULATION

Consider a multivariate population consisting of \( N \) units which is divided into \( L \) disjoint strata of sizes \( N_1, N_2, \ldots, N_L, N = \sum_{h=1}^{L} N_h \). Suppose that \( p \) characteristics are to be measured on each selected unit of the sample and the estimation of the \( p \)-overall population means \( \bar{Y}_j \); \( j = 1, \ldots, p \) is of interest. Assume that the number of strata and the strata boundaries are fixed in advance. Let \( n_h \) units be drawn without replacement from the \( h \)-th stratum. For \( j \)-th character, an unbiased estimate of the population mean \( \bar{Y}_j \) is

\[
\bar{y}_{jst} = \sum_{h=1}^{L} w_h \bar{y}_{jh}
\]  

(4.3)

With a sampling variance

\[
V(\bar{Y}_{jst}) = \sum_{h=1}^{L} \left( \frac{1}{n_h} - \frac{1}{N_h} \right) w_h^2 S_{jh}^2 \]  

(4.4)
The problem of determining the optimum sample allocations involves the
determination of sample sizes \(n_1, n_2, \ldots, n_L\) that minimize the variances of
various characters under the given sampling budget \(C\). Usually, in stratified
sampling a linear cost function given in chapter 1 expression (1.4), is used.
If travel costs between the units selected in the sample from a given stratum
are substantial, empirical and mathematical studies indicate that the travel
cost may be expressed by "\(\sum_{h=1}^{L} t_h \sqrt{n_h}\)" where \(t_h\) is the travel cost incurred
in approaching a sampled unit in the \(h\)-th stratum for measurement (see
Cochran (1977)). The total cost of the survey will then be

\[
C = c_0 + \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} \tag{4.5}
\]

where \(c_h; h = 1, 2, \ldots, L\) denote the per unit cost of measurement in the \(h^{th}\)
stratum and \(c_0\) is the overhead cost. The cost function (4.5) is nonlinear.

Thus the problem (4.2) with cost function (4.5) can be expressed as

\[
\begin{align*}
\text{Minimize} & \quad \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_h} - \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{N_h} \\
\text{Subject to} & \quad \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \leq C \quad \text{for } j = 1, 2, \ldots, p \\
\text{and} & \quad 2 \leq n_h \leq N_h; \quad h = 1, 2, \ldots, L
\end{align*} \tag{4.6}
\]
Ignoring the summation independent of \( n_h \) in the objective function, the allocation problems in (4.6) may be written as the following \( p \)-Nonlinear Programming Problems (NLPP):

\[
\begin{align*}
\text{Minimize} & \quad \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_h} \\
\text{Subject to} & \quad \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \leq C, \quad j = 1, 2, \ldots, p \\
\text{and} & \quad 2 \leq n_h \leq N_h; \quad h = 1, 2, \ldots, L
\end{align*}
\]

In many practical situations the measurement cost \( c_h \) and the travel cost \( t_h \) in the various strata are not fixed and may be considered as random variables.

In such a situation the NLPP (4.7) can be expressed as the following chance constrained NLPP:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_h} \\
\text{Subject to} & \quad P \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \leq C \right) \geq p_0, \quad j = 1, 2, \ldots, p \\
\text{and} & \quad 2 \leq n_h \leq N_h; \quad h = 1, 2, \ldots, L
\end{align*}
\]

where \( p_0, \ 0 \leq p_0 \leq 1 \) is a specified probability close to 1.
4.3 SOLUTION USING CHANCE CONSTRAINED PROGRAMMING

The costs \( c_h \) and \( t_h, \ h = 1, \ldots, L \) have been assumed to be independently and normally distributed random variables as \( N(\mu_{c_h}, \sigma_{c_h}^2) \) and \( N(\mu_{t_h}, \sigma_{t_h}^2) \) respectively.

The function \( \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right) \), as a linear combination of random variables, will also be normally distributed with mean

\[
E \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right)
\]

and variance

\[
V \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right).
\]

Now

\[
E \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right) = E \left( \sum_{h=1}^{L} c_h n_h \right) + E \left( \sum_{h=1}^{L} t_h \sqrt{n_h} \right) + c_0
\]

\[
= \sum_{h=1}^{L} n_h E(c_h) + \sum_{h=1}^{L} \sqrt{n_h} E(t_h) + c_0
\]

\[
= \sum_{h=1}^{L} n_h \mu_{c_h} + \sum_{h=1}^{L} \sqrt{n_h} \mu_{t_h} + c_0
\]

(4.9)
\[ V \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right) = V \left( \sum_{h=1}^{L} c_h n_h \right) + V \left( \sum_{h=1}^{L} t_h \sqrt{n_h} \right) \]
\[ = \sum_{h=1}^{L} n_h^2 V(c_h) + \sum_{h=1}^{L} n_h V(t_h) \]
\[ = \sum_{h=1}^{L} n_h^2 \sigma_{c_h}^2 + \sum_{h=1}^{L} n_h \sigma_{t_h}^2 \]  \hspace{1cm} (4.10)

Now let \( f(t) = \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \), then the chance constraint in (4.8) is given by
\[ P \left( f(t) \leq C \right) \geq p_0 \]

or
\[ P \left\{ \frac{f(t) - E(f(t))}{\sqrt{V(f(t))}} \leq \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right\} \geq p_0 \]

or
\[ P \left\{ Z \leq \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right\} \geq p_0 \]

where \( Z = \left[ \frac{f(t) - E(f(t))}{\sqrt{V(f(t))}} \right] \) is standard normal variate. Thus the probability of realizing \( \{ f(t) \} \) less than or equal to \( C \) can be expressed as
\[ P \left( f(t) \leq C \right) = \phi \left[ \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right], \]  \hspace{1cm} (4.11)
where \( \phi(z) \) represents the cumulative density function of the standard normal variable evaluated at \( z \). If \( K_\alpha \) represents the value of the standard normal variate at which \( \phi(K_\alpha) = \rho_0 \), then the constraint (4.11) can be expressed as

\[
\phi \left[ \frac{C - \mathcal{E}\{f(t)\}}{\sqrt{\mathcal{V}\{f(t)\}}} \right] \geq \phi(K_\alpha) \tag{4.12}
\]

The inequality will be satisfied if and only if

\[
\left[ \frac{C - \mathcal{E}\{f(t)\}}{\sqrt{\mathcal{V}\{f(t)\}}} \right] \geq K_\alpha
\]

or equivalently, \( \mathcal{E}\{f(t)\} + K_\alpha \sqrt{\mathcal{V}\{f(t)\}} \leq C \) \tag{4.13}

Substituting the values from (4.9) and (4.10) in (4.13), we get

\[
\left( \sum_{h=1}^{L} n_h \mu_{ch} + \sum_{h=1}^{L} \sqrt{n_h \mu_{th} + c_0} \right) + K_\alpha \sqrt{\sum_{h=1}^{L} n_h^2 \sigma_{ch}^2 + \sum_{h=1}^{L} n_h \sigma_{th}^2} \leq C \tag{4.14}
\]

The constants \( \mu_{ch}, \mu_{th}, \sigma_{ch}, \) and \( \sigma_{th} \) in (4.14) are unknown.

Now

\[
\mathcal{E}\left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right) = \sum_{h=1}^{L} n_h \bar{c}_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \tag{4.15}
\]

and
\[ \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 = \sum_{h=1}^{L} n_h^2 \hat{\sigma}_{c_h}^2 + \sum_{h=1}^{L} n_h \hat{\sigma}_{t_h}^2, \] respectively. (4.16)

where \( \bar{c}_h, \bar{t}_h, \hat{\sigma}_{c_h}^2 \) and \( \hat{\sigma}_{t_h}^2 \) are the estimated means and variances from the sample.

Thus an equivalent deterministic constraint is obtained as:

\[ \left( \sum_{h=1}^{L} \bar{c}_h n_h + \sum_{h=1}^{L} \bar{t}_h \sqrt{n_h} + c_0 \right) + K_\alpha \sqrt{\sum_{h=1}^{L} \hat{\sigma}_{c_h}^2 n_h^2 + \sum_{h=1}^{L} \hat{\sigma}_{t_h}^2 n_h \leq C } \] (4.17)

The problem of obtaining a compromise allocation may be given as a Multiobjective-NLPP. The ‘p’ objectives are to minimize the individual variances of the estimates of the population means of the \( p \)-characteristics simultaneously, subject to the cost constraint (4.17), that is,

\[
\begin{align*}
\text{Minimize} & \quad \begin{bmatrix} V(\bar{y}_{1st}) \\ \vdots \\ V(\bar{y}_{pst}) \end{bmatrix} \\
\text{Subject to} & \quad \sum_{h=1}^{L} \bar{c}_h n_h + \sum_{h=1}^{L} \bar{t}_h \sqrt{n_h} + c_0 + \\
& \quad K_\alpha \sqrt{\sum_{h=1}^{L} \hat{\sigma}_{c_h}^2 n_h^2 + \sum_{h=1}^{L} \hat{\sigma}_{t_h}^2 n_h \leq C } \\
\text{and} & \quad 2 \leq n_h \leq N_h; \ h=1,2,\ldots,L 
\end{align*}
\tag{4.18}
\]

Using a suitable compromise criterion the Multiobjective-NLPP (4.18) may be converted into a single objective NLPP.
A reasonable criterion given by Yates (1960) is to workout a compromise allocation that "Minimizes the sum of the variances $V_j$; $j = 1, 2, ..., p$".

Using this criterion we may express the multiobjective-NLPP (4.18) as a single objective NLPP as:

$$\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{p} V_j \\
\text{Subject to} & \quad \left( \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} \bar{t}_h \sqrt{n_h} + c_0 \right) + \\
& \quad K_\alpha \left( \sum_{h=1}^{L} \hat{\sigma}_{c_h}^2 n_h^2 + \sum_{h=1}^{L} \hat{\sigma}_{t_h}^2 n_h \right) \leq C \\
& \quad 2 \leq n_h \leq N_h; \quad h = 1, 2, ..., L
\end{align*}$$

(4.19)

Where $V_j = V(\bar{v}_{jst}); j = 1, 2, ..., p$.

When the numerical data regarding $\bar{c}_h$, $\bar{t}_h$, $\hat{\sigma}_{c_h}^2$ and $\hat{\sigma}_{t_h}^2$, are available the NLPP (4.19) may be solved by using a suitable nonlinear programming technique. However, we used the optimization software LINGO (2001) to obtain a solution to NLPP (4.19).

4.4 A NUMERICAL ILLUSTRATION

The values of $N_h$, $W_h$, $S_h$, expected measurement costs $E(c_h)$, and the expected travel costs within stratum $E(t_h)$ and their variances are given for four different characteristics in a population stratified in five strata. The
data are mainly from Chatterjee (1968). The values of strata sizes are added assuming the population size as 6000. The total budget of the survey is assumed to be 1500 units with an overhead cost \( c_0 = 300 \) units.

**Table 4.1**  
Values of \( N_h, W_h \) and \( S_{jh} \) for five strata and four characteristics

<table>
<thead>
<tr>
<th>( h )</th>
<th>( N_h )</th>
<th>( W_h )</th>
<th>( S_{1h} )</th>
<th>( S_{2h} )</th>
<th>( S_{3h} )</th>
<th>( S_{4h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1500</td>
<td>0.25</td>
<td>28</td>
<td>206</td>
<td>38</td>
<td>120</td>
</tr>
<tr>
<td>2</td>
<td>1920</td>
<td>0.32</td>
<td>24</td>
<td>133</td>
<td>26</td>
<td>184</td>
</tr>
<tr>
<td>3</td>
<td>1260</td>
<td>0.21</td>
<td>32</td>
<td>48</td>
<td>44</td>
<td>173</td>
</tr>
<tr>
<td>4</td>
<td>480</td>
<td>0.08</td>
<td>54</td>
<td>37</td>
<td>78</td>
<td>92</td>
</tr>
<tr>
<td>5</td>
<td>840</td>
<td>0.14</td>
<td>67</td>
<td>9</td>
<td>76</td>
<td>117</td>
</tr>
</tbody>
</table>

The measurement costs \( c_1, c_2, c_3, c_4, c_5 \) and the travel cost \( t_1, t_2, t_3, t_4, t_5 \) are assumed to be independently and normally distributed random variables with means and variances as

\[
E(c_1) = 1, \ E(c_2) = 1, \ E(c_3) = 1.5, \ E(c_4) = 1.5 \text{ and } E(c_5) = 2
\]

\[
E(t_1) = 0.5, \ E(t_2) = 0.5, \ E(t_3) = 1, \ E(t_4) = 1 \text{ and } E(t_5) = 1.5
\]

\[
V(c_1) = 0.25, \ V(c_2) = 0.25, \ V(c_3) = 0.35, \ V(c_4) = 0.35 \text{ and } V(c_5) = 0.45.
\]

\[
V(t_1) = 0.125, \ V(t_2) = 0.125, \ V(t_3) = 0.175, \ V(t_4) = 0.175 \text{ and } V(t_5) = 0.225.
\]

Using the given data values NLPP (4.19) takes the form:
**Multivariate Optimum Allocation...**

**Minimize** \[
\frac{1377.5081}{n_1} + \frac{2449.5358}{n_2} + \frac{740.9373}{n_3} + \frac{35.7884}{n_4} + \frac{56.2973}{n_5}
\]

**Subject to** \[
\begin{align*}
& 1n_1 + 1.5n_2 + 1.5n_3 + 1.5n_4 + 2n_5 + \\
& (0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} + 300) \\
& + 2.33 \left[ (0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) + \\
& (0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5) \right]^{1/2} \leq 1500
\end{align*}
\]  \( (4.20) \)

\[2 \leq n_1 \leq 1500 \]
\[2 \leq n_2 \leq 1920 \]
\[2 \leq n_3 \leq 1260 \]
\[2 \leq n_4 \leq 480 \]
\[2 \leq n_5 \leq 840 \]

The optimum compromise allocation which is the solution to the NLPP (4.20) given by the optimization software LINGO is

\[n_{1c}^* = 194.9769, \quad n_{2c}^* = 228.4192, \quad n_{3c}^* = 113.5010, \quad n_{4c}^* = 34.5660 \quad \text{and} \quad n_{5c}^* = 57.7153\]

With the corresponding value of the objective function as 67.92505.

Where ‘c’ stands for ‘compromise’.

The rounded off values of the optimum compromise allocation is

\[n_{1c}^* = 195, \quad n_{2c}^* = 228, \quad n_{3c}^* = 114, \quad n_{4c}^* = 35, \quad n_{5c}^* = 58\]

with total sample size

\[n_c = \sum_{h=1}^{5} n_{hc} = 630.\]

Under the compromise allocation the individual variances (ignoring fpc) for the four characteristics are given by

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\[ V_j(c) = \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_{hc}}, \quad j = 1, 2, \ldots, 4 \]  

(4.21)

This gives

\[ V_{1(c)} = 2.956286, \quad V_{2(c)} = 22.714802, \quad V_{3(c)} = 4.579747, \quad V_{4(c)} = 37.572325 \]

with the trace value \( T(c) = \sum_{j=1}^{4} V_j(c) = 67.82316, \)

where 'c' stands for the proposed compromise allocation.

4.5 DISCUSSION

In multivariate stratified surveys when the use of individual optimum allocations is not possible, we need to work out an allocation that is optimum for all characteristics in some sense, using a compromise criterion. In this chapter a study is carried out to obtain an optimum compromise allocation with a chance cost constraint involving random parameters with known means and variances.

In the following the relative efficiencies of the proposed compromise allocation as compared to the individual optimum allocations are studied.

If \( \text{Trace}(\mathbf{n}) \) denotes the trace of the variance-covariance matrix of \( \bar{Y}_{jst} \) corresponding to allocation \( \mathbf{n} \), the relative efficiency of an allocation \( \mathbf{n} \) with respect to another allocation \( \mathbf{n'} \), may be defined as the ratio,

\[ \frac{\text{Trace}(\mathbf{n'})}{\text{Trace}(\mathbf{n})} \]  

(see Sukhatme et al. (1984)). For the sake of
comparison of the proposed allocation with proportional allocation and Cochran’s average compromise allocation (Cochran (1977)) in the following, these allocations and the corresponding traces of the variance-covariance matrices are worked out.

The rounded off proportional allocation for a total sample of size of 630 is given as:

\[ n_1(p) = 158, \quad n_2(p) = 202, \quad n_3(p) = 132, \quad n_4(p) = 50 \quad \text{and} \quad n_5(p) = 88. \]

The variance for the four characteristics are:

\[ V_1(p) = 2.317299, \quad V_2(p) = 26.716507, \quad V_3(p) = 3.625913 \quad \text{and} \quad V_4(p) = 36.990162 \]

with a trace value \( T(p) = \sum_{j=1}^{4} V_j(p) = 69.649881 \),

where 'p' stands for proportional allocation.

The Cochran’s average allocation is given by the character-wise average of the individual optimum allocations that is

\[ n_{h(a)} = \frac{1}{p} \sum_{j=1}^{p} n_{j|h} \quad \text{(4.22)} \]

where \( n_{h(a)}; h=1,2,\ldots,L \) denote the Cochran’s average allocation and \( n_{j|h}; h=1,2,\ldots,L; j=1,2,\ldots,p \) denote the optimum allocation for the \( j^{th} \) characteristic that are the solution to the NLPP.
Minimize \[ V_j = \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_h} \]

Subject to \[ \left( \sum_{h=1}^{L} \hat{\sigma}_h n_h + \sum_{h=1}^{L} t_h \sqrt{n_h} + c_0 \right) + \left( \sum_{h=1}^{L} \hat{\sigma}_{c_h}^2 n_h^2 + \sum_{h=1}^{L} \hat{\sigma}_{t_h}^2 n_h \right) \leq C \]

\[ j = 1, 2, ..., 4. \] (4.23)

and \[ 2 \leq n_h \leq N_h; \ h = 1, 2, ..., L \]

Let \( \mathbf{n}_j = (n_{j1}^*, n_{j2}^*, ..., n_{jL}^*) \) denote the solution to the \( j \)-th NLPP in (4.23) and \( V_j^* \) be the corresponding optimum value of the objective function given by

\[ V_j^* = \sum_{h=1}^{L} \frac{W_h^2 S_{jh}^2}{n_{j,h}^*}; \ j = 1, 2, ..., 4 \] (4.24)

For the given data:
For \( j = 1 \)

Minimize \( \frac{49}{n_1} + \frac{58.9824}{n_2} + \frac{45.1584}{n_3} + \frac{18.6624}{n_4} + \frac{87.9844}{n_5} \)

Subject to \( (1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5) + \)

\[ 0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} + 300 + \]

\[ 2.33\{(0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) \}

\(+ (0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5)\}^{1/2} \leq 1500 \]

\[ 2 \leq n_1 \leq 1500 \]

\[ 2 \leq n_2 \leq 1920 \]

\[ 2 \leq n_3 \leq 1260 \]

\[ 2 \leq n_4 \leq 480 \]

and

\[ 2 \leq n_5 \leq 840 \]

(4.25)

The optimum allocation \( n_1^* = (n_{11}^*, n_{12}^*, n_{13}^*, n_{14}^*, n_{15}^*) \) is

\[ n_{11}^* = 132.999, \ n_{12}^* = 143.2324, \ n_{13}^* = 107.7228, \ n_{14}^* = 72.3840, \ n_{15}^* = 127.6964. \]

The corresponding value of the variance ignoring fpc is \( V_1^* = 2.148212 \).
For \( j = 2 \)

\[
\begin{align*}
\text{Minimize} & \quad \frac{2652.25}{n_1} + \frac{1811.3536}{n_2} + \frac{101.6064}{n_3} + \frac{8.7616}{n_4} + \frac{1.5876}{n_5} \\
\text{Subject to} & \quad (1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5 + \\
& \quad 0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} + 300) + \\
& \quad 2.33((0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) \\
& \quad + (0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5))^{1/2} \leq 1500 \\
& \quad 2 \leq n_1 \leq 1500 \\
& \quad 2 \leq n_2 \leq 1920 \\
& \quad 2 \leq n_3 \leq 1260 \\
& \quad 2 \leq n_4 \leq 480 \\
& \quad 2 \leq n_5 \leq 840 \\
\text{and} & \quad 2 \leq n_5 \leq 840
\end{align*}
\]

(4.26)

The optimum allocation \( n_2^* = (n_{21}^*, n_{22}^*, n_{23}^*, n_{24}^*, n_{25}^*) \) is

\[
 n_{21}^* = 303.1810, \; n_{22}^* = 259.2840, \; n_{23}^* = 60.5848, \; n_{24}^* = 18.3975, \; n_{25}^* = 6.6782.
\]

The corresponding value of the variance ignoring fpc is \( V_2^* = 18.12507 \).
For \( j = 3 \)

\[
\text{Minimize } \frac{90.25}{n_1} + \frac{69.2224}{n_2} + \frac{85.3776}{n_3} + \frac{38.9376}{n_4} + \frac{113.2096}{n_5}
\]

\[
\text{Subject to } (1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5 +
\]
\[
0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} + 300) +
\]
\[
2.33((0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2)
\]
\[
+(0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5))^{1/2} \leq 1500
\]
\[
2 \leq n_1 \leq 1500
\]
\[
2 \leq n_2 \leq 1920
\]
\[
2 \leq n_3 \leq 1260
\]
\[
2 \leq n_4 \leq 480
\]
\[
2 \leq n_5 \leq 840
\]

The optimum allocation \( n_3^* = (n_{31}^*, n_{32}^*, n_{33}^*, n_{34}^*, n_{35}^*) \) is

\[
n_{31}^* = 142.0023, \; n_{32}^* = 126.7286, \; n_{33}^* = 117.2123, \; n_{34}^* = 82.6231, \; n_{35}^* = 117.3308
\]

The corresponding value of the variance ignoring fpc is \( V_3^* = 3.346324 \).
For \( j = 4 \)

Minimize \[
\frac{900}{n_1} + \frac{3466.8544}{n_2} + \frac{1319.8689}{n_3} + \frac{54.1696}{n_4} + \frac{268.3044}{n_5}
\]

Subject to \[
(1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5 + \\
0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} + 300) + \\
2.33\{(0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) \\
+ (0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5)^{1/2} \leq 1500
\]

\[
2 \leq n_1 \leq 1500
\]

\[
2 \leq n_2 \leq 1920
\]

\[
2 \leq n_3 \leq 1260
\]

\[
2 \leq n_4 \leq 480
\]

(4.28)

and

\[
2 \leq n_5 \leq 840
\]

The optimum allocation \( \mathbf{n}^* = (n_{41}^*, n_{42}^*, n_{43}^*, n_{44}^*, n_{45}^*) \) is

\[
n_{41}^* = 139.7336, \ n_{42}^* = 246.2649, \ n_{43}^* = 139.3793, \ n_{44}^* = 31.8239, \ n_{45}^* = 59.5315.
\]

The corresponding value of the variance ignoring fpc is \( V_4^* = 36.19729 \).

These results (after rounding off) are summarized in Table 4.2 with a trace

value \( T_{(a)} = \sum_{j=1}^{4} V_j(a) = 71.535548 \).
### Table 4.2
Individual Optimum Allocations and the Average Allocation with Variances

<table>
<thead>
<tr>
<th>Characteristic $j$</th>
<th>Individual Optimum Allocations</th>
<th>$V_j^*$</th>
<th>$V_{j(a)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j = 1$</td>
<td>$n^*_1$</td>
<td>$n^*_2$</td>
<td>$n^*_3$</td>
</tr>
<tr>
<td>$j = 2$</td>
<td>303</td>
<td>259</td>
<td>61</td>
</tr>
<tr>
<td>$j = 3$</td>
<td>142</td>
<td>127</td>
<td>117</td>
</tr>
<tr>
<td>$j = 4$</td>
<td>140</td>
<td>246</td>
<td>139</td>
</tr>
<tr>
<td>Cochran's Average Allocation (rounded off)</td>
<td>179</td>
<td>194</td>
<td>106</td>
</tr>
</tbody>
</table>

### 4.6 CONCLUSION:

The results of the study are summarized in the Table 4.3

### Table 4.3
Summary of the results

<table>
<thead>
<tr>
<th>Allocations</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$n_5$</th>
<th>Trace</th>
<th>R.E.w.r.t. Proportional Allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportional</td>
<td>158</td>
<td>202</td>
<td>132</td>
<td>50</td>
<td>88</td>
<td>69.649881</td>
<td>1.000000</td>
</tr>
<tr>
<td>Cochran's Average</td>
<td>179</td>
<td>194</td>
<td>106</td>
<td>51</td>
<td>78</td>
<td>71.535548</td>
<td>0.973640</td>
</tr>
<tr>
<td>Proposed</td>
<td>195</td>
<td>228</td>
<td>114</td>
<td>35</td>
<td>58</td>
<td>67.82316</td>
<td>1.026933</td>
</tr>
</tbody>
</table>
The last column of Table 4.3 gives the relative efficiencies of the Cochran's Average allocation and the proposed allocation as compared to the proportional allocation. It is evident that the proposed allocation is the most efficient allocation among the discussed allocations under the given circumstances.