CHAPTER 2

FAST-SCALE INSTABILITY IN POWER ELECTRONIC CONVERTERS

2.1 INTRODUCTION

In this chapter, the fast-scale instability in a ZAD controlled DC-DC Čuk converter is investigated. Effects of varying the control parameters on the qualitative behaviour of the system are studied in detail. To reduce the complexity in deriving the map dynamics and computing the ZAD control parameters, a reduced order model is derived using moment matching technique. The dynamics of this converter system is mathematically described and analysed with a simple discrete map. Computer simulations as well as experimental investigations are performed to study the qualitative behaviour of the system for variations of different parameters. It is found that for even small control parameter variations, the system exhibits period-doubling bifurcation.

2.2 ČUK CONVERTER MODELING AND ORDER REDUCTION

The Čuk converter which is a widely used derived DC-DC converter, provides an output voltage lesser or greater than the input voltage depending on the duty cycle, but with polarity opposite to that of the input voltage. Figure 2.1 shows the schematic of a ZAD controlled Čuk converter. The average state model of the converter is derived by considering the
equivalent circuits during the switch conduction subinterval and the diode conduction subinterval.

Figure 2.1 Schematic diagram of a ZAD controlled Ćuk converter

The system dynamics is given as:

\[ \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}, \]  
(2.1)

where \( \mathbf{x} = \begin{bmatrix} v_{c1} & v_{c2} & i_{L1} & i_{L2} \end{bmatrix}^T \),

\[
\mathbf{A} = \begin{bmatrix} 0 & 0 & \frac{1-D}{C_1} & \frac{-D}{C_1} \\
0 & -\frac{1}{RC_2} & 0 & \frac{1}{C_2} \\
-\frac{(1-D)}{L_1} & 0 & 0 & 0 \\
\frac{D}{L_2} & -\frac{1}{L_2} & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\
0 \\
\frac{E}{L_1} \\
0 \end{bmatrix}.
\]
where $v_{C1}$ and $v_{C2}$ are the voltages across the transfer and output capacitances respectively. $i_{L1}$ is the current through the input inductance $L_1$, $i_{L2}$ is the current through the output inductance $L_2$ and $D$ is the steady-state duty cycle. For algebraic brevity, applying the following non-dimensionalisation,

$$
x_1 = \frac{v_{C1}}{E}, \quad x_2 = \frac{v_{C2}}{E}, \quad x_3 = \frac{1}{E} \sqrt{\frac{L}{C}i_{L1}},
$$

$$
x_4 = \frac{1}{E} \sqrt{\frac{L}{C}i_{L2}}, \quad \gamma = \frac{1}{R} \sqrt{\frac{L}{C}}, \quad T = \frac{T_s}{\sqrt{LC}},
$$

the averaged system matrices in the dimensionless form are given as follows:

$$
A = \begin{bmatrix}
0 & 0 & 1-D & -D \\
0 & -\gamma & 0 & 1 \\
-(1-D) & 0 & 0 & 0 \\
D & -1 & 0 & 0 
\end{bmatrix}
\quad \begin{bmatrix}
0 \\
0 \\
0 \\
1 
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
0 \\
0 \\
0 
\end{bmatrix}, \quad (2.2)
$$

where $\gamma$ is a variable used for non-dimensionalisation, $T_s$ is the switching time period and $T$ is the switching time period in dimensionless form.

To reduce the complexity in the controller design, a reduced order model of the Čuk converter derived using moment matching technique is used wherein the two dominant poles of the system (i.e. the input inductor current $i_{L1}$ and the output capacitor voltage $v_{C2}$) are retained and the effects of the transfer capacitor $C_1$ and the output inductor $L_2$ are neglected. Hence it becomes sufficient to regulate these two variables.
The DC-DC Čuk converter operating in CCM is designed with the specifications $E = (5 - 10) \, \text{V}$, $V_o = 7.5 \, \text{V}$ and $I_o = (0-1) \, \text{A}$, $T_s = 400 \, \mu\text{s}$, $\Delta I_{L1} = 4.8 \%$ of the source current, $\Delta I_{L2} = 9.6 \%$ of the source current, $\Delta V_{C1} = 4.2 \%$ of the output voltage, $\Delta V_{C2} = 0.00125 \%$ of the output voltage and the component values are calculated as $L_1=L_2=L=5 \, \text{mH}$, $C_1=C_2=C=1.2 \, \mu\text{F}$. The reduced order state space matrices of the Čuk converter with the designed parameters are given as:

$$
A = \begin{bmatrix} 0 & 1 \\ 0.1947 & 0.75 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

(2.3)

### 2.3 ZAD CONTROL SCHEME

This section describes the principle and design of ZAD strategy in detail, which involves the direct calculation of duty cycle and the derivation of time constants associated with the error dynamics.

#### 2.3.1 Derivation of Time Constants $G_1$ and $G_2$

This subsection discusses a simple reduced order sliding surface design approach for a DC-DC Čuk converter, wherein a new and systematic technique for the selection of sliding surface coefficients to implement ZAD is attempted. In order to meet the objective, the switching surface $s(x(t))$, is defined as a linear combination of the errors of the dominant state variables of the system given as:

$$
s(x(t)) = G_1(x_2 - x_{2ref}) + G_2(x_3 - x_{3ref}).
$$

(2.4)

This ensures that the output capacitor voltage $x_2$ follows the reference $x_{2ref}$, where $x_{2ref} = v_{c2ref}/E$ and the input inductor current $x_3$ also
tracks the set value $x_{3\text{ref}}$, where $x_{3\text{ref}} = \frac{i_{x\text{ref}}}{E} \sqrt{\frac{L}{C}}$. Here $G_1$ and $G_2$ are the time constants associated with the first-order error dynamics on the surface $s(x(t))$.

Let $r$ be the vector with reference dynamic variables, $x$ be the vector with actual state variables and $e$ be the error vector given as:

$$
\begin{align*}
  r &= \begin{bmatrix} x_{2\text{ref}} & x_{3\text{ref}} \end{bmatrix}^T, \\
  x &= \begin{bmatrix} x_2 & x_3 \end{bmatrix}^T, \\
  e &= \begin{bmatrix} e_2 & e_3 \end{bmatrix}^T, \\
\end{align*}
$$

(2.5)

where $e_2 = x_{2\text{ref}} - x_2$ and $e_3 = x_{3\text{ref}} - x_3$.

To derive the sliding surface $s(x(t))$, the phase-variable canonical form representation of the system as given in Equation (2.3) is considered. The sliding surface $s(x(t))$ defined as a linear function of the coordinates of tracking error vector in Equation (2.4) is also expressed as:

$$
  s(x(t)) = G.e, \quad \text{where } G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} G_1, G_2 > 0.
$$

The objective of the tracking error problem is to keep the error vector $e$ on the surface $s(x(t)) = 0$, which implies that $e$ converges exponentially to zero and is given by:

$$
  s(x(t)) = G.e = 0 \quad \text{and} \quad s(x(t)) = \Delta s(x(t)) = G.\dot{e} = 0.
$$

(2.6)

On the sliding surface, a second order system is reduced to first order system with stable linear differential equation. Also the system dynamics on the sliding surface is determined only by the coefficient vector $G$. Hence, the control is insensitive to parameter variations. To determine the control law, the error state equation using the accessible states is derived as follows:

$$
  \dot{e} = \ddot{r} - \ddot{x} = \ddot{r} - Ar + Ae - bu.
$$

(2.7)
The Filippov’s average equivalent switch control $u_{eq}$ that guarantees $\dot{s}(x(t)) = 0$ is obtained as:

$$\dot{s}(x(t)) = G \dot{e} = G(\dot{r} - Ar + Ae - bu_{eq}) = 0, \quad (2.8)$$

$$u_{eq} = (Gb)^{-1} G[\dot{r} - Ar + Ae]. \quad (2.9)$$

Substitution of this control law in Equation (2.7), gives the error dynamics as:

$$\dot{e} = [I - b(Gb)^{-1} G] \dot{r} - Ar + Ae. \quad (2.10)$$

By applying the invariance condition, $\dot{r} - Ar = 0$, the above equation is modified as:

$$\dot{e} = [I - b(Gb)^{-1} G] Ae = A_{eq} e. \quad (2.11)$$

If $(Gb)^{-1}$ exists, the vector $G$ is derived by selecting the eigenvalues of $A_{eq}$ such that it guarantees the asymptotic convergence of error to zero at the desired rate. The matrix $A_{eq}$ is chosen as:

$$A_{eq} = \begin{bmatrix} -1 & 0 \\ 0 & -2.4272 \end{bmatrix}.$$ 

The vector $G$ is then obtained using Equation (2.11) as:

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.206 \end{bmatrix}. \quad (2.12)$$

### 2.3.2 Principle of ZAD Controller

The ZAD strategy involves controlling the system by achieving the zero average of the switching surface $s(x(t))$, in every switching cycle $T$. Of
the various possible PWM implementations, such as leading, trailing and centered pulse, the ZAD control is implemented in a single, updated centered PWM. The switching pulse $u$ shown in Figure 2.2 is mathematically defined as:

$$
u = \begin{cases} 1, & nT \leq t \leq nT + \frac{d_n}{2}, \\ 0, & 0, nT + \frac{d_n}{2} < t < nT + (T - \frac{d_n}{2}), \\ 1, & nT + (T - \frac{d_n}{2}) \leq t \leq nT + T, \end{cases} \quad (2.13)$$

where $d_n$ refers to the time for which the switch remains ON in the $n^{th}$ period. So the state $x(nT)$ at $t = nT$ is the sampling state and the states $x(nT + \frac{d_n}{2})$ and $x((n+1)T - \frac{d_n}{2})$ are the switching states.

Figure 2.2 Scheme of control variable ‘$u$’

Being a linear, time invariant system under unitary pulses, the more general solution of the states is obtained through direct integration and yields:

$$x((n+1)T) = e^{\lambda T} x(nT) + \left( e^{\lambda (T-d_n/2)} + 1 \right) A^{-1} \left( e^{\lambda d_n/2} - I \right) b$$

$$- e^{\lambda d_n/2} A^{-1} \left( e^{\lambda (T-d_n)} - I \right) b,$$

(2.14)
where $d_n$ is calculated from the following equation.

$$
\int_{nT}^{(n+1)T} s(x(t))dt = 0.
$$

(2.15)

As the ZAD control algorithm imposes the control variable $u$ to force a zero average dynamics of the switching surface at each sampling period, calculating the duty cycle involves the online solution of the transcendental Equation (2.15), which is unviable from a practical point of view. Therefore, the practical implementation of the ZAD control strategy is achieved by considering a piecewise-linear approximation of the error surface which can be used to avoid the exact solution of Equation (2.15). Thus, the surface $s(x(t))$ is divided into three intervals: $s_1$ between 0 and $d_n/2$, $s_2$ between $d_n/2$ and $(T - d_n/2)$, and $s_1$ again between $(T - d_n/2)$ and $T$ over a switching period. With this approach, the general expression for the surface integral in each sampling period is given as:

$$
\int_{nT}^{(n+1)T} s(x(t))dt = \int_{nT}^{nT + d_n/2} \left( s(x(nT)) + t \dot{s}_1(x(nT)) \right)dt
$$

$$
+ \int_{nT + d_n/2}^{(n+1)T - d_n/2} \left( s(x(nT)) + \frac{d_n}{2} \dot{s}_1(x(nT)) + \left( t - \frac{d_n}{2} \right) \dot{s}_2(x(nT)) \right)dt
$$

$$
+ \int_{(n+1)T - d_n/2}^{(n+1)T} \left( s(x(nT)) + \frac{d_n}{2} \dot{s}_1(x(nT)) + \left( T - \frac{d_n}{2} \right) \dot{s}_2(x(nT)) \right)dt
$$

$$
+ \left( t - T + \frac{d_n}{2} \right) \dot{s}_1(x(nT)) \right)dt
$$

(2.16)

where $s(x(nT))$ corresponds to the value of the switching surface at the sampling instant, $\dot{s}_1(x(nT))$ and $\dot{s}_2(x(nT))$ are the slopes when $u = 1$ and $u = 0$ respectively. Now on solving Equation (2.16), the duty cycle is computed in terms of the system states as $d_n/T$ where
From Equation (2.17), it is concluded that the duty cycle corresponding to the ZAD control algorithm can be determined provided \( s(x(t)) \) at instant \( nT \), \( \dot{s}(x(t)) \) when \( u = 1 \) and \( u = 0 \) as well as switching time period \( T \) are known. Using the reduced order model derived in Section 2.2 and upon substitution of Equation (2.4) in Equation (2.17), the duty cycle \( d_n/T \) for the ZAD controller is derived after some algebra as:

\[
d_n = \frac{2s(x(nT)) + T\dot{s}(x(nT))}{\dot{s}(x(nT)) - \dot{s}(x(nT))} \tag{2.17}
\]

\[
d_n = \frac{(2G_x - \frac{2TG_z}{4-\gamma^2})x_2(0) + (2G_x + T(G_i - \frac{G_z\gamma}{4-\gamma^2})x_3(0) - 2G_x x_{2ref} - 2G_z x_{3ref} + T G_z}{G_i ((l - \frac{2}{4-\gamma^2})x_2(0) - (1 - \frac{\gamma}{4-\gamma^2})x_3(0))} \tag{2.18}
\]

Here, \( x_2(0) \) and \( x_3(0) \) are the initial state values at the beginning of each cycle. Since \( d_n \) should take values between \([0, T]\), if \( d_n \geq T \), it is taken as \( d_n = T \), while if \( d_n \leq 0 \), then \( d_n \) is taken as 0. The block diagram of the closed-loop ZAD controlled Ćuk converter is illustrated in Figure 2.3.
2.4  **BIFURCATION ANALYSIS OF ZAD CONTROLLED ČUK CONVERTER**

In this section, the qualitative behaviour (the behaviour of the system as it goes from a stable region to an unstable region for variations in system parameters) is investigated mathematically and also by means of numerical simulations and experimental measurements.

2.4.1  **Mathematical Bifurcation Analysis**

The Jacobian plays an important role in the study of dynamical systems. The essence of using a Jacobian in the analysis of dynamical systems lies in capturing the dynamics in a small neighborhood of an equilibrium point or orbit (stable or unstable). This conventional method is used here to examine the bifurcation phenomena. In this section, the discrete-time model is first derived for the ZAD controlled Čuk converter and then the dynamical behaviour is investigated by tracking the movements of the characteristic multipliers evaluated at the appropriate operating states. To avoid the complexity in designing the ZAD control scheme, a second order model is used. However, the mathematical bifurcation analysis is performed with the fourth order model of the converter.

Using the duty cycle Equation (2.18) and by solving the state model of Čuk converter derived in Equation (2.1), the Poincaré map $P_T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is obtained as:
where $A_{\text{on}}$ and $A_{\text{off}}$ are the system matrices corresponding to the switch and diode conduction subinterval respectively given as:

$$
A_{\text{on}} = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & -\gamma & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0
\end{bmatrix}, \\
A_{\text{off}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & -\gamma & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}.
$$

Since the system must be in a balance, it must satisfy Equation (2.18) and the states are calculated by solving Equation (2.19). The Jacobian of the system and the eigenvalues are calculated using a numerical algorithm for practical values of $G_2$. As the value of $G_2$ increases, a characteristic multiplier grows more negatively and leaves the unit circle through -1, while the other multipliers remain within the unit circle indicating a period-doubling bifurcation. Figure 2.4 shows the loci of the characteristic multiplier of the system that leaves the unit circle against the dynamics of the error constant.
Figure 2.4 Loci of the characteristic multiplier for variations in error dynamics time constant ‘$G_2$’ of a ZAD controlled Ćuk converter

2.4.2 Stability Analysis by Computer Simulation

A MATLAB simulation of the ZAD controlled Ćuk converter has been carried out prior to the experimental verification and the Poincaré map given in Equation (2.19) is used to update the four state variables of the Ćuk converter for every switching cycle. The value of gain $G_1$ is fixed at 0.5 as derived in Equation (2.12) and the system behaviour is studied for variation in gain $G_2$. It is observed that for $G_2 = 0.206$, the output capacitor voltage shows the expected period-1 behaviour as seen from Figure 2.5(a). When $G_2$ is slightly increased to 0.2097, the system undergoes a bifurcation and exhibits period-2 behaviour as shown in Figure 2.5(b). Upon further increasing $G_2$ to 0.216, the waveform becomes apparently aperiodic indicating the onset of chaos as can be seen from Figure 2.5(c).
Figure 2.5  Simulated waveforms of a ZAD controlled Čuk converter showing (a) period-1 operation for $G_2 = 0.206$; (b) period-2 operation for $G_2 = 0.2097$; (c) chaotic operation for $G_2 = 0.216$.

The phase portraits which are very useful in uncovering subtle periodicity are also plotted between the capacitor voltage and the inductor current as shown in Figures 2.6(a) and 2.6(b). Figure 2.6(a) clearly shows the double-path which indicates the period-2 operation of the system and Figure 2.6(b) shows the chaotic behavior which is characterized by an aperiodic and apparently random trajectory. Figure 2.7 shows the bifurcation diagrams for variation in gain $G_2$. 
2.4.3 Stability Analysis by Experimental Studies

An experimental setup with the same design parameter values used in the simulation has been built to validate the numerical results. The constructed prototype is controlled using DS1104 R&D controller board of dSPACE that can be plugged into a PCI slot of PC. This system allows developing digital controllers using the Rapid Control Prototyping (RCP) tool and is specifically designed for the development of high speed multivariable
digital controllers and real time simulations. It is a complete real time control system based on a 603 PowerPC floating point processor running at 250 MHz. Figures 2.8(a) and 2.8(b) show the basic schematic and photograph of the experimental setup based on a digital PWM, built for validating the simulation studies.

![Figure 2.8](image)

**Figure 2.8** (a) Schematic diagram depicts the hardware implementation of a ZAD controlled Ćuk converter; (b) snapshot of the experimental setup

The input and output inductors are of ferrite core type and the capacitors are of plain polyester type. Power MOSFET IRF540N is used as a switch and IN5408 is used as a diode. The output capacitor voltage and the input inductor current are sensed and given to the Analog to Digital Converter (ADC) channels of a Digital Signal Processor (DSP) controller, where the ZAD control algorithm is implemented. The bipolar 12 bit ADC unit in the dSPACE accepts a maximum of ±10V. For compatibility, the sensed signals
are scaled down to less than ±10V using voltage divider circuits. The ADC converts these signals into digital samples, which are then used by the controller to provide the necessary switching signals to the driver circuit. The controller provides the necessary switching signals to the driver circuit. The driver circuit uses IR2110 driver IC with the necessary circuit, to provide isolation and amplification, to subsequently drive the switch S.

The value of gain $G_2$ is varied with the gain $G_1$ fixed at 0.6 and the corresponding waveforms of the output capacitor voltage and the input inductor current are shown in Figures 2.9(a) – 2.9(c). When $G_1 = 0.6$ and $G_2 = 0.3$, it is observed from Figures 2.9(a(i)) and 2.9(a(ii)) (correspond to simulated waveform shown in Figure 2.5(a)) that the system possesses period-1 behavior. It is also inferred that during the normal period-1 operation, the system settles to the reference value of output voltage. When $G_2$ is reduced to 0.29, it can be seen from Figures 2.9(b(i)) and 2.9(b(ii)) (correspond to simulated waveform shown in Figure 2.5(b)) that the system possesses period-2 behaviour and the output voltage has decreased, implying that the system has lost its regulation property as it enters the period-2 region of operation. On further reducing $G_2$, the system enters into chaotic behaviour as can be seen from Figures 2.9(c(i)) and 2.9(c(ii)) (correspond to simulated waveform shown in Figure 2.5(c)).
Figure 2.9 Experimental waveforms of a ZAD controlled Ćuk converter showing (a) stable period-1 operation; (b) period-2 operation; (c) chaotic operation
2.5 CONCLUSION

In this chapter, the existence of fast-scale instability in a DC-DC Čuk converter under ZAD control, as the control parameters are varied has been investigated. The ZAD control has the advantages of fixed frequency implementation and robust performance and is worth studying to a greater depth. It has been revealed that as the time constant associated with the first-order error dynamics of the sliding surface is varied, the normal stable periodic behaviour undergoes a period-doubling bifurcation and enters the chaotic regime. The results obtained through numerical simulation and mathematical analyses were validated with the experimental results. Numerical analysis has been done by examining the movement of the complex eigenvalues of Jacobian matrix at the equilibrium point, as some chosen parameters are changed. The same simulated dynamics also has been verified using a suitable experimental setup based on DSP.