Chapter I
Introduction

Logical reasoning occupies highly significant position in Indian philosophy. It is also evident in western philosophy particularly Greek philosophy. All schools of Indian philosophy have shown great concern about logical reasoning as an integral part of philosophical discourse.

The word ‘Logic’ comes from classical Greek language ‘logos’ originally meaning ‘to say’ something significant. Logos developed a wide variety of senses, including ‘description’, ‘theory’ (sometimes as opposed to ‘fact’), ‘explanation’, ‘reason’, ‘reasoning power’, ‘principle’, ‘ratio’, ‘prose’ or ‘the science of thinking about or explaining the reason for something using formal method’ is most often said to be the study of criteria for the evaluation of arguments, although the exact definition of logic is a matter of controversy among philosophers. However the subject is grounded, the task of the logician is the same: to advance an account of valid and fallacious inference to allow one to distinguish good from bad arguments.

Logic is the science of reasoning. The aim of logic is to provide methods, techniques and devices which help in differentiating right reasoning from wrong and good reasoning from bad. But it does not mean that those who study logic can reason correctly. However, it is true that those who study logic certainly make less error while arguments. Knowledge of logic help one to face a problem in more

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orderly and systematic way, and in most of the cases makes the solution less difficult and more specific.

Logic is commonly thought to deal with the way people use their reasoning faculty. Indeed some familiar logical principles have traditionally been called “law of thought” in this connection; it is important to guard against confusion normally occurring to people. Logic does not provide an accurate factual description of human thought process, nor does it attempt to do so, for people are often found with their thinking patterns confused, fallacious, and inconsistent—the very opposite of logical; if we seek laws to describe processing of such patterns we have to take resort to psychology, not logic.

It has sometimes been suggested that since logic does not describe the way people actually think, then perhaps logic provides rules of correct or reasoned thinking—rules prescribing the ways we ought to think. This view also seems untenable, for the imposition of strict logical rules to guide thinking patterns because it would very likely lead to intellectual sterility rather than a higher degree of reasoning, since thinking is a natural process while logic tries to clarify or justify them.

But the logicians are not interested merely with the study of methods or techniques of differentiating right reasoning from wrong; it is equally important for them to acquire skill to apply those methods in illustrating the nature of truth experienced through reasoning and discourse as well as the value of the method adopted as correct and/or incorrect. How efficiently or how skillfully one makes use of these methods in practical life is nothing but demonstrating the technical aptitude in an artificial but artistic way.
All arts are concerned with ‘doing’, and ‘practicing’. Also in Buddhism, Buddhist teaching is full of artistic expressions, can be understood and tested with methodology of logical reasoning by anyone. Buddhism teaches that the solutions to our problems are within us not outside. The Buddha asked all his followers not to take his word as true, but rather to test the teachings for themselves. This way, each person decides for him or herself and takes responsibility for their own actions. Then, they will be able to understand the ultimate Truth, and also their identity with Nāma, Rūpa, Karma, etc. ‘Practice makes a man perfect’ is true for all arts, and it is equally true for logic as well.

Philosophical investigations in Buddhism are in practice for time unknown. It is necessary to establish that the philosophical reasoning in Buddhism can be logically construed and be shown as universal principle.

It is also accepted in Buddhism that reasoning is necessary for propagation of religious teachings at a certain level, but it is not acceptable that every reasoning must be reliable and trustworthy. Therefore, the Buddha used logical reasoning in order to negate old beliefs of the contemporary cult leaders and showed them the path to ‘Truth’ and ‘Knowledge’.

The formal study of Western logic was pioneered by the ancient Greek philosophers, discussions of some elements of logic and a focus on methods of inference can be traced back to the late 5th century BCE. The Sophists, and later Plato (early 4th c.) displayed an interest in sentence analysis, truth, and fallacies, and Eubulides of Miletus (mid-4th c.) is on record as the inventor of both the Liar and the Sorites paradox. But logic as a fully systematic discipline begins by
Aristotle, who systematized much of the logical inquiry of his predecessors. His main achievements were his theory of the logical interrelation of affirmative and negative existential and universal statements and, based on this theory, his syllogistic, which can be interpreted as a system of deductive inference. Aristotle's logic is known as term-logic, since it is concerned with the logical relations between terms, such as 'human being', 'animal', 'white'. It shares elements with both set theory and predicate logic. Aristotle's successors in his school, the Peripatos, notably Theophrastus and Eudemus, widened the scope of deductive inference and improved some aspects of Aristotle's logic.3

In the Hellenistic period, and apparently independent of Aristotle's achievements, the logicians Diodorus Cronus and his pupil Philo worked out the beginnings of a logic that took propositions, rather than terms, as its basic elements. They influenced the second major theorist of logic in antiquity, the Stoic Chrysippus (mid-3rd c.), whose main achievement is the development of a propositional logic, crowned by a deductive system. Considered by many in antiquity as the greatest logician, he was innovative in a large number of topics that are central to contemporary formal and philosophical logic. The many close similarities between Chrysippus' philosophical logic and that of Gottlob Frege are especially striking. Chrysippus' Stoic successors systematized his logic, and made some additions.

The development of logic from c. 100 BCE to c. 250 CE is mostly in the dark, but there can be no doubt that logic was one of the

topics regularly studied and researched. At some point Peripatetics and Stoics began taking notice of the logical systems of each other, and we witness some conflation of both terminologies and theories. Aristotelian syllogistic became known as ‘categorical syllogistic’ and the Peripatetic adaptation of Stoic syllogistic as ‘hypothetical syllogistic’. In the 2nd century CE, Galen attempted to synthesize the two traditions; he also professed to have introduced a third kind of syllogism, the ‘relational syllogism’, which apparently was meant to help formalize mathematical reasoning. The attempt of some Middle Platonists (1st c. BCE–2nd c. CE) to claim a specifically Platonic logic failed, and in its stead, the Neo-Platonists (3rd–6th c. CE) adopted a scholasticized version of Aristotelian logic as their own. In the monumental—if rarely creative—volumes of the Greek commentators on Aristotle’s logical works we find elements of Stoic and later Peripatetic logic. For this chapter, we will see the development of methodology and logical reasoning of the well-known philosopher will be told here under heading as: Ancient Logic, Indian Logic, Medieval (European) Logic, Precursors of Modern Logic and Modern Logic to the Present.4

Ancient Logic

Zeno of Elea (c. 490-430 B.C.)5

Zeno was born in the Greek colony of Elea in southern Italy around 490 B.C. He was a pupil and friend of the philosopher Parmenides and studied with him in Elea. The Eleatic School, one of


the leading pre-Socratic schools of Greek philosophy, had been founded by Parmenides in Elea in southern Italy. His philosophy of monism claimed that the many things which appear to exist are merely a single eternal reality which he called Being. His principle was that "all is one" and that change or non-Being are impossible. Certainly Zeno was greatly influenced by the arguments of Parmenides and Plato tells us that the two philosophers visited Athens together in around 450 BC.

Zeno had already written a work on philosophy before his visit to Athens and Plato reports that Zeno's book meant that he had achieved certain fame in Athens before his visit there. Unfortunately no work by Zeno has survived, but there is very little evidence to suggest that he wrote more than one book. The book Zeno wrote before his visit to Athens was his famous work which, according to Proclus, contained forty paradoxes concerning the continuum. Four of the paradoxes, which we shall discuss in detail below, were to have a profound influence on the development of mathematics.

‘Dialectic’, the technique of arguing for or against a position by careful logical reasoning and in particular the technique of arguing against a view by showing that it entails unacceptable consequences was a crucial innovation, which has governed philosophical methodology ever since. In the absence of such a methodology one can only defend a position by mystical revelation say, or by rhetorical rather than rational appeal, or by force perhaps. Zeno was celebrated for his paradoxes until Aristotle called him the “founder of dialectic”. (In philosophy of least, for such an approach has been a part of
mathematics for even longer). Later philosophers however, especially Plato and Aristotle were far finer exponents of the approach.\(^6\)

The four most famous paradoxes are the *Dichotomy*, the *Achilles*, the *Arrow*, and the *Stadium*.\(^7\)

1. The *Dichotomy*:

   Suppose Homer wants to catch a stationary bus. Before he can get there, he must get halfway there. Before he can get halfway there, he must get a quarter of the way there. Before traveling a fourth, he must travel one-eighth; before an eighth, one-sixteenth; and so on.

   \[
   H - \frac{B}{8} - \frac{B}{4} - \frac{B}{2} - \cdots
   \]

   The resulting sequence can be represented as:

   \[
   \left\{ \ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1 \right\}
   \]

   This description requires one to complete an infinite number of tasks, which Zeno maintains is impossibility.

   This sequence also presents a second problem in that it contains no first distance to run, for any possible (finite) first distance could be divided in half, and hence would not be first after all. Hence, the trip cannot even begin. The paradoxical conclusion then would be that travel over any finite distance can neither be completed nor begun, and so all motion must be an illusion.

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This argument is called the *Dichotomy* because it involves repeatedly splitting a distance into two parts. It contains some of the same elements as the *Achilles and the Tortoise* paradox, but with a more apparent conclusion of motionlessness. It is also known as the **Race Course** paradox. Some, like Aristotle, regard the Dichotomy as really just another version of *Achilles and the Tortoise*. However, they emphasize different points. In the *Achilles and the Tortoise*, the focus is that movement by multiple objects is just an illusion whereas in the *Dichotomy* the focus is that movement is actually impossible.

2. The *Achilles*: The running Achilles can never catch a crawling tortoise ahead of him because he must first reach where the tortoise started. However, when he reaches there, the tortoise has moved ahead, and Achilles must now run to the new position, which by the time he reaches the tortoise has moved ahead, etc. Hence the tortoise will always be ahead.

3. The *Arrow*: Time is made up of instants, which are the smallest measure and indivisible. An arrow is either in motion or at rest. An arrow cannot move, because for motion to occur, the arrow would have to be in one position at the start of an instant and at another at the end of the instant. However, this means that the instant is divisible which is impossible because by definition, instants are indivisible. Hence, the arrow is always at rest.

4. The *Stadium*: Half the time is equal to twice the time.

<table>
<thead>
<tr>
<th>First Position</th>
<th>Second Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>O O O (A)</td>
<td>O O O (A)</td>
</tr>
<tr>
<td>O O O (E)</td>
<td>O O O (B)</td>
</tr>
<tr>
<td>O O O (C)</td>
<td>O O O (C)</td>
</tr>
</tbody>
</table>

They start at the first position. Row A stays stationary while rows B & C move at equal speeds in opposite directions. When
they have reached the second position, each B has passed twice as many C's as A's. Thus it takes row B twice as long to pass row A as it does to pass row C. However, the time for rows B & C to reach the position of row A is the same. So half the time is equal to twice the time.

Though all four arguments seem illogical, not to mention confusing, they are not that simple to explain away and lead to some very serious problems for mathematics. To the Greek mathematicians, who had no real concept of convergence or infinity, these reasoning were incomprehensible. Aristotle discarded them as "fallacies" without really showing why and Zeno's paradoxes were hidden away in the mathematical closet for the next 2500 years. For that time, they were reduced mainly as novelties of philosophy. However, they were revived mathematically in the twentieth century by the efforts of people like Bertrand Russell and Lewis Carroll. Today, armed with the tools of converging series and Cantor's theories on infinite sets, these paradoxes can be explained to some satisfaction. However, even today the debate continues on the validity of both the paradoxes and the rationalizations.

**Socrates (470 BC-399 BC)**

Socrates is the ancient Greek thinker who laid the early foundations for Western philosophical thought. His "Socratic Method" involved asking probing questions in a give-and-take which would eventually lead to the truth. Socrates was born in Athens and fought as a foot soldier in the Peloponnesian War with Sparta, but in

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later years became a devotee of philosophy and argument. He spent years in the public places of Athens, engaging his fellow citizens in philosophical discussions and urging them to greater self-analysis. Socrates's iconoclastic attitude didn't sit well with everyone, and at age 70 he was charged with heresy and corruption of local youth. Convicted, he carried out the death sentence by drinking hemlock, becoming one of history's earliest martyrs of conscience. Socrates's most famous pupil was Plato, who in turn instructed the philosopher Aristotle.

Socrates's contributions to philosophy were a new method of approaching knowledge, a conception of the soul as the seat both of normal waking consciousness and of moral character, and a sense of the universe as purposively mind-ordered. His method, called dialectic, consisted in examining statements by pursuing their implications, on the assumption that if a statement were true it could not lead to false consequences. The method may have been suggested by Zeno of Elea, but Socrates refined it and applied it to ethical problems.

In Plato's dialogues and other Socratic dialogues, Socrates attempts to examine first principles or premises by which we all reason and argue. Socrates typically argues by cross-examining someone's claims and premises in order to draw out a contradiction or inconsistency among them. For example, in the Euthyphro, Socrates asks Euthyphro to provide a definition of piety. Euthyphro replies that the pious is that which is loved by the gods. But, Socrates also has Euthyphro agreeing that the gods are quarrelsome and their quarrels, like human quarrels, concern objects of love or hatred.
Therefore, Socrates reasons, at least one thing exists which certain
gods love but other gods hate. Again, Euthyphro agrees. Socrates
concludes that if Euthyphro's definition of piety is acceptable, then
there must exist at least one thing which is both pious and impious (as
it is both loved and hated by the gods) — which, Euthyphro admits, is
absurd. Thus, Euthyphro is brought to a realization by this dialectical
method that his definition of piety is not sufficiently elaborate, thus
wrong. This is to shown that Socrates use dialectical methodology for
debatation.

**Plato (428-347 B.C.)**

The Logical Principles

1. The principle of identity: thinking must agree with itself, all our convictions must agree with one another.
2. The principle of contradiction: contradictory determinations cannot suit at the same time one and the same thing.
3. The principle of sufficient reason: only that knowledge is scientific in character whose grounds are known.
4. The relations between general ideas, to use modern terms, of inclusion, of appurtenance, of reunion, of equivalence, etc., were quite familiar to him, but he applied them, implicitly, without formulating them as explicit relations, such as emerges – for instance – from Protagoras.
5. Plato states that every sentence expresses the relation between two concepts, the predicate and the subject, and that our thinking consists in asserting of denying this relation, as

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demonstrated in Theaetetus, or The Sophist which represents, in fact, the beginning of a theory of judgment.

**Aristotle (384-322 B.C.)**

Greek philosopher and scientist whose thought determined the course of Western intellectual history for two millennia. He was the son of the court physician to Amyntas III, grandfather of Alexander the Great. In 367 he became a student at the Academy of Plato in Athens; he remained there for 20 years. After Plato's death in 347 BC, he returned to Macedonia, where he became tutor to the young Alexander. In 335 he founded his own school in Athens, the Lyceum. His intellectual range was vast, covering most of the sciences and many of the arts. He invented the study of formal logic, devising for it a finished system, known as syllogistic, that was considered the sum of the discipline until the 19th century; his work in zoology, both observational and theoretical, also was not surpassed until the 19th century.

Aristotle theorized philosophy was the foundation of the ability to understand the basic axioms that comprise knowledge. For study and question completely, Aristotle viewed logic as the basic means of reasoning. To think logically, one had to apply the syllogism, which was a form of thought comprised of two premises that led to a conclusion; Aristotle taught that this form can be applied to all logical reasoning.

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Aristotle's methodology and logical reasoning, especially his theory of the syllogism, has had an unparalleled influence on the history of Western thought. It did not always hold this position: in the Hellenistic period, Stoic logic, and in particular the work of Chrysippus, was much more celebrated. However, in later antiquity, following the work of Aristotelian Commentators, Aristotle's logic became dominant, and Aristotelian logic was what was transmitted to the Arabic and the Latin medieval traditions, while the works of Chrysippus have not survived.

This article is written from the latter perspective. As such, it is about Aristotle's logic, which is not always the same thing as what has been called "Aristotelian" logic. The ancient commentators grouped together several of Aristotle's treatises under the title Organon12 ("Instrument") and regarded them as comprising his logical works:

1. *Categories*
2. *On Interpretation*
3. *Prior Analytics*
4. *Posterior Analytics*
5. *Topics*
6. *On Sophistical Refutations*

The Categories: is text from Aristotle’s Organon that enumerates all the possible kinds of thing which can be the subject or the predicate of a proposition. The ten categories, or classes, are

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- Substance: As mentioned above the notion of "substance" is defined as that which can be said to be predicated of nothing nor be said to be within anything. Hence, "this particular man" or "that particular tree" are substances. Later in the text, Aristotle calls these particulars "primary substances," to distinguish them from "secondary substances," which are universals. Hence, "Socrates" is a primary Substance, while "man" is a secondary substance. ("a human being," "a horse")

- quantity This is the spatial extension of an object. All medieval discussions about the nature of the continuum, of the infinite and the infinitely divisible, are a long footnote to this text. It is of great importance in the development of mathematical ideas in the medieval and late scholastic period. ("two cubic meters," "100 kilometers")

- quality This is a determination which characterizes the nature of an object. ("red," "taught," "bold," "dry")

- relation This is the way in which one object may be related to another. ("triple," "one-third")

- place Position in relation to the surrounding environment. ("on the acropolis," "in the Pireaus")

- Time: Position in relation to the course of events. ("is lying down," "is standing upright")

- position The examples Aristotle gives indicate that he meant a condition of rest resulting from an action: 'Lying', 'sitting'. Thus position may be taken as the end point for the corresponding action. The term is, however, frequently taken to mean the relative position of the parts of an object (usually a living object), given that the position of the parts is inseparable from the state of rest implied.
• State: The examples Aristotle gives indicate that he meant a condition of rest resulting from an affection (i.e. being acted on): ‘shod’, ‘armed’. The term is, however, frequently taken to mean the determination arising from the physical accoutrements of an object: one's shoes, one's arms, etc. ("is clothed," "is armed")

• Action: The production of change in some other object. ("burns," "cuts")

• Affection: The reception of change from some other object. It is also known as passivity. It is clear from the examples Aristotle gave for action and for affection that action is to affection as the active voice is to the passive. Thus for action he gave the example, ‘to lance’, ‘to cauterize’; for affection, ‘to be lanced’, ‘to be cauterized.’ The term is frequently misinterpreted to mean a kind of emotion or passion. being acted upon ("is burned," "is cut")

• logical properties of words in each category (substance, for example, has not opposite; most qualities admit of variation in degree, etc.)

• the chief meanings of certain crucial terms, like "opposite," "prior," and "motion"

  o On Interpretation

Aristotle's account of simple sentences (what later come to be called "propositions." Deals with quantifiers ("all," "some," "none") and their logical relations.

  o Prior Analytics

Aristotle's analysis of the simplest form of argument: the three-term Syllogism. The standard example in philosophy has always been:
• All men are mortal. [Premise 1 in the form: All B's are C's.]
• Socrates is a man. [Premise 2 in the form: (All) A is B.]
• Therefore, Socrates is mortal. [Conclusion in the form: All A's are C's.]

This example is somewhat misleading, despite the fact that it is the standard one, since it treats a proper name ("Socrates") as a term (or class name.) One of the fundamental departures of modern (19th & 20th Century C.E.) symbolic logic is that it treats sentences about individuals differently from the way it treats sentences about classes. But with this first figure form of the syllogism Aristotle arrives at a clear and explicit distinction between truth and validity, where the latter is a property of argument forms. (If the premises of a valid argument are true, the conclusion must be true.)

- Posterior Analytics

Here Aristotle identifies the valid forms of the syllogism. He identifies the formal key to valid syllogistic forms in the middle term (identified in the form above by "B.") The middle term must be "distributed" (quantified) if an argument form is to be valid. (Of course this is a necessary but not sufficient condition. Not every argument form with a distributed middle term is valid.) For a syllogism to achieve the status of a demonstration the argument form must be valid and the premises must be true, and must be known to be true unconditionally. The premises must, therefore, either be themselves derivable as conclusions of other demonstrations following necessarily from necessarily true premises or they must be known by "intuition"
The *Topics* provide a manual for participants in the contests of dialectical argument as instituted in the Academy by Plato. Books 2–7 provide general procedures or rules (*topoi*) about how to find an argument to establish or refute a given thesis. The descriptions of these procedures—some of which are so general that they resemble logical laws—clearly presuppose a notion of logical form, and Aristotle's *Topics* may thus count as the earliest surviving logical treatise.

**On Sophistical Refutations**

Deals with a variety of bad or invalid argument forms: "fallacies". The *Sophistical Refutations* are the first systematic classification of fallacies, sorted by what logical flaw each type manifests (e.g. equivocation, begging the question, affirming the consequent, *secundum quid*) and how to expose them.

**Variables.** One of Aristotle’s most revolutionary contributions to logic was his introduction of variables into logical discourse. Variables enabled Aristotle of express logical principles directly instead of describing them metalogically or illustrating them by means of standard examples. He used these indirect methods in his earlier treatises (including the *De Interpretatione*), but it is in the Prior Analytics that the new method becomes prominent. For instance, one of the laws of conversion is first described metalogically: “It is necessary that

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the assertoric universal negative proposition should convert in respect of its terms.” This law is then illustrated with the aid of standard example: “If no pleasure is good, then no good will be pleasure” finally it id expressed directly: “If A belongs to none of the B’s, then B belongs to none of the A’s” On other occasions, Aristotle used concrete inferences or inference schemata as a means of referring to a logic principle. Thus, the law of conversion could have been exemplified by an inference “No pleasure is good; therefore, no good is pleasure,” or represented by an inference schema, “A belongs to no B: therefore, B belongs to no A,” It is a moot point whether or not Aristotle appreciated the difference between “if-then” propositions, implications, and inferences or inference schemata. In fact, in his exposition of syllogistic (in the Prior Analytics) did formulate the syllogistic laws directly, in preference to representing them by means of inference schemata. Strangely enough, Aristotle made no attempt to explain his use of variables. This may have merely because he was accustomed to using them in his lectures in the Lyceum. Possibly they were used at an early stage simply for the purpose of abbreviating concrete terms in the standard examples that he employed to illustrate logical principles.

**Affirmation and denial.**

The various affirmations and denials we can make by using the copula with singular and common nouns, which latter include adjectives, fall into classes or types of propositions:

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(1) Singular, e.g., “Socrates is white,” “Socrates is not white.”

(2) Universal, e.g., “Every man is white,” “No man is white.”

(3) Particular, e.g., “Some man is white,” “Some man is not white.”

(4) Indefinite, e.g., “Man is white,” “Man is not white.”

However, for the purposes of syllogistic only universal and particular propositions are required. These propositions are called categorical. And Aristotle’s logic they exhibit a slightly different from if variables occur in them. If, for instance, we let “B” and “A” stand for “man” and “white” the four types of categorical propositions read: “A belongs to all B,” “A belongs to some B,” A belongs to no B,” A belongs to some B,” and “A does not belong to some B.” Variants such as “A is predicated of all B” and “A is predicated of some B” are also common in Aristotle. In all these examples A is of course, the predicate, and B is the subject.

**Syllogism**\(^{15}\). Aristotle’s definition of a syllogism as a “propositional expression in which, certain things having been laid down, something other than what has been laid down follows of necessity from their being so “fails to distinguish from other types of logical principles. An Aristotelian syllogism is, in fact, (1) an “if – then” proposition-i.e., an implication, of the form “If a and B then y, Where the Greek letters stand for categorical propositions with variable terms – (2) that is true for all values of the variables involved. (3) with

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“premises” (represented by \( a \) and \( B \)) with have at least one term in common, while the remaining two terms are the ones occurring in the “conclusion” (represented by \( y \)). In an implication satisfying these results of substituting concrete nouns for variables, equiform nouns for equiform variables, is also a syllogism. Thus, the Aristotelian version of what is known in traditional logic as syllogism in Barbara has the from “If \( A \) is predicated of all \( B \) and \( B \) is predicated of all \( C \), then is Predicated of all \( C \),” The syllogism in Darii reads “If \( A \) belong to all \( B \) and \( B \) belongs to al \( B \) and \( B \) belongs to some \( C \)’ then \( A \) belong to some \( C \).”

\[
\begin{align*}
\text{All } B \text{ is } A; \\
\text{(1).} \quad \text{All } C \text{ is } B; \\
\text{Therefore, } C \text{ is } A \end{align*}
\]

\[
\begin{align*}
\text{All } B \text{ is } A; \\
\text{(2).} \quad \text{Some } C \text{ is } B; \\
\text{Therefore, some } C \text{ is } A
\end{align*}
\]

**The early Peripatetics: Theophrastus and Eudemus**16

Aristotle’s pupil and successor Theophrastus of Eresus (c. 371–c. 287 BCE) wrote more logical treatises than his teacher, with a large overlap in topics. Eudemus of Rhodes (later 4th cent. BCE) wrote books entitled *Categories*, *Analytics* and *On Speech*. Of all these works only a number of fragments and later testimonies survive, mostly in Aristotle commentators. Theophrastus and Eudemus

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simplified some aspects of Aristotle's logic, and developed others where Aristotle left us only hints.

**Improvements and Modifications of Aristotle's Logic**

The two Peripatetics try to redefined Aristotle's first figure, so that it includes every syllogism in which the middle term is subject of one premise and predicate of the other. In this way, five types of non-modal syllogisms only intimated by Aristotle later in his *Prior Analytics* (Baralipton, Celantes, Dabitis, Fapesmo and Frisesomorum) are included, but Aristotle's criterion that first figure syllogisms are evident is given up (Theophrastus fr. 91, Fortenbaugh). Theophrastus replaced Aristotle's two-sided contingency by one-sided possibility, so that possibility no longer entails non-necessity. Both recognized that the problematic universal negative (‘A possibly holds of no B’) is simply convertible (Theophrastus fr. 102A Fortenbaugh). Moreover, they introduced the principle that in mixed modal syllogisms the conclusion always has the same modal character as the weaker of the premises (Theophrastus frs. 106 and 107 Fortenbaugh), where possibility is weaker than actuality, and actuality than necessity. In this way Aristotle's modal syllogistic is notably simplified and many unsatisfactory theses, like the one mentioned above (that from ‘Necessarily AaB’ and ‘BaC’ one can infer ‘Necessarily AaC’) disappear.
Prosleptic Syllogisms

Theophrastus introduced the so-called prosleptic premises and syllogisms. A prosleptic premise is of the form:

For all $X$, if $\Phi(X)$, then $\Psi(X)$

where $\Phi(X)$ and $\Psi(X)$ stand for categorical sentences in which the variable $X$ occurs in place of one of the terms. For example:

(1) $A$ [holds] of all of that of all of which $B$ [holds].
(2) $A$ [holds] of none of that which [holds] of all $B$.

Theophrastus considered such premises to contain three terms, two of which are definite ($A$, $B$), one indefinite (‘that’, or the bound variable $X$). We can represent (1) and (2) as

$\forall X (BaX \rightarrow AaX)$
$\forall X (XaB \rightarrow AeX)$

Prosleptic syllogisms then come about as follows: They are composed of a prosleptic premise and the categorical premise obtained by instantiating a term ($C$) in the antecedent ‘open categorical sentence’ as premises, and the categorical sentences one obtains by putting in the same term ($C$) in the consequent ‘open categorical sentence’ as conclusion. For example:

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A [holds] of all of that of all of which B [holds].  
B holds of all C.  
Therefore, A holds of all C.

Theophrastus distinguished three figures of these syllogisms, depending on the position of the indefinite term (also called ‘middle term’) in the prosleptic premise; for example (1) produces a third figure syllogism, (2) a first figure syllogism. The number of prosleptic syllogisms was presumably equal to that of types of prosleptic sentences: with Theophrastus' concept of the first figure these would be sixty-four (i.e. 32 + 16 + 16). Theophrastus held that certain prosleptic premises were equivalent to certain categorical sentences, e.g. (1) to ‘A is predicated of all B’. However, for many, including (2), no such equivalent can be found, and prosleptic syllogisms thus increased the inferential power of Peripatetic logic.

**Forerunners of Modus Ponens and Modus Tollens**¹⁸

Theophrastus and Eudemus considered complex premises which they called ‘hypothetical premises’ and which had one of the following two (or similar) forms:

If something is $F$, it is $G$  
Either something is $F$ or it is $G$  (with exclusive ‘or’)

They developed arguments with them which they called ‘mixed from a hypothetical premise and a probative premise’. These arguments were inspired by Aristotle's syllogisms ‘from a hypothesis’;

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they were forerunners of *modus ponens* and *modus tollens* and had the following forms, employing the exclusive ‘or’:

If something is $F$, it is $G$.  If something is $F$, it is $G$.  
$a$ is $F$.  $a$ is not $G$.  
Therefore, $a$ is $G$.  Therefore, $a$ is not $F$.  

Either something is $F$ or it is $G$.  Either something is $F$ or it is $G$.  
$a$ is $F$.  $a$ is not $F$.  
Therefore, $a$ is not $G$.  Therefore, $a$ is $G$.  

Theophrastus also recognized that the connective particle ‘or’ can be inclusive; and he considered relative quantified sentences such as those containing ‘more’, ‘fewer’, and ‘the same’, and seems to have discussed syllogisms built from such sentences, again following up upon what Aristotle said about syllogisms from a hypothesis.

**Wholly Hypothetical Syllogisms**

These syllogisms were originally abbreviated term-logical arguments of the kind

If [something is] $A$, [it is] $B$.  
If [something is] $B$, [it is] $C$.  
Therefore, if [something is] $A$, [it is] $C$.  

and at least some of them were regarded as reducible to Aristotle's categorical syllogisms, presumably by way of the equivalences to

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‘Every $A$ is $B$’, etc. In parallel to Aristotle’s syllogistic, Theophrastus distinguished three figures; each had sixteen modes. The first eight modes of the first figure are obtained by going through all permutations with ‘not $X$’ instead of ‘$X$’ (with $X$ for $A$, $B$, $C$); the second eight modes are obtained by using a rule of contraposition on the conclusion:

(CR) From ‘if $X$, $Y$’ infer ‘if the contradictory of $Y$ then the contradictory of $X$’

The sixteen modes of the second figure were obtained by using (CR) on the schema of the first premise of the first figure arguments, e.g.

If [something is] not $B$, [it is] not $A$.
If [something is] $B$, [it is] $C$.
Therefore, if [something is] $A$, [it is] $C$.

The sixteen modes of the third figure were obtained by using (CR) on the schema of the second premise of the first figure arguments, e.g.

If [something is] $A$, [it is] $B$.
If [something is] not $C$, [it is] not $B$.
Therefore, if [something is] $A$, [it is] $C$.

Theophrastus claimed that all second and third figure syllogisms could be reduced to first figure syllogisms. If Alexander of Aphrodisias (2nd c. CE Peripatetic) reports faithfully, any use of (CR) which transforms a syllogism into a first figure syllogism was such a reduction. The large number of modes and reductions can be explained by the fact that Theophrastus did not have the logical means for substituting negative for positive components in an
argument. In later antiquity, after some intermediate stages, and possibly under Stoic influence, the wholly hypothetical syllogisms were interpreted as propositional-logical arguments of the kind

If $p$, then $q$.
If $q$, then $r$.
Therefore, if $p$, then $r$.

The Stoics$^{20}$

The founder of the Stoa, Zeno of Citium (335–263 BCE), studied with Diodorus. His successor Cleanthes (331–232) tried to solve the Master Argument by denying that every past truth is necessary and wrote books—now lost—on paradoxes, dialectics, argument modes and predicates. Both philosophers considered knowledge of logic as a virtue and held it in high esteem, but they seem not to have been creative logicians. By contrast, Cleanthes' successor Chrysippus of Soli (c. 280–207) is without doubt the second great logician in the history of logic. It was said of him that if the gods used any logic, it would be that of Chrysippus (D. L. 7.180), and his reputation as a brilliant logician is amply testified. Chrysippus wrote over 300 books on logic, on virtually any topic logic today concerns itself with, including speech act theory, sentence analysis, singular and plural expressions, types of predicates, indexicals, existential propositions, sentential connectives, negations, disjunctions, conditionals, logical consequence, valid argument forms, theory of deduction, propositional logic, modal logic, tense logic, epistemic logic, logic of suppositions, logic of imperatives, ambiguity and logical

paradoxes, in particular the Liar and the Sorites (D. L. 7.189–199). Of all these, only two badly damaged papyri have survived, luckily supplemented by a considerable number of fragments and testimonies in later texts, in particular in Diogenes Laertius (D. L.) book 7, sections 55–83, and Sextus Empiricus *Outlines of Pyrrhonism* (S. E. *PH*) book 2 and *Against the Mathematicians* (S. E. *M*) book 8. Chrysippus' successors, including Diogenes of Babylon (c. 240–152) and Antipater of Tarsus (2nd cent. BCE), appear to have systematized and simplified some of his ideas, but their original contributions to logic seem small. Many testimonies of Stoic logic do not name any particular Stoic. Hence the following paragraphs simply talk about ‘the Stoics’ in general; but we can be confident that a large part of what has survived goes back to Chrysippus.

**Logical Achievements Besides Propositional Logic**

The subject matter of Stoic logic are called sayables (*lekta*): they are the underlying meanings in everything we say and think, but, Frege's senses, subsist also independently of us. They are distinguished from spoken and written linguistic expressions: what we *utter* are those expressions, but what we *say* is the sayables. There are complete and deficient sayables. Deficient sayables, if said, make the hearer feel prompted to ask for a completion; e.g. when someone says ‘writes’ we enquire ‘who?’. Complete sayables, if said, do not make the hearer ask for a completion. They include assertibles (the Stoic equivalent for propositions), imperativals, interrogatives, inquiries, exclamatives, hypotheses or suppositions, stipulations,

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oaths, curses and more. The accounts of the different complete sayables all had the general form ‘a so-and-so sayable is one in saying which we perform an act of such-and-such’. For instance: ‘an imperatival sayable is one in saying which we issue a command’, ‘an interrogative sayable is one in saying which we ask a question’, ‘a declaratory sayable (i.e. an assertible) is one in saying which we make an assertion’. Thus, according to the Stoics, each time we say a complete sayable, we perform three different acts: we utter a linguistic expression; we say the sayable; and we perform a speech-act. Chrysippus was aware of the use-mention distinction. He seems to have held that every denoting expression is ambiguous in that it denotes both its denotation and itself. Thus the expression ‘a wagon’ would denote both a wagon and the expression ‘a wagon’.

Assertibles (axiōmata) differ from all other complete sayables in their having a truth-value: at any one time they are either true or false. Truth is temporal and assertibles may change their truth-value. The Stoic principle of bivalence is hence temporalized, too. Truth is introduced by example: the assertible ‘it is day’ is true when it is day, and at all other times false. This suggests some kind of deflationist view of truth, as does the fact that the Stoics identify true assertibles with facts, but define false assertibles simply as the contradictories of true ones.

Assertibles are simple or non-simple. A simple predicative assertible like ‘Tiger is walking’ is generated from the predicate ‘is walking’, which is a deficient assertible since it elicits the question ‘who?’, together with a nominative case (Tiger's individual quality or the correlated sayable), which the assertible presents as falling under
the predicate. There is thus no interchangeability of predicate and subject terms as in Aristotle; rather, predicates but not the things that fall under them are defined as deficient, and thus resemble propositional functions. It seems that whereas some Stoics took the Fregean approach that singular terms had correlated sayables, others anticipated the notion of direct reference. Concerning indexicals, the Stoics took a simple definite assertible like ‘this one is walking’ to be true when the person pointed at by the speaker is walking. When the thing pointed at ceases to be, so does the assertible, though the sentence used to express it remains. A simple indefinite assertible like ‘someone is walking’ is said to be true when a corresponding definite assertible is true. Aristotelian universal affirmatives (‘Every A is B’) were to be rephrased as conditionals: ‘If something is A, it is B’. Negations of simple assertibles are themselves simple assertibles. The Stoic negation of ‘Tiger is walking’ is (It is) not (the case that) Tiger is walking’, and not ‘Tiger is not walking’. The latter is analyzed in a Russellian manner as ‘Both Tiger exists and not: Tiger is walking’. There are present tense, past tense and future tense assertibles. The temporalized principle of bivalence holds for them all. The past tense assertible ‘Tiger walked’ is true when there is at least one past time at which ‘Tiger is walking’ was true.

Syntax and Semantics of Complex Propositions

The Stoics concerned themselves with several issues we would place under the heading of predicate logic; but their main achievement was the development of a propositional logic, i.e. of a

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system of deduction in which the smallest substantial unanalyzed expressions are propositions, or rather, assertibles.

The Stoics defined negations as assertibles that consist of a negative particle and an assertible controlled by this particle. Similarly, non-simple assertibles were defined as assertibles that either consist of more than one assertible or of one assertible taken more than once and that are controlled by a connective particle. Both definitions can be understood as being recursive and allow for assertibles of indeterminate complexity. Three types of non-simple assertibles feature in Stoic syllogistic. Conjunctions are non-simple assertibles put together by the conjunctive connective ‘both ... and ...’. They have two conjuncts. Disjunctions are non-simple assertibles put together by the disjunctive connective ‘either ... or ... or ...’. They have two or more disjuncts, all on a par. Conditionals are non-simple assertibles formed with the connective ‘if ..., ...’; they consist of antecedent and consequent. What type of assertible an assertible is, is determined by the connective or logical particle that controls it, i.e. that has the largest scope. ‘Both not p and q’ is a conjunction, ‘Not both p and q’ a negation. Stoic language regimentation asks that sentences expressing assertibles always start with the logical particle or expression characteristic for the assertible. Thus, the Stoics invented an implicit bracketing device similar to that used in Łukasiewicz' Polish notation.

Stoic negations and conjunctions are truth-functional. Stoic (or at least Chrysippean) conditionals are true when the contradictory of the consequent is incompatible with its antecedent. Two assertibles are contradictories of each other if one is the negation of the other;
that is, when one exceeds the other by a prefixed negation particle. The truth functional Philonian conditional was expressed as a negation of a conjunction: that is, not as ‘if \( p, q \)’ but as ‘not both \( p \) and not \( q \)’. Stoic disjunction is exclusive and non-truth-functional. It is true when necessarily precisely one of its disjuncts is true. Later Stoics introduced a non-truth-functional inclusive disjunction.

Same as Philo and Diodorus, Chrysippus distinguished four modalities and considered them as modal values of propositions rather than modal operators; they satisfy the same standard requirements of modal logic. Chrysippus' definitions are: An assertible is possible when it is both capable of being true and not hindered by external things from being true. An assertible is impossible when it is [either] not capable of being true [or is capable of being true, but hindered by external things from being true]. An assertible is necessary when, being true, it either is not capable of being false or is capable of being false, but hindered by external things from being false. An assertible is non-necessary when it is both capable of being false and not hindered by external things [from being false]. Chrysippus' modal notions differ from Diodorus' in that they allow for future contingents and from Philo's in that they go beyond mere conceptual possibility.

**Arguments**

Arguments are normally compounds of assertibles. They are defined as a system of at least two premises and a conclusion. Syntactically, every premise but the first is introduced by ‘now’ or

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23 **Op. cit.**
An argument is valid if the (Chrysippean) conditional formed with the conjunction of its premises as antecedent and its conclusion as consequent is correct. An argument is ‘sound’ (literally: ‘true’), when in addition to being valid it has true premises. The Stoics defined so-called argument modes as a sort of schema of an argument. A mode of an argument differs from the argument itself by having ordinal numbers taking the place of assertibles. A mode of the argument

If it is day, it is light.
But it is not the case that it is light.
Therefore it is not the case that it is day.

is

If the 1st, the 2nd.
But not: the 2nd.
Therefore not: the 1st.

The modes functioned first as abbreviations of arguments that brought out their logically relevant form; and second, it seems, as representatives of the form of a class of arguments.

**Stoic Syllogistic**

Stoic syllogistic is an argumental deductive system consisting of five types of indemonstrables or axiomatic arguments and four inference rules, called *themata*. An argument is a syllogism precisely if it either is an indemonstrable or can be reduced to one by means of
the *themata*. Syllogisms are thus certain types of formally valid arguments. The Stoics explicitly acknowledged that there are valid arguments that are not syllogisms; but assumed that these could be somehow transformed into syllogisms.

All basic indemonstrables consist of a non-simple assertible as leading premiss and a simple assertible as co-assumption, and have another simple assertible as conclusion. They were defined by five standardized meta-linguistic descriptions of the forms of the arguments:

- A first indemonstrable is an argument composed of a conditional and its antecedent as premises, having the consequent of the conditional as conclusion.
- A second indemonstrable is an argument composed of a conditional and the contradictory of its consequent as premises, having the contradictory of its antecedent as conclusion.
- A third indemonstrable is an argument composed of a negated conjunction and one of its conjuncts as premises, having the contradictory of the other conjunct as conclusion.
- A fourth indemonstrable is an argument composed of a disjunctive assertible and one of its disjuncts as premises, having the contradictory of the remaining disjunct as conclusion.
- A fifth indemonstrable, finally, is an argument composed of a disjunctive assertible and the contradictory of one of its disjuncts as premises, having the remaining disjunct as conclusion.
Whether an argument is an indemonstrable can be tested by comparing it with these meta-linguistic descriptions. For instance,

If it is day, it is not the case that it is night.  
But it is night.  
Therefore it is not the case that it is day.

comes out as a second indemonstrable, and

If five is a number, then either five is odd or five is even.  
But five is a number.  
Therefore either five is odd or five is even.

as a first indemonstrable. For testing, a suitable mode of an argument can also be used as a stand-in. A mode is syllogistic, if a corresponding argument with the same form is a syllogism (because of that form). However in Stoic logic there are no five modes that can be used as inference schemata that represent the five types of indemonstrables. For example, the following are two of the many modes of fourth indemonstrables:

Either the 1\textsuperscript{st} or the 2\textsuperscript{nd}. 
But the 2\textsuperscript{nd}.  
Therefore not the 1\textsuperscript{st}.

Either the 1\textsuperscript{st} or not the 2\textsuperscript{nd}.  
But the 1\textsuperscript{st}.  
Therefore the 2\textsuperscript{nd}.

Although both are covered by the meta-linguistic description, neither could be singled out as the mode of the fourth indemonstrables: If we disregard complex arguments, there are thirty-two modes corresponding to the five meta-linguistic
descriptions; the latter thus prove noticeably more economical. The almost universal assumption among historians of logic that the Stoics represented their five (types of) indemonstrables by five modes is false and not supported by textual evidence.

Of the four themata, only the first and third are extant. They, too, were meta-linguistically formulated. The first thema, in its basic form, was:

- When from two [assertibles] a third follows, then from either of them together with the contradictory of the conclusion the contradictory of the other follows.

This is an inference rule of the kind today called antilogism. The third thema, in one formulation, was:

- When from two [assertibles] a third follows, and from the one that follows [i.e. the third] together with another, external assumption, another follows, then this other follows from the first two and the externally co-assumed one.

This is an inference rule of the kind today called cut-rule. It is used to reduce chain-syllogisms. The second and fourth themata were also cut-rules, and reconstructions of them can be provided, since we know what arguments they together with the third thema were thought to reduce, and we have some of the arguments said to be reducible by the second thema. A possible reconstruction of the second thema is:
- When from two assertibles a third follows, and from the third and one (or both) of the two another follows, then this other follows from the first two.

A possible reconstruction of the fourth thema is:

- When from two assertibles a third follows, and from the third and one (or both) of the two and one (or more) external assertible(s) another follows, then this other follows from the first two and the external(s).

A Stoic reduction shows the formal validity of an argument by applying to it the themata in one or more steps in such a way that all resultant arguments are indemonstrables. This can be done either with the arguments or their modes. For instance, the argument mode

If the 1st and the 2nd, the 3rd.
But not the 3rd.
Moreover, the 1st.
Therefore not: the 2nd.

can be reduced by the third thema to (the modes of) a second and a third indemonstrable as follows:

When from two assertibles (‘If the 1st and the 2nd, the 3rd’ and ‘But not the 3rd’) a third follows (‘Not: both the 1st and the 2nd’—this follows by a second indemonstrable) and from the third and an external one (‘The 1st’) another follows (‘Not: the 2nd’—this follows by a third indemonstrable), then this other (‘Not: the 2nd’) also follows from the two assertibles and the external one.
The second *thema* reduced, among others, arguments with the following modes:

Either the 1\textsuperscript{st} or not the 1\textsuperscript{st}. If the 1\textsuperscript{st}, if the 1\textsuperscript{st}, the 2\textsuperscript{nd}.
But the 1\textsuperscript{st}. But the 1\textsuperscript{st}.
Therefore the 1\textsuperscript{st}. Therefore the 2\textsuperscript{nd}.

The Peripatetics chided the Stoics for allowing such useless arguments, but the Stoics rightly insisted that if they can be reduced, they are valid.

The four *themata* can be used repeatedly and in any combination in a reduction. Thus propositional arguments of indeterminate length and complexity can be reduced. Stoic syllogistic has been formalized, and it has been shown that the Stoic deductive system shows strong similarities with relevance logical systems like those by McCall. Like Aristotle, the Stoics aimed at proving non-evident formally valid *arguments* by reducing them by means of accepted inference rules to evidently valid *arguments*. Thus, although their logic is a propositional logic, they did not intend to provide a system that allows for the deduction of all propositional-logical truths, but rather a system of valid propositional-logical arguments with at least two premises and a conclusion. Nonetheless, we have evidence that the Stoics expressly recognized many simple logical truths. For example, they accepted the following logical principles: the principle of double negation, stating that a double negation (‘not: not: $p$’) is equivalent to the assertible that is doubly negated (i.e. $p$); the principle that any conditional formed by using the same assertible as
antecedent and as consequent (‘if \( p, p' \)) is true; the principle that any two-place disjunctions formed by using contradictory disjuncts (‘either \( p \) or not: \( p' \)) is true; and the principle of contraposition, that if ‘if \( p, q' \) then ‘if not: \( q, \) not: \( p' \).

**Indian Logic**

The term "Indian logic" may be used to refer to the system of logic (Nyaya) that forms one of the six principal schools of Hindu philosophy. The history of Indian logic covers at least 23 centuries, and the number of works by Indian logicians, published and unpublished, is vast. Those available in Western languages or accessible in good editions constitute only a fraction of this material. Attention here is confined to material from some of the published Sanskrit texts.

This survey falls into five parts:

(1) Grammar, which was well developed by the time of the Sanskrit grammar of Panini. Its sophisticated logical rules and techniques influenced almost all later scholarly developments in India.

(2) *Mimāmsā* – investigation – (its root *man* means “thinking”) is made up of two texts: (a) *Pūrva-Mimāmsā* which means “Anterior Investigation”, sometimes also called *Karma-Mimāmsā* “Investigation of the Writings”; (b) *Uttara-Mimāmsā* “Posterior Investigation”, or, as it is also called, *Brahma-Mimāmsā* “Investigation of the Universal Principle” or *Sariraka Mimāmsā* “Investigation of the Incorporeal Soul”, or *Vedānta*, its current name,
meaning “End o the Veda”.25 It dealt largely with problems of textual interpretation and faced a variety of logical problems in the course of its history.

(3) Vaisesika and Old Nyaya. Vaisesika, which also embodied a system of natural philosophy, provided a list of categories that set the framework within which logicians of the Old Nyaya School developed their systematic analyses of perception and inference.

(4) Buddhist logic, partly a reaction against the old Nyaya. Some branches of Buddhist logic laid a foundation for formal logic and began to exclude extraneous considerations of ontology, epistemology, and psychology. The aim and scope of Buddhist logic is defined by Dharmakīrti as follow: “All successful human action is (necessarily) preceded by right knowledge, therefore we are going to investigate it”.26 Error and doubt are the opposite of right knowledge and it is then necessary to indicate how one can avoid error and doubt and how is it possible to reach right knowledge. In Stcherbatsky’s opinion, the definition and the aim of Buddhist logic are very similar to the modern psychologists’ conception of logic.27

The object of Dinnāga’s logical study is Pramāṇa, the way of true knowledge, Pramāṇa is defined as “The knowledge that does not disagree with reality; it is also knowledge of what is not known”.28

The Pramāṇa problems are four: (1) The number of the modes of knowledge; (2) Their nature; (3) Their object; (4) The result of the modes of knowledge.

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27 Ibid., p. 60.
T. R. V. Murti specifies that the whole Buddhist logic is based on acceptance of two modes of knowledge: immediate – pratyaksa, and mediate – anumāna. These two modes of knowledge are the only possible ones and are mutually exclusive, exhausting the entire sphere of knowledge. From the first mode of knowledge is “received” – grāhāna, the datum of knowledge; the second has the function of “thinking” – adhyāvasāya, the datum according to certain forms.29

(5) New Nyaya, The Navya-Nyāya or Neo-Logical darśana (school) of Indian philosophy was founded in the 13th century CE by the philosopher Gangeśa Upādhyāya of Mithila. It was a development of the classical Nyāya darśana. Other influences on Navya-Nyāya were the work of earlier philosophers Vācaspati Miśra (900–980 CE) and Udayana (late 10th century).

Gangeśa's book Tattvacintāmani ("Thought-Jewel of Reality") was written partly in response to Śrīharśa's Khandanakhandakhādya, a defence of Advaita Vedānta, which had offered a set of thorough criticisms of Nyāya theories of thought and language. In his book, Gangeśa both addressed some of those criticisms and – more importantly – critically examined the Nyāya darśana himself. He held that, while Śrīharśa had failed successfully to challenge the Nyāya realist ontology, his and Gangeśa's own criticisms brought out a need to improve and refine the logical and linguistic tools of Nyāya thought, to make them more rigorous and precise.

Tattvacintāmani dealt with all the important aspects of Indian philosophy, logic, set theory, and especially epistemology,

which Gangeśa examined rigorously, developing and improving the Nyāya scheme, and offering examples. The results, especially his analysis of cognition, were taken up and used by other darśanas.

Navya-Nyāya developed a sophisticated language and conceptual scheme that allowed it to raise, analyse, and solve problems in logic and epistemology. It systematised all the Nyāya concepts into four main categories: sense or perception (pratyakṣa), inference (anumāna), comparison or similarity (upamāna), and testimony (sound or word; śabda).30

Lack of space prevents discussion of the role of logic in the sciences and of the philosophical schools of the Vedanta, which dealt with logical topics, especially in the later developments within Advaita and Dvaita.

Since more attention will be paid to doctrines than to individual logicians and their works and dates, a chronological table may be helpful in providing a rough outline of their historical context. (In the table, names of writings are in italic.) The table makes it clear that some of the schools which developed simultaneously were in a position to influence each other.31

Medieval Logic

The term ‘Medieval Methodology and Logical Reasoning’ is used to designate the structure of Medieval Methodology and Logical Reasoning developed in the schools and universities of Western Europe between the eleventh and fifteenth centuries. A characteristic of Medieval Methodology and Logical Reasoning were its meta-linguistic formulation as a quasi-prescriptive systematization of the syntax and semantics of natural language, specifically; of scholastic Latin. Within this frame-work medieval logicians developed;

(1) a general theory of reference (suppositio terminorum) that was applied both to the semantical problem of use and mention of expressions and to the formulation of what is now called quantification theory,

(2) a general theory of implication (consequentia) governing a logic of propositions,

(3) a well-articulated logic of modalities, and
some sophisticated treatments of problems in the philosophy of logic and language.  

For this chapter, we will study only the important medieval logician such as Peter Abelard (1079-1142) and William of Sherwood (1190–1249).

**Peter Abelard (1079-1142)**

Peter Abelard ['Abailard' or 'Abaelard' or 'Habalaarz' and so on] was the pre-eminent philosopher and theologian of the twelfth century. Abelard was the greatest logician since Antiquity: he devised a purely truth-functional propositional logic, recognizing the distinction between force and content we associate with Frege, and worked out a complete theory of entailment as it functions in argument (which we now take as the theory of logical consequence). His logical structure is flawed in its handling of topical inference, but that should not prevent our recognition of Abelard's achievements.

Abelard observes that the same propositional content can be expressed with different force in different contexts: the content *that Socrates is in the house* is expressed in an assertion in “Socrates is in the house”; in a question in “Is Socrates in the house?”; in a wish in “If only Socrates were in the house!” and so on. Hence Abelard can distinguish in particular the assertive force of a sentence from its propositional content, a distinction that allows him to point out that the component sentences in a conditional statement are not asserted, though they have the same content they do when asserted – "If Socrates is in the kitchen, then Socrates is in the house" does not assert that Socrates is in the kitchen or that he is in the house, nor do

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the antecedent or the consequent, although the same form of words could be used outside the scope of the conditional to make such assertions. Likewise, the distinction allows Abelard to define negation, and other propositional connectives, purely truth-functionally in terms of content, so that negation, for instance, is treated as follows: not- \( p \) is false/true if and only if \( p \) is true/false.

The key to the theory of argument, for Abelard, is found in inferentia, best rendered as ‘entailment’ since Abelard requires the connection between the propositions involved to be both necessary and relevant. That is, the conclusion — more exactly, the sense of the final statement — is required by the sense of the preceding statement(s), so that it cannot be otherwise. Abelard often speaks of the sense of the final statement being ‘contained’ in the sense of the preceding statement(s), much as we speak of the conclusion being contained in the premisses. An entailment is complete (perfecta) when it holds in virtue of the logical form (complexio) of the propositions involved. By this, Abelard tells us, he means that the entailment holds under any uniform substitution in its terms, the criterion now associated with Bolzano. The traditional four figures and moods of the categorical syllogism derived from Aristotle, and the doctrine of the hypothetical syllogism derived from Boethius, are all instances of complete entailments, or as we should say, valid inference.

There is another way in which conclusions can be necessary and relevant to their premisses, yet not be formally valid (not be a complete entailment). The necessary connection among the propositions, and the link among their senses, might be a function of non-formal metaphysical truths holding in all possible worlds. For
instance, human beings are a kind of animal, so the consequence “If Socrates is a human being, Socrates is an animal” holds of necessity and the sense of the antecedent compels that of the consequent, but it is not formally valid under uniform substitution. Abelard takes such incomplete entailments to hold according to the theory of the topics (to be forms of so-called topical inference). The sample inference above is validated by the topic “from the species”, a set of metaphysical relations one of which is expressible in the rule “Whatever the species is predicated of, so too is the genus” which grounds the inferential force of the entailment. Against Boethius, Abelard maintained that topical rules were only needed for incomplete entailment, and in particular are not required to validate the classical moods of the categorical and hypothetical syllogism mentioned in the preceding paragraph.

Abelard spends a great deal of effort to explore the complexities of the theory of topical inference, especially charting the precise relations among conditional sentences, arguments, and what he calls ‘argumentation’ (roughly what follows from conceded premisses). One of his investigation is that he denies that a correlate of the Deduction Theorem holds, maintaining that a valid argument need not correspond to an acceptable conditional sentence, nor conversely, since the requirements on arguments and conditionals differ.

In the end, it seems that Abelard's principles of topical inference do not work, a fact that became evident with regard to the topic “from opposites”: Abelard's principles lead to inconsistent results, a result noted by Alberic of Paris. This led to a crisis in the
theory of inference in the twelfth century, since Abelard unsuccessfully tried to evade the difficulty.33

**William of Sherwood** (1190–1249)34

He was the author of two books which were an important influence on the development of Scholastic logic: *Introductiones in Logicam* (Introduction to Logic), and *Syncategoremata*. These are the first known works to deal in a systematic way with what is now called supposition theory, known in William's time as the *logica moderna*.

William's main work is a small logic manual, *Introductiones in logicam*. It survives in a single manuscript probably written in the late thirteenth century, headed 'Introductiones Magistri Guilli. De Shyreswode in Logicam', *(Bibliotheque Nationale, Cod. Lat. 16617, formerly Codex Sorbonnensis 1797)*. It did not appear fully in print until 1937, in Grabman's Latin edition, and was not translated into English until 1966, by Kretzmann. No other works than are definitely by him have ever been printed.

The book consists of Six Chapters. Five of these are expositions of Aristotle's main logical works, as follows: 1. 'Statements', corresponding to *De Interpretatione*, 2. 'The Predicables', corresponding to *Categories*, 3. 'Syllogism', corresponding to *Prior Analytics*, 4. 'Dialectical Reasoning' corresponding to *Topics*, and 6. 'Sophistical Reasoning' corresponding to *Sophistical Refutations*. However, Chapter 5,

'Properties of Terms', contains material that is not in Aristotle, but is a distinctively medieval development, (Supposition theory) that deals with the semantics of propositions. The theory attempts to explain how the truth of simple sentences, expressed schematically, depend on how the terms 'supposit' or stand for certain extra-linguistic items, and tries to address the problem of sentential forms, like 'I promise you a horse', which do not appear to fit the standard syllogistic forms.

In this chapter William introduces what was to become a standard division of supposition into 'material', 'formal' and 'personal'. In material supposition, a term stands for itself, as when we say that 'Socrates' is a name (note that medieval Latin did not use quotation marks as in modern English). In formal supposition, the word signifies its meaning, as in man is a species. Formal supposition is similar to what is indicated in modern philosophical logic by italicising a common noun, as when we refer to the concept horse. Personal supposition is approximately the relation we now call 'satisfied by', or 'denotes', as in 'the term 'man' denotes Socrates, Aristotle, &c'.

He discusses a number of problem cases. For example, the sentence 'every man sees a man' is true when there is a single man that every man sees (for example if 'every man sees Socrates' is true). But the sentence is also true when every man sees a different man, or when some men see a single man (such as Socrates), other men see another man, and innumerable cases in between. This is called 'confused' supposition. This instance of the problem of multiple generality, is now thought to be insoluble using the fixed schema of Aristotle's semantics.
William's work spurred a development of logic in the thirteenth century under the general designation *De Proprietibus Terminorum*. Those who engaged in this part of logic were called the *Moderni*, or *Terministae*. Its most detailed treatment is found in Ockham, and in the works of those who followed him.

**Precursors of Modern Logic**

**Gottfried Wilhelm Leibniz (1646-1716)**

**Formal logic**

Leibniz is the most important logician between Aristotle and 1847, when George Boole and Augustus De Morgan each published books that began modern formal logic. Leibniz enunciated the principal properties of what we now call conjunction, disjunction, negation, identity, set inclusion, and the empty set. The principles of Leibniz's logic and, arguably, of his whole philosophy, reduce to two:

1. All our ideas are compounded from a very small number of simple ideas, which form the alphabet of human thought.
2. Complex ideas proceed from these simple ideas by a uniform and symmetrical combination, analogous to arithmetical multiplication.

With regard to (1), the number of simple ideas is much greater than Leibniz thought. As for (2), logic can indeed be grounded in a symmetrical combining operation, but that operation is analogous to either of addition or multiplication. The formal logic that

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emerged early in the 20th century also requires, at minimum, unary negation and quantified variables ranging over some universe of discourse.

Leibniz published nothing on formal logic in his lifetime; most of what he wrote on the subject consists of working drafts.

In his book *History of Western Philosophy* Bertrand Russell went as far as claiming that Leibniz had developed logic in his unpublished writings to a level which was reached only 200 years later.

The two conclusions of Leibniz’s logic.

(5) Leibniz’s logic is a sort of universal mathematics, in which one concludes *via formae* in a much ampler way than by virtue of pure syllogistic form.

(6) Liebniz’s logic, considering the invention of syllogisms as one of the most wonderful inventions of the spirit, without denying thus traditional logic, is a more complex one.\(^{36}\)

Without quoting other excerpts, one can draw the following conclusions:

(1) In Leibniz’s conception logic was the Encyclopedia that means the harmonious structure of the whole knowledge. It did not differ from metaphysics, and had the character of a *Mathesis universalis*.

(2) Logic was not only a method of derivation of ideas or propositions, and of inventing them, but also a method which determined the primitive material composed of ideas and primitive

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propositions, based on which and starting from which the *ars inveniendi ac judicandi* could be applied.

(3) The logic Leibniz dreamed of was “a more sublime one: (as Philalethe stated on his *Nouveaux Essais*), so it could neither be mistaken for the “rudiments” of classic logic nor with mathematical logic, which in any case was not an *ars inveniendi*.

(4) This vast conception of logic, which would have harmoniously linked all our present knowledges, from which the future ones could have been derived, is only the formal explanation of Leibniz’s principle of “pre-established harmony”.

(5) In consequence, logic is not derivable from Leibniz’s philosophical conceptions, nor were other disciplines derivable from it: *Leibniz’s logic is his philosophy itself*.37

**Charles Sanders Peirce (1839-1914)**38

Charles Peirce's contributions to logical theory are numerous and profound. His work on relations building on ideas of De Morgan influenced Schroder, and through Schroder, Peano, Russell, Lowenheim and much of contemporary logical theory. Although Frege anticipated much of Peirce's work on relations and quantification theory, and to some extent developed it to a greater extent, Frege's work remained out of the mainstream until the twentieth century. Thus it is plausible that Peirce's influence on the development of logic has been of the same order as Frege's. Further discussion of Peirce's influence can be found in Dipert (1995).

In contrast to Frege's highly systematic and thoroughly developed work in logic, Peirce's work remains fragmentary and extensive, rich with profound ideas but most of them left in a rough and incomplete form. Three of the Peirce's contributions to logic that are not as well-known as others are described below:

- 1. Three-Valued Logic
- 2. Calculus of Relations
- 3. Existential Graphs

1. Three-Valued Logic

In three unnumbered pages from his unpublished notes written before 1910, Peirce developed what amounts to a semantics for three-valued logic. This is at least ten years before Emil Post's dissertation, which is usually cited as the origin of three-valued logic. A good source of information about these three pages is Fisch and Turquette (1966), which also includes reproductions of the three pages from Peirce's notes.

In his notes, Peirce experiments with three symbols representing truth values: $V$, $L$, and $F$. He associates $V$ with “1” and “$T$”, indicating truth. He associates $F$ with “0” and “$F$”, indicating falsehood. He associates $L$ with “1/2” and “$N$”, indicating perhaps an intermediate or unknown value.

Peirce defines a large number of unary and binary operators on these three truth values. The semantics for the operators is indicated.

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by truth tables. Two examples are given here. First, the bar operator
(indicated here by a minus sign) is defined as follows:

\[
\begin{array}{c|ccc}
    x & V & L & F \\
\hline
    \neg x & F & L & V \\
\end{array}
\]

Applied to truth the bar operator yields falsehood, applied to
unknown it yields unknown and applied to falsehood it yields truth.

The \textit{Z} operator is a binary operator which Peirce defines as follows:

\[
\begin{array}{c|ccc}
    V & L & F \\
\hline
    V & V & L & F \\
    L & L & L & F \\
    F & F & F & F \\
\end{array}
\]

Thus, the \textit{Z} operator applied to a falsehood and anything else
yields a falsehood. The \textit{Z} operator applied to an unknown and
anything but a falsehood yields an unknown. And the \textit{Z} operator
applied to a truth and some other value yields the other value.

The bar operator and the \textit{Z} operator provide the essentials of a
truth-functionally complete strong Kleene semantics for three-valued
logic. In addition to these two strong Kleene operators, Peirce defines
several other forms of negation, conjunction, and disjunction. The
notes also provide some basic properties of some of the operators,
such as being symmetric and being associative.

2. Calculus of Relations
Building on ideas of De Morgan, Peirce fruitfully applied the concepts of Boolean algebra to relations. Boolean algebra is concerned with operations on general or class terms. Peirce applied the same idea to what he called “relatives” or “relative terms.” While his ideas evolved continually over time on this subject, fairly definitive presentations are found in Peirce (1870) and Peirce (1883). The calculus of relatives is developed further in Tarski (1941). A history of work on the subject is Maddux (1990).

Given relative terms such as “friend of” and “enemy of” (more briefly “f” and “e”), Peirce studied various operations on these terms such as the following:

(union) friend of or enemy of
A pair <a, b> stands in this relation if and only if if stands in one or both of the relations. In symbols “f + e”.

(intersection) friend of and enemy of
A pair <a, b> stands in this relation if and only if if stands in both of the relations. In symbols “f . e”.

(relative product) friend of an enemy of
A pair <a, b> stands in this relation if and only if there is a c such a is a friend of c and c is an enemy of b. In symbols “f ; e”.

(relative sum) friend of every enemy of
A pair <a, b> stands in this relation if and only if a is the friend of every object c that is the enemy of b. In symbols “f , e” (Peirce uses a dagger rather than a comma)
A pair \(<a, b>\) stands in this relation if and only if \(<a, b>\) does not stand in the friend-of relation. In symbols “\(\neg f\)” (Peirce places a bar over the relative term).

A pair \(<a, b>\) stands in this relation if and only if \(b\) is a friend of \(a\). In symbols “\(\sim f\)” (Peirce places an upwards facing semi-circle over the relative term).

Peirce presented numerous theorems involving his operations on relative terms. Examples of the numerous such laws identified by Peirce are:

\[
\neg (r + s) = \neg r + \neg s \\
\neg (r ; s) = \neg r , \neg s \\
(r . s) , t = (r , s) . (r , t)
\]

Peirce's calculus of relations has been criticized for remaining unnecessarily tied to previous work on Boolean algebra and the equational paradigm in mathematics. It has been frequently claimed that real progress in logic was only realized in the work of Frege and later work of Peirce in which the equational paradigm was dropped and the powerful expressive ability of quantification theory was realized.

Nevertheless, Peirce's calculus of relations has remained a topic of interest to this day as an alternative, algebraic approach to the logic of relations. It has been studied by Lowenheim, Tarski and others. Lowenheim's famous theorem was originally a result about the
calculus of relations rather than quantification theory, as it is usually presented today. Some of the subsequent work on the calculus of relations is outlined in Maddux (1990).

3. Existential Graphs

Following his development of quantification theory, Peirce developed a graphical system for analyzing logical reasoning that he felt was superior in analytical power to his algebraic and quantificational notations. A large portion of this material is reprinted as volume 4, book 2 of Peirce (1933) and is discussed, for example, in Roberts (1964), Roberts (1973), Zeman (1964) and Hammer (1996). This system of “existential graphs” encompassed propositional logic, first-order logic with identity, higher-order logic, and modal logic.

Peirce’s Pragmatism is centered on logic and was therefore called “logical pragmatism”.

Peirce’s logic is the study of signs by division for two kinds are;

(1) The natural sign has a “denotative” function and
(2) The arbitrary sign has a “representative” function.

For instance, smoke, as the sign of fire, is a natural sign, as the property of a sign is physically linked to the thing signified. But an olive branch as a sign of peace is accepted by convention. Such a sign has significance in connection with somebody, who is interpreting it. There is such an interpretation likewise for the natural sign, but this does not necessarily imply knowledge of the convention which ascribes significance to it.
It follows that any idea, being a sign of an object, presupposes another idea by which it is interpreted, the latter however in its turn presupposes another idea, and so on, infinitely.

Peirce divides logic into two parts:

(1) Logic of the sign (formal logic); we shall revert to Peirce’s contribution in the chapter “Mathematic logic”. This conception logic is a game of signs which as a matter of fact, was a necessary consequence of the thesis he proceeded from.

(2) Logic of the truth (material logic) brings about the intervention of a determining psychological element: truth is a belief. And, any assertion is a belief, but a reflected belief, excluding any doubt. This belief, says Pierce, determines human action, because man believes in the efficacy of his action. This idea becomes a rule: what determines the truth of an idea is its efficacy. An idea is true, if the action, to which the faith in its truth leads, results in achievements propitious and useful to man, otherwise it is false. He expounds a vast theory of abstraction and degrees of abstraction in a study entitled *Phaneroscopy*, in which he explains the general notions as being only psychological entities. The foundation of a true idea does not reside in the thing whose sign it is, but in the power it exerts on nature: “utility makes the truth”.40

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Modern Logic: Frege To The Present

Friedrich Ludwig Gottlob Frege (1848-1925) 41

German mathematician who became a logician and philosopher, helped found both modern mathematical logic and analytic philosophy. His work has exerted a fundamental and far-reaching influence on 20th century philosophy, especially in English-speaking countries.

Though his education and early work were mathematical, and especially geometrical, Frege's thought soon turned to logic. His book, namely Begriffsschrift (Concept Script), marked a turning point in the history of logic. It broke much new ground, including a clean treatment of functions and variables. Frege wanted to show that mathematics grew out of logic, but in so doing devised techniques that took him far beyond the Aristotelian syllogistic and Stoic propositional logic that had come down to him in the logical tradition. In effect, he invented axiomatic predicate logic, in large part thanks to his invention of quantified variables, which eventually became ubiquitous in mathematics and logic, and solved the problem of multiple generality. Though previous logic had dealt with the logical constants and, or, if...then..., not, and some and all, iterations of these operations were little understood; even the distinction between a pair of sentences like "every boy loves some girl" and "some girl is loved by every boy" could not be represented. It is sometimes noted that Aristotle's logic would not be able to represent even the most elementary inferences in Euclid's geometry, but Frege's "conceptual

notation" could represent inferences involving indefinitely complex mathematical statements. Hence the analysis of logical concepts and the machinery of formalization that is essential to Bertrand Russell's theory of descriptions and *Principia Mathematica* (with Alfred North Whitehead), and to Gödel's incompleteness theorems, and to Alfred Tarski's theory of truth, is ultimately due to Frege.

Frege's purpose was to defend the view that arithmetic is a branch of logic, a view known as logicism. Already in the 1879 Begriffschrift important preliminary theorems related to mathematical induction were derived within pure logic.

In his later *Grundgesetze der Arithmetik* (1893, 1903), published at its author's expense, he attempted to derive all of the laws of arithmetic from axioms he asserted as logical reasoning. Most of these axioms were carried over from his *Begriffsschrift*, though not without some significant changes. The one truly new principle was one he called the Basic Law V: the "value-range" of the function \( f(x) \) is the same as the "value-range" of the function \( g(x) \) if and only if \( \forall x[f(x) = g(x)] \). In modern notation and terminology, let \( \{x|Fx\} \) denote the extension of the predicate \( Fx \), and similarly for \( Gx \). Then Basic Law V says that the predicates \( Fx \) and \( Gx \) have the same extension iff \( \forall x[Fx \leftrightarrow Gx] \).

In a famous episode, Bertrand Russell wrote to Frege, just as Vol. 2 of the *Grundgesetze* was about to go to press in 1903, showing that Russell's paradox could be derived from Frege's Basic Law V. (This letter and Frege's reply thereto are translated in Jean van Heijenoort 1967.) Hence the system of the *Grundgesetze* was
inconsistent. Frege wrote a hasty last-minute appendix to vol. 2, deriving the contradiction and proposing to eliminate it by modifying Basic Law V.

Frege's proposed remedy was subsequently shown to imply that there is but one object in the universe of discourse, and hence is worthless (indeed this would make for a contradiction in Frege's system if he had axiomatized the idea, fundamental to his discussion, that the True and the False are distinct objects; see e.g. Dummett 1973). But recent work has shown that much of the program of the Grundgesetze might be salvaged in other ways:

- Basic Law V can be weakened in other ways. The best-known way is due to George Boolos. A "concept" $F$ is "small" if the objects falling under $F$ cannot be put in 1-to-1 correspondence with the universe of discourse, that is, if: $\neg \exists R[R \text{ is 1-to-1 } \& \forall x \exists y(xRy \& Fy)]$. Now weaken V to $V^*$: a "concept" $F$ and a "concept" $G$ have the same "extension" if and only if neither $F$ nor $G$ is small or $\forall x(Fx \leftrightarrow Gx)$. $V^*$ is consistent if second-order arithmetic is, and suffices to prove the axioms of second-order arithmetic.

- Basic Law V can simply be replaced with Hume's Principle, which says that the number of $F$s is the same as the number of $G$s if and only if the $F$s can be put into a one-to-one correspondence with the $G$s. This principle too is consistent if second-order arithmetic is, and suffices to prove the axioms of second-order arithmetic. This result is termed Frege's Theorem because it was noticed that in developing arithmetic, Frege's use of Basic Law V is restricted to a proof of Hume's Principle; it is
from this in turn that arithmetical principles are derived. On
Hume's Principle and Frege's Theorem, see.

- Frege's logic, now known as second-order logic, can be
  weakened to so-called predicative second-order logic. However,
  this logic, although provably consistent by finitistic or
  constructive methods, can interpret only very weak fragments of
  arithmetic.

Frege's logical work is *Principia Mathematica*, appeared in
1910-13, the dominant approach to mathematical logic was still that of
George Boole and his descendants, especially Ernst Schröder. Frege's
logical reasoning nevertheless spread through the writings of his
student Rudolph Carnap and other admirers, particularly Russell and
Wittgenstein.


Russell is generally recognized as one of the founders of analytic
philosophy, even of its several branches. At the beginning of the 20th
century, alongside G. E. Moore, Russell was largely responsible for the
British "revolt against Idealism", a philosophy greatly influenced by
G. W. F. Hegel and his British apostle, F. H. Bradley. This revolt was
echoed 30 years later in Vienna by the logical positivists' "revolt
against metaphysics". Russell was particularly critical of a doctrine he
ascribed to idealism and coherentism, this he dubbed the doctrine of
internal relations, which, Russell suggested, held that in order to
know any particular thing, we must know all of its relations. Based on
this Russell attempted to show that this would make space, time,

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science and the concept of number not fully intelligible. Russell's logical work with Whitehead continued this project.

Russell's first mathematical book, *An Essay on the Foundations of Geometry*, was published in 1897. This work was heavily influenced by Immanuel Kant. Russell soon realized that the conception it laid out would have made Albert Einstein's schema of space-time impossible, which he understood to be superior to his own system. Thenceforth, he rejected the entire Kantian program as it related to mathematics and geometry, and he maintained that his own earliest work on the subject was nearly without value.

Russell interested in the definition of number and studied the work of George Boole, Georg Cantor, and Augustus De Morgan, while materials in the Bertrand Russell Archives at McMaster University include notes of his reading in algebraic logic by Charles S. Peirce and Ernst Schröder. He became convinced that the foundations of mathematics were to be found in logic, and following Gottlob Frege took an extensionalist approach in which logic was in turn based upon set theory. In 1900, he attended the first International Congress of Philosophy in Paris, where he became familiar with the work of the Italian mathematician, Giuseppe Peano. He mastered Peano's new symbolism and his set of axioms for arithmetic. Peano defined logically all of the terms of these axioms with the exception of *o*, *number*, *successor*, and the singular term, *the*, which were the primitives of his system. Russell took it upon himself to find logical definitions for each of these. Between 1897 and 1903 he published several articles applying Peano's notation to the classical Boole-Schröder algebra of relations, among them *On the Notion of Order*,
Russell eventually discovered that Gottlob Frege had independently arrived at equivalent definitions for \(0\), \(\text{successor}\), and \(\text{number}\), and the definition of number is now usually referred to as the Frege-Russell definition. It was largely Russell who brought Frege to the attention of the English-speaking world. He did this in 1903, when he published *The Principles of Mathematics*, in which the concept of class is inextricably tied to the definition of number. The appendix to this work detailed a paradox arising in Frege's application of second- and higher-order functions which took first-order functions as their arguments, and he offered his first effort to resolve what would henceforth come to be known as the Russell Paradox.

Before writing *Principles*, Russell became aware of Cantor's proof that there was no greatest cardinal number, which Russell believed was mistaken. The Cantor Paradox in turn was shown (for example by Crossley) to be a special case of the Russell Paradox. This caused Russell to analyze classes, for it was known that given any number of elements, the number of classes they result in is greater than their number. This in turn led to the discovery of a very interesting class - namely, the class of all classes. It contains two kinds of classes: those classes that contain themselves, and those that do not. Consideration of this class led him to find a fatal flaw in the so-called principle of comprehension, which had been taken for granted by logicians of the time. He showed that it resulted in a contradiction, whereby \(Y\) is a member of \(Y\), if and only if, \(Y\) is not a member of \(Y\). This has become known as Russell's paradox, the solution to which he outlined in an appendix to *Principles*, and which he later developed into a complete
theory, the Theory of types. Aside from exposing a major inconsistency in naive set theory, Russell's work led directly to the creation of modern axiomatic set theory. It also crippled Frege's project of reducing arithmetic to logic. The Theory of Types and much of Russell's subsequent work have also found practical applications with computer science and information technology.

Russell continued to defend logicism, the view that mathematics is in some important sense reducible to logic, and along with his former teacher, Alfred North Whitehead, wrote the monumental *Principia Mathematica*, an axiomatic system on which all of mathematics can be built. The first volume of the *Principia* was published in 1910, and is largely ascribed to Russell. More than any other single work, it established the specialty of mathematical or symbolic logic. Two more volumes were published, but their original plan to incorporate geometry in a fourth volume was never realized, and Russell never felt up to improving the original works, though he referenced new developments and problems in his preface to the second edition. Upon completing the *Principia*, three volumes of extraordinarily abstract and complex reasoning, Russell was exhausted, and he never felt his intellectual faculties fully recovered from the effort. Although the *Principia* did not fall prey to the paradoxes in Frege's approach, it was later proven by Kurt Gödel that neither *Principia Mathematica*, nor any other consistent system of primitive recursive arithmetic, could, within that system, determine that every proposition that could be formulated within that system was decidable, i.e. could decide whether that proposition or its negation was provable within the system (Gödel's incompleteness theorem).
Russell's last significant work in mathematics and logic, *Introduction to Mathematical Philosophy*, was written by hand while he was in jail for his anti-war activities during World War I. This was largely an explication of his previous work and its philosophical significance.

In much the same way that Russell used logic in an attempt to clarify issues in the foundations of mathematics, he also used logic in an attempt to clarify issues in philosophy. As one of the founders of analytic philosophy, Russell made significant contributions to a wide variety of areas, including metaphysics, epistemology, ethics and political theory, as well as to the history of philosophy. Underlying these various projects was not only Russell's use of logical analysis, but also his long-standing aim of discovering whether, and to what extent, knowledge is possible.

More than this, Russell's various contributions were also unified by his views concerning both the centrality of scientific knowledge and the importance of an underlying scientific methodology that is common to both philosophy and science. In the case of philosophy, this methodology expressed itself through Russell's use of logical analysis. In fact, Russell often claimed that he had more confidence in his methodology than in any particular philosophical conclusion.

Russell's conception of philosophy arose in part from his idealist origins. This is so, even though he believed that his one, true revolution in philosophy came about as a result of his break from idealism. Russell saw that the idealist doctrine of internal relations led to a series of contradictions regarding asymmetrical (and other) relations necessary for mathematics. Thus, in 1898, he abandoned the
idealism that he had encountered as a student at Cambridge, together with his Kantian methodology, in favour of a pluralistic realism. As a result, he soon became famous as an advocate of the "new realism" and for his "new philosophy of logic," emphasizing as it did the importance of modern logic for philosophical analysis. The underlying themes of this "revolution," including his belief in pluralism, his emphasis upon anti-psychologism, and the importance of science, remained central to Russell's philosophy for the remainder of his life.

Russell's methodology consisted of the making and testing of hypotheses through the weighing of evidence (hence Russell's comment that he wished to emphasize the "scientific method" in philosophy), together with a rigorous analysis of problematic propositions using the machinery of first-order logic. It was Russell's belief that by using the new logic of his day, philosophers would be able to exhibit the underlying "logical form" of natural language statements. A statement's logical form, in turn, would help philosophers resolve problems of reference associated with the ambiguity and vagueness of natural language. Thus, just as we distinguish three separate sense of "is" (the is of predication, the is of identity, and the is of existence) and exhibit these three senses by using three separate logical notations ($P$, $x=y$, and $x$ respectively) we will also discover other ontologically significant distinctions by being aware of a sentence's correct logical form. On Russell's view, the subject matter of philosophy is then distinguished from that of the sciences only by the generality and the a prioricity of philosophical statements, not by the underlying methodology of the discipline. In philosophy, as in mathematics, Russell believed that it was by
applying logical machinery and insights that advances would be made.

Russell’s most famous example of his "analytic" method concerns denoting phrases such as descriptions and proper names. In his *Principles of Mathematics*, Russell had adopted the view that every denoting phrase (for example, "Scott," "blue," "the number two," "the golden mountain") denoted, or referred to, an existing entity. By the time his landmark article, "On Denoting," appeared two years later, in 1905, Russell had modified this extreme realism and had instead become convinced that denoting phrases need not possess a theoretical unity.

While logically proper names (words such as "this" or "that" which refer to sensations of which an agent is immediately aware) do have referents associated with them, descriptive phrases (such as "the smallest number less than pi") should be viewed as a collection of quantifiers (such as "all" and "some") and propositional functions (such as "x is a number"). As such, they are not to be viewed as referring terms but, rather, as "incomplete symbols." In other words, they should be viewed as symbols that take on meaning within appropriate contexts, but that are meaningless in isolation.

Thus, in the sentence

(1) The present King of France is bald,

the definite description "The present King of France" plays a role quite different from that of a proper name such as "Scott" in the sentence

(2) Scott is bald.
Letting $K$ abbreviate the predicate "is a present King of France" and $B$ abbreviate the predicate "is bald," Russell assigns sentence (1) the logical form

$$(1') \text{There is an } x \text{ such that (i) } Kx, \text{ (ii) for any } y, \text{ if } Ky \text{ then } y=x, \text{ and (iii) } Bx.$$ 

Alternatively, in the notation of the predicate calculus, we have

$$(1'') \exists x [(Kx & \forall y (Ky \to y=x)) & Bx].$$

In contrast, by allowing $s$ to abbreviate the name "Scott," Russell assigns sentence (2) the very different logical form

$$(2') Bs.$$ 

This distinction between various logical forms allows Russell to explain three important puzzles. The first concerns the operation of the Law of Excluded Middle and how this law relates to denoting terms. According to one reading of the Law of Excluded Middle, it must be the case that either "The present King of France is bald" is true or "The present King of France is not bald" is true. But if so, both sentences appear to entail the existence of a present King of France, clearly an undesirable result. Russell's analysis shows how this conclusion can be avoided. By appealing to analysis $(1')$, it follows that there is a way to deny $(1)$ without being committed to the existence of a present King of France, namely by accepting that "It is not the case that there exists a present King of France who is bald" is true.

The second puzzle concerns the Law of Identity as it operates in (so-called) opaque contexts. Even though "Scott is the author of Waverley" is true, it does not follow that the two referring terms "Scott" and "the author of Waverley" are interchangeable in every situation. Thus although "George IV wanted to know whether Scott was the the author of Waverley" is true, "George IV wanted to know
whether Scott was Scott" is, presumably, false. Russell's distinction between the logical forms associated with the use of proper names and definite descriptions shows why this is so.

To see this we once again let \( s \) abbreviate the name "Scott." We also let \( w \) abbreviate "Waverley" and \( A \) abbreviate the two-place predicate "is the author of." It then follows that the sentence

\[
(3) \ s = s
\]

is not at all equivalent to the sentence

\[
(4) \ \exists x \ [Axw \land \forall y (Ayw \rightarrow y=x) \land x=s].
\]

The third puzzle relates to true negative existential claims, such as the claim "The golden mountain does not exist." Here, once again, by treating definite descriptions as having a logical form distinct from that of proper names, Russell is able to give an account of how a speaker may be committed to the truth of a negative existential without also being committed to the belief that the subject term has reference. That is, the claim that Scott does not exist is false since

\[
(5) \ \sim \ \exists x (x = s)
\]

is self-contradictory. (After all, there must exist at least one thing that is identical to \( s \) since it is a logical truth that \( s \) is identical to itself!) In contrast, the claim that a golden mountain does not exist may be true since, assuming that \( G \) abbreviates the predicate "is golden" and \( M \) abbreviates the predicate "is a mountain," there is nothing contradictory about

\[
(6) \ \sim \ \exists x (Gx \land Mx).
\]

Russell's emphasis upon logical analysis also had consequences for his metaphysics. In response to the traditional problem of the external world which, it is claimed, arises since the external world can be known only by inference, Russell developed his famous 1910
distinction between "knowledge by acquaintance and knowledge by
derscription." He then went on, in his 1918 lectures on logical
atomism, to argue that the world itself consists of a complex of logical
atoms (such as "little patches of colour") and their properties.
Together they form the atomic facts which, in turn, are combined to
form logically complex objects. What we normally take to be inferred
entities (for example, enduring physical objects) are then understood
to be "logical constructions" formed from the immediately given
entities of sensation, viz., "sensibilia." It is only these latter entities
that are known non-inferentially and with certainty.

**Ludwig Wittgenstein (1889-1951)**

Wittgenstein was under the influence of Moore and Russell,
and he in turn influenced Russell and many contemporary
philosophers and logicians, especially the neo-positivists and those
belonging to the philosophy of language. He took up duty as professor
at Cambridge, succeeding to the chair of Moore (from 1939 to 1947).

The first period of his activity, Wittgenstein asserted himself
as logician with metaphysical and even mystical tendencies. The
second period, he remained the analyst philosopher of language upon
whom were based the analyst schools of Oxford and Cambridge.

However, it cannot be stated that the second period had
nothing in common with the first; many of his propositions in
*Tractatus* lend support to his position in *Philosophische
Untersuchungen*.

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I. *Tractatus Logico-Philosophicus*\(^{44}\)

The entire treatise is divided into seven principal propositions, six of which have various explicative propositions.

Wittgenstein’s main idea regarding logic is that natural language is exposed to errors, because the same word may represent several things, or because several words may stand for the same thing. For remedy this defect a rigorous language must be established, made up of signs, signs that have no content and are governed by rules that indicate the way in which they are linked. The ensemble of the rules for manipulating the symbols is called by him logical syntax. Wittgenstein states that the “conceptual writing” – Begriffsschrift – of Frege and Russell is such a “logical language” or syntax, but does not exclude all errors. Here are the indications given by Wittgenstein to establish a symbolism which would “avoid the error”:

1. In order to recognize a symbol in a sign, its significant use must be taken into consideration.

2. The sign determines a logical form only together with its syntactic logical application.

3. If a sign is not used then it has no sense.

4. In logical syntax, the signification of a sign must never play any role; it must be possible to establish the syntax without any question of the signification of a sign, it must only presuppose the description of expressions.

5. Russell’s error stands in the fact that on establishing the rules of the signs he spoke of their signification.

As any symbol is empty of content, the whole of logic which is a combination of these symbols refers to nothing, and in this

connection he writes: “This sheds light because logical propositions cannot be confirmed by experience, just as they cannot be invalidated by experience”. “It is clear, he continues, why logic has been called the theory of forms and deduction”.

According to Wittgenstein the result is a fact, which the mathematical logicians have not taken into account, namely: no reference to a domain of facts, no convention can determine a logical law. Such a law is, he says, a tautology, and this says nothing and shows in itself that it is a tautology. He give definition of logic is ‘scaffolding of the world’ and gives an ontological fundamental of a purely formal (symbolic) order to reality, which says nothing and is s priori.

I. *Logisch Untersuchungen*45

In the second period he asserts that not only every natural language, but even each of the linguistic sectors possesses a particular way of signifying that cannot be understood by the construction of a linguistic game or linguistic model – *Sprachbild*, to show what signs signify.

The analysis of natural language alone is able to clarify the philosophical problems, which arise especially from the common error “of trying to find explanations to facts that should be considered as original”, All these problems spring from a confuse understanding of language and only represent “linguistic diseases”. A single philosophical therapy exists: logical analysis of natural language.

Rudolph Carnap (1891-1970)\textsuperscript{46}

Carnap was one of the most significant philosophers of the twentieth century, and made important contributions to logic, philosophy of science, semantics, modal theory and probability. In the 1930s he developed a daring pragmatic conventionalism according to which many traditional philosophical disputes are viewed as the expression of different linguistic frameworks, not genuine disagreements. This distinction between a language (framework) and what can be said within it was central to Carnap’s philosophy, reconciling the apparently a priori domains such as logic and mathematics with a thoroughgoing empiricism: basic logical and mathematical commitments partially constitute the choice of language. There is no uniquely correct choice among alternative logics or foundations for mathematics; it is a question of practical expedience, not truth. Thereafter, the logic and mathematics may be taken as true in virtue of that language. The remaining substantive questions, those not settled by the language alone, should be addressed only by empirical means. There is no other source of news. Beyond pure logic and mathematics, Carnap’s approach recognized within the sciences commitments aptly called a priori - those not tested straightforwardly by observable evidence, but, rather, presupposed in the gathering and manipulation of evidence. This a priori, too, is relativized to a framework and thus comports well with empiricism. By the time of his death, philosophers were widely rejecting what they saw as logical empiricism, though often both their arguments and the views offered as improvements had been pioneered by Carnap and his associates. By his centenary, however,

there emerged a new and fuller understanding of his ideas and of their importance for twentieth-century philosophy.

Syntax

Carnap’s most impressive logical methodology was the meta-language that was adequate for discussing logic and the foundations of mathematics. In the process he worked out a new pragmatic and conventionalist epistemology with consequences for the character of observation and the interpretation of scientific theories. In the background were disputes among logicists, formalists and intuitionists over the foundations of mathematics, and between classicists and intuitionists over the character of logic. While Wittgenstein seemed to be saying that one could not talk about logical form, only show it, Carnap tried to develop a way of talking about logic in which its structure or form could be expressed. If the original language were suitably restricted, its structure could be described, even within that language itself, thanks to Godel’s device of arithmetization. Other object languages could be described in this meta-language, and still more powerful object languages could be described in distinct meta-languages.

This required a choice which could be settled neither by empirical means, nor, on pain of circularity, by logical ones. The issue resembled that which had faced geometers at the end of the nineteenth century, and Carnap met the new problem as Hilbert had met the old, with a version of a theory of implicit definition. The alternative logical structures are conventions defining the terms they contain, and we should tolerate alternatives. Hence, Carnap’s famous

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principle of tolerance: ‘In logic there are no morals’, meaning roughly that there is no uniquely correct logic. The alternatives can be described, though not always in the language itself, and their consequences investigated and compared. Some may be more useful (more convenient, simpler, more powerful), and so a choice might be made on these grounds, but not because one of them is the right one. This is the central doctrine of Logische Syntax der Sprache (The Logical Syntax of Language) (1934a). The English translation included several sections containing a virtual semantics originally omitted for lack of space.

The doctrine was illustrated by presenting two languages.

Language I is a coordinate language (using symbols of position rather than names of objects) and is highly restrictive, allowing, either in its formulas or in the definitions of terms, only quantifiers that can be replaced with finite conjunctions or finite disjunctions. Carnap thought this the language preferred by intuitionists, though they never entirely agreed.

Language II is classical in using unlimited quantifiers and is sufficient for expressing classical mathematics. In Language II is to develop a consequence relation much more powerful than the traditional idea of derivation, and to show that mathematics could be complete in this new sense.

Most logical discussions, including those of Languages I and II, begin with lists of grammatical categories and of terms therein, and go on to specify what shall count as sentences, derivations, proofs and so on. But this provides no general terms which are useful for the discussion of other languages. Part IV of Logical Syntax provides definitions of grammatical categories and everything else to discuss
and compare alternative languages, including translation. This is done with just two primitive terms: ‘is a sentence’ and ‘is a direct consequence of’. The definitions do not always work, but, even so, are an amazing achievement. He tried to distinguish the logical from the descriptive vocabulary and specifically allowed rules of inference which were not wholly logical in character. He also developed a general notion of analyticity, which he later modified. This discussion was essentially of a sort later called semantical, and in limited ways surpassed even Tarski’s concurrent discussions. Carnap denied only that a syntactical discussion of truth was possible and, hence, that a truth theory was part of logic.

Carnap generalizes this observation. All philosophic questions, properly understood, are really about language. The traditional practice of philosophy is a mixture of empirical (psychological and sociological) questions with ones about the logical structure of language. This is not bad, but the concoction allowed metaphysicians, who certainly did not think their enterprise an empirical one, to think they were talking about extra-linguistic objects to which they somehow had a transcendent non-empirical access. To this Carnap says: Humbug! When the empirical component is assigned to science, where it belongs, only the logic of the language of science remains. Properly speaking, this is what philosophy is and all that it is. Such logic of science Carnap calls ‘syntax’, but his sense is far broader than that of later generations. He supposed that the grammatical form of observation sentences could be determined or at least marked syntactically and also that the question of which sentences implied which others could likewise be answered syntactically. Such an account of the structure of observation and
inference would be the non-psychological core of a proper philosophic epistemology; it would be the logical structure of science.

**Probability**

This notion of probability is essentially a generalization of deductive or logical implication, and, as such, a relation between sentences or propositions. Elementary statements of such probabilities are analytic or contradictory (just as claims of logical entailment would be). For Carnap, these two probability notions are not rivals, not even rival explications of the same ordinary concept. They are explications of two entirely different notions, both useful and both exhibiting the mathematical structure required to merit the term ‘probability’.

Carnap developed this notion most extensively in Logical Foundations of Probability (1950a), which discusses what it is to explicate an ordinary concept and proceeds to provide the required analysis. Some results were initially implausible, such as that the degree of confirmation for any scientific law on any evidence is zero. This is mitigated, however, by the result that the instances of such a law need not have zero confirmation values. Kemeny and Bar-Hillel showed independently that Carnap’s formulations exclude primitive predicates (such as disposition terms) having meaning relations to other primitive terms. Kemeny proposed a solution which Carnap adopted in ‘Meaning Postulates’ (1952), but still the account applied to a restricted range of languages, and Carnap’s later years were devoted to broadening the theory’s range and adequacy. Much of this new work appeared posthumously.

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While the issue was raised privately much earlier, in 1951 W.V. Quine launched a public attack on the intelligibility of Carnap’s notion of analyticity (or truth in virtue of meaning). Quine’s view came to be very influential, though not universal. Whatever the merits of the case, Quine came to accept a notion of analyticity that applied to what had been the central examples under debate, namely, elementary logic and the so-called truths of essential predication (for example, ‘All bachelors are unmarried’). This did not entirely resolve all of the issues, but it was unclear just how much any unresolvable disagreements should affect the evaluation of Carnap’s philosophy.

From the early 1950s until its appearance in 1964 Carnap worked on the massive The Philosophy of Rudolf Carnap. It included an intellectual autobiography, a then-complete bibliography, and twenty-six critical essays with Carnap’s extensive replies covering every aspect of his work.

In 1956 Carnap moved to UCLA (after two years at the Institute for Advanced Study in Princeton) to fill the post left vacant by Reichenbach’s death. He continued to work on probability and confirmation, and also resumed work on topics which were more explicitly part of general philosophy of science. In ‘The Methodological Character of Theoretical Concepts’ (1956) he ventured a sophisticated criterion of empirical significance, and in other papers he advanced an account of analyticity for theoretical concepts. The last book published in his lifetime was Philosophical Foundations of Physics: An Introduction to the Philosophy of Science (1966), which was essentially a reworked transcript of a course, and reissued under its less formidable subtitle a few years later. It contains wonderfully accessible treatments of quantitative concepts and measurement,
non-Euclidean geometry, relativity and a variety of other issues, such as determinism and scientific realism.

Carnap left a legacy of clarity of thought, philosophic achievement and personal kindness that has rarely been equalled. After a period of eclipse, his work has been partially ‘rediscovered’, and it seems likely to inform and inspire succeeding generations of philosophers much as it had done throughout the middle third of the twentieth century.

**Willard Van Orman Quine (born 1908)**

Quine is the foremost representative of naturalism in the second half of the twentieth century. His naturalism consists of an insistence upon a close connection or alliance between philosophical views and those of the natural sciences. This contrasts with views which distinguish philosophy from science and place philosophy in a special transcendent position for gaining special knowledge. The methods of science are empirical; so Quine, who operates within a scientific perspective, is an empiricist, but with a difference. The unit of empirical significance is not simple impressions (ideas) or even isolated individual observation sentences, but systems of beliefs. The broad theoretical constraints for choice between theories, such as explanatory power, parsimony, precision and so on, are foremost in this empiricism. He is a fallibilist, since he holds that each individual belief in a system is in principle revisable. Quine proposes a new conception of observation sentences, a naturalized account of our knowledge of the external world, including a rejection of a priori knowledge, and he extends the same empiricist and fallibilist account to our knowledge of logic and mathematics.

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Quine confines logic to first-order logic and clearly demarcates it from set theory and mathematics. The language of first-order logic serves as a canonical notation in which to express our ontological commitments. The slogan ‘To be is to be the value of a variable’ ([1953] 1961: 15) encapsulates this project. Deciding which ontology to accept is also carried out within the naturalistic constraints of empirical science - our ontological commitments should be to those objects to which the best scientific theories commit us. On this basis Quine’s own commitments are to physical objects and sets. Quine is a physicalist and a Platonist, since the best sciences require physical objects and the mathematics involved in the sciences requires abstract objects, namely, sets.

The theory of reference (which includes notions such as reference, truth and logical truth) is sharply demarcated from the theory of meaning (which includes notions such as meaning, synonymy, the analytic-synthetic distinction and necessity). Quine is the leading critic of notions from the theory of meaning, arguing that attempts to make the distinction between merely linguistic (analytic) truths and more substantive (synthetic) truths has failed. They do not meet the standards of precision which scientific and philosophical theories adhere to and which are adhered to in the theory of reference. He explores the limits of an empirical theory of language and offers a thesis of the indeterminacy of translation as further criticism of the theory of meaning.
Logic as first-order logic

Quine distinguishes the theory of reference from the theory of meaning. He is skeptic of notions associated with the theory of meaning, such as those of meaning, intension, synonymy, analyticity and necessity. But he relies on and makes contributions to the theory of reference, for example, to the understanding of logical truth, truth, reference and ontological commitment.

The notion of logical truth falls squarely in the theory of reference. His most characteristic definition of logical truth is that a sentence is a logical truth if it is true and if it remains true when one uniformly replaces its non-logical parts. The logical parts are the logical constants, signs for negation, disjunction, quantification and identity. ‘Brutus killed Caesar or it is not the case that Brutus killed Caesar’ is such a logical truth. In Quine’s terminology the logical constants ‘or’ and ‘it is not the case that’ occur essentially, while the non-logical part ‘Brutus killed Caesar’ can be uniformly varied and the resulting sentence will still be true. In other words a logical truth cannot be changed into a falsehood by varying the non-logical expressions, whereas an ordinary truth can be.

One of concept of logical truth virtues lies in its being parsimonious, that is, in what it does not say. Logical truth and related notions are often explained in modal terms. That is, logical truths are said to be distinguished by being ‘necessary’ or ‘true in all possible worlds’, and a valid argument is defined as one in which, if the premises are true, the conclusion ‘must be true’. These accounts make logic presuppose modal notions. Quine’s definition leaves logic

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autonomous in this respect. Indeed Quine is a critic of modal logic, challenging various attempts to explain the notion of necessity. Quine defined logical truth as, is a precisely explained species of truth, fitting squarely inside the theory of reference. He relies on the concept of truth, which he construes along the lines of Tarski’s theory.

Quine said the logical constants the truth-functional connectives ‘not’, ‘and’, ‘or’, ‘if... then’, ‘if and only if’; the quantifiers ‘all’ and ‘some’; and the identity predicate ‘a=b’. The language of logic so construed is that of sentences formed out of truth-functional connectives, quantifiers, identity, schematic predicate letters and individual variables. Quantificational logic of this sort is also known as first-order logic. For Quine, logic is first-order logic with identity, ruled out as logic on this construal are modal logic, because ‘necessity’ is not taken as a logical constant. Also excluded is higher-order logic, which has quantifiers for predicate positions. On other grounds other proposals such as intuitionist logic are also ineligible. Set theory, and with it mathematics, are not logic.

Quine holds to set theory that is the theory of the ‘is a member of’ predicate and is stated in the language of first-order logic. Given the theory of membership and logic as first-order logic plus identity, Quine introduces mathematical notions as definitional abbreviations: for example, a number is defined as a special set, addition as a special function on these sets and so on. He argues that logic does not include set theory because membership should not be considered a logical constant for the following reasons:

(1) There is a general consensus about elementary logic, which, given paradoxes such as Russell’s, is lacking in the case of set theory. Alternative set theories have the status of so many tentative
hypotheses. This lends credence to Quine’s view that mathematics based on set theory is not very different from other sciences.

(2) The incompleteness of set theory contrasts sharply with the completeness of elementary logic.

(3) The ontology of set theory is not as topic neutral as that of logic. The second item in the membership relation is restricted to sets. Logic is the most general of subjects, since the variables of logic are not restricted to any one category of objects.

The theory of meaning: its myths and dogmas

In the 1930s, Quine’s reaction to Carnap’s work was a skeptical critique of the prevalent view among analytic philosophers that logic and mathematics are justified in some distinctively linguistic way, that is, that they are based on merely analytic-linguistic truths - truths by convention. The skepticism then and later questions proposed accounts of the distinction. Linguistic truth or truth by convention, as opposed to non-conventionally based empirical truths, appears in the end to be a purported distinction without a real difference. The importance is the failure to satisfy the requirement of precision in explaining the distinction. In ‘Truth By Convention’ (1936) and ‘Carnap on Logical Truth’ (1960a), Quine takes up different attempts to characterize such truths and finds that for the most part they are either too broad – not distinctive of logic or mathematics - or require non-linguistically based truth. There are as many of these characterizations as there are different senses of ‘convention’. Quine considers truth by convention as based on the following: the arbitrary factor in axiomatization; formalization-disinterpretation; the arbitrary element in hypothesizing; and

definition. But neither logic nor mathematics is distinguished by being axiomatized, or formalized-disinterpreted. The somewhat arbitrary choice of which sentences to take as axioms, so long as we can prove the right sentences, is also not distinctive of them. If truth by convention is taken as the somewhat arbitrary element in framing hypotheses, then this too is not distinctive of logic or mathematics. Nor are the formulas involved distinguished by being true by definition. Thus, if \( p \to p \) is defined in terms of \( \neg p \lor p \), then the truth of the defined formula depends on the truth of the defining formula, and that formula's truth is not a matter of definition or convention.

Quine's logic and mathematics can be precisely characterized in terms from the theory of reference. Logic is described in terms of truth, the logical constants and interchange of the extralogical elements; mathematics can be characterized in terms of set theory. But neither of these subjects is distinct in having a different epistemological basis which results in their being in some interesting sense 'analytic' or mere 'linguistic truths'.

The 'Two Dogmas' of Quine's 1951 essay were the dogma of reduction and the dogma of the analytic/synthetic distinction. In discussing the dogma of analyticity, Quine questions, as he did in 'Truth by Convention', whether the distinction can be well made. Here he rejects five ways of explaining analyticity. These involve appeals to: (1) meanings, (2) definition, (3) interchangeability, (4) semantic rules and (5) the verifiability theory of meaning.

(1) 'Bachelors are unmarried men' might be regarded as analytic, where that notion is explained as truth in virtue of the meanings of its words. One suggestion is that the meaning of
‘unmarried man’ is included in the meaning of ‘bachelor’. Another approach would hypothesize the existence of meanings to explain synonymy and then use synonymy in turn to show how the above sentence is a synonymous instance of a logical truth. The success of the above explanations requires assuming that there are such things as precisely characterizable meanings. Quine is sceptical of this assumption. He rejects three accounts of meanings:

(a) referential theories - meanings as referents;
(b) mentalism - meanings as ideas; and
(c) intensionalism - meanings as intensional entities.

Meaning (or sense), as Frege taught, must be distinguished from reference.

The notion of meaning that is to explain synonymy and analyticity cannot simply be reference, because co-referential terms need not be synonymous and so will not distinguish analytic sentences from true non-analytic ones. Truth in virtue of meaning where meaning is simply reference is too broad as all truths would then be analytic. As to meanings as ideas, Quine brings to bear empiricist and behaviourist qualms. He maintains that such intensional objects are neither required as posits by our theories of language, nor are they precisely accounted for. The attempt to explain intensional notions is either circular or unhelpful. There is a circle of intensional notions, meaning, synonymy, analyticity, necessity, and we can define one in terms of another. His criticism is that if we do not break out of this intensional circle, then the account has failed to clarify the matter. For example, if the meaning of a predicate ‘is human’ is the property of being human, how would one go about identifying whether ‘being a rational animal’ or ‘being a featherless
biped' stood for the same property or had the same meaning? One answer is that the sentence ‘humans are rational animals’ is analytic, while the sentence ‘humans are featherless bipeds’ is not. But this relies on the notion of ‘analytic’ - which we haven’t yet defined. Another approach at giving an identity condition uses modal notions and says that the first sentence is a necessary truth while the second is not. This, however, raises the problem of giving a precise account of modal notions. Explaining modal claims in terms of analyticity, for example, ‘Necessarily humans are rational’ as explained by “'Humans are rational" is analytic’, will not do - since we have not defined ‘analytic’. Quine’s challenge is that one break out of this intensional circle and explain notions from the theory of meaning in more acceptable terms.

(2) He next rejects accounts of analyticity in terms of logical truth and synonymy. On this account a sentence is analytic if it is a synonymous instance of a logical truth, that is, ‘All bachelors are unmarried men’ is analytic in that it is derivable from the first-order logical truth ‘All bachelors are bachelors’ by putting a synonym ‘unmarried man’ for the second occurrence of ‘bachelor’. One account attempts to explain synonymy in terms of definition. However, the various forms of definition either presuppose synonymy or stipulate it; none explain it. Quine is skeptical of definitions or philosophical analysis when thought of as capturing or analyzing some concept or meaning. Instead philosophical explication is thought of in terms of the theory of reference and scientific hypothesizing and thus again embodying the naturalistic theme of the continuity of science and philosophy. One does not capture ‘the meaning’ of an expression; one
explicates or proposes a theory of the referential features one is interested in preserving.

(3) Another attempt to define synonymy asserts that two expressions are synonymous if they are interchangeable. But it is not enough to say expressions are synonymous when the interchange of the one with the other within extensional contexts does not change the truth value of the sentences involved. This has the unacceptable consequence that merely coextensive terms would be synonyms. To do better one has to require interchangeability within intensional contexts. However, this raises the problem of breaking out of the circle of intensional notions.

(4) It consists in constructing an artificial language and then defining ‘analytic’ for it. While it is possible to construct a language and specify that relative to it logic, mathematics and such truths as ‘All bachelors are unmarried men’ and ‘Nothing is larger than itself’ are analytic, this language-relative specification of analyticity does not clarify matters. It does not help to be told that in one language, language 1 (artificial or otherwise), we have a list of sentences that are analytic 1, and that in another language, language 2, we have the list analytic 2 and so on. What we want of an explication of analyticity is an account of what analytic 1, analytic 2 and so on have in common. The appeal to artificial languages fails to provide this characterization. Moreover, the problem is precisely why ‘All bachelors are unmarried’ is on the list and ‘No bachelors are six-legged’ is not. To be told that a sentence is analytic because it is on a list (even the list of an artificial language) provides no real distinction.

(5) The last attempt to define analyticity that Quine considers appeals to the verification theory of meaning. According to
this theory, ‘the meaning of a statement is the method of empirically confirming or infirming it’; ‘statements are synonymous if and only if they are alike in point of method of empirical confirmation or infirmation’. Though sympathetic towards the empiricist thrust of this theory Quine does not think it survives the holistic criticism of the dogma of reductionism.

Quine’s critique of the theory of meaning has amounted to a challenge to provide precise accounts of its notions. What counts as precise could take the form of reducing intensional notions to extensional ones. His criticisms of modal concepts has spurred a generation of responses in what is known as possible world semantics, which in one of its variations can be seen as trying to provide a reduction of intensional modal notions via extensional metalinguistic truth conditions for modal statements. The success of this reduction is still challenged by Quinians. More in keeping with Quine’s challenge to explicate the theory of meaning is Davidson’s work on letting a Tarskian theory of truth serve as surrogate for a theory of meaning. Another way in which scepticism about the theory of meaning might be overcome would be by an empirical and behaviouristically constrained account of such notions. Carnap took up this challenge and sketched a programme for empirically identifying meanings by testing translation hypotheses, for example, a linguist’s hypotheses for translating the terms ‘Pferd’ from German to English as ‘horse’. Quine’s response was the topic of radical translation and his thesis of the indeterminacy of translation.