6. OPTIMAL RESERVE INVENTORY FROM ONE OUTPUT MACHINE TO TWO INPUT MACHINES WHEN INTER-ARRIVAL BREAKDOWN IS A RANDOM VARIABLE

6.1 INTRODUCTION

Many authors have addressed the optimal reserve inventory of semi finished products between two machines in series. The very first person who has derived a mathematical model for the optimal reserve inventory between two machines in series was Hanssmann F [20]. This model has been considered as basic model. But this basic model has some constraints and not suitable for all situations. This constraint is eliminated by Ramachandran V et. al [34] and have derived a modified mathematical model for the basic model. This modified basic model is the root cause for the development of so many mathematical models in the determination of optimal reserve inventory between two machines in series.

Following the modified basic model many authors have introduced the concept of treating the repair time or failure time of first machine as exponential distribution, erlang 2 distribution, gamma 2 distribution etc. Sachithanantham S et. al [39] have developed another model in which the repair time satisfies the Setting the Clock Back to Zero Property (SCBZ property). The SCBZ property was introduced by Raja Rao B et. al [32] and is the extension of memoryless property. The concept of order statistics for the repair time is applied and a formula is derived for optimal reserve inventory by Sehik Uduman P.S et. al [47] and Srinivasan S et. al [52] have established the possession of SCBZ property in the interarrival periods between breakdowns and have obtained the formula for optimal reserve between the machines. Srinivasan S et. al [50] have considered the breakdown time random variable undergoing change of distribution after a truncation point and have obtained the optimal reserve inventory by considering the various distributions for the repair time.
In a manufacturing industry, a semi finished product may undergo two or more stages for further processing to reach the stage of the finished product. So, there is an imperative need to keep reserve the inventory to ensure uninterrupted production and obviate losses arising from idle time during subsequent stages of production. In this chapter, a specific case is discussed in which two stages are considered with one machine in the first stage and two machines in the second stage such that the output of first stage machine is the input for second stage machines. A mathematical model is derived for a situation while a first stage machine is under halt or breakdown and to run the machines in the second stage. Generalized mathematical model is also obtained when there is one machine in the first stage whose output is in the input for ‘n’ machines in the second stage.

6.2 THE MODEL

A system with two stages is considered. In the first stage it is assumed that there is only machine $M_1$ and in the second stage as two machines say $M'_2$ and $M''_2$. The output of $M_1$ is the input for $M'_2$ and $M''_2$. The machines in the second stage may have same or different process types. During the breakdown time of the machine in the first stage a reserve inventory is maintained to ensure uninterrupted production in the next stage. This reserve inventory is needed as otherwise; the machines in the second stage may become idle which will impact not only the profits but also bring loss due to non-functioning of machines. The breakdown may occur due to many reasons like, equipment failure, quality related matters, material shortage, manpower shortage, maintenance problems etc. In this model, occurrence of breakdown due to equipment failure alone is considered. A mathematical model has been derived for obtaining reserve inventory by treating repair time and interarrival time as random variables.
6.3 DEPICTION OF THE MODEL

Figure 6.1: Reserve Inventory between Machines

6.4 NOTATIONS EMPLOYED IN THE MODEL

$S$ - level of reserve inventory

$\hat{S}$ - optimal reserve inventory

$X$ - a continuous random variable denotes the interarrival between breakdown time of $M_1$ whose pdf is $f(x)$ with cdf $F(x)$

$\mu$ - mean time interval between the successive breakdowns of machine $M_1$

$\frac{1}{\mu}$ - average number of breakdowns per unit time

$\tau$ - a random variable denotes the duration of breakdown / repair time $M_1$ and its pdf $g(.)$ with cdf $G(.)$

$h$ - holding cost per unit of reserve inventory

$d_i$ - cost per unit of idle time of machine $M_2^i$ for $i = 1$ to $2$

$r_i$ - consumption rate per unit time of machine $M_2^i$ for $i = 1$ to $2$
6.5 ASSUMPTIONS

- It is assumed that the reserve inventory level is at a constant level $S$ as long as no breakdowns occur, and shortages will be replenished to that level within a short time after each breakdown, perhaps by overtime production of machine $M_1$.

- It is assumed that the duration of the breakdown and the replenishment time taken together are small in comparison with the mean time $\mu$ between breakdowns.

- When $M_1$ breakdown then the supply to $M_2'$ and $M_2''$ are from the reserve inventory.

- The consumption rate of $M_2'$ and $M_2''$ are constant.

- The repair time of $M_1$ is a random variable.

6.6 IDLE TIME OF $M_2'$ AND $M_2''$

The idle time of machines $M_2'$ and $M_2''$ during a breakdown of machine $M_1$ will be,

$$
t = \begin{cases} 
0 & \text{if } \tau > \frac{S}{r_1 + r_2} \\
\tau - \frac{S}{r_1 + r_2} & \text{if } \tau > \frac{S}{r_1 + r_2}
\end{cases}
$$

Consequently, the expected cost of idle time per breakdown is

$$
(d_1 + d_2) \int_{\frac{S}{r_1 + r_2}}^{\infty} (\tau - \frac{S}{r_1 + r_2}) g(\tau) d\tau
$$

As there are $\frac{1}{\mu}$ breakdowns per unit time, the expected cost per unit time becomes
Differentiating (6.1) both sides w.r.t $S$ by using the Leibnitz’s rule of Differentiation of Integrals, we get

\[
\frac{dE(C)}{dS} = h - \frac{(d_1 + d_2)}{\mu(r_1 + r_2)} \int_{\frac{S}{r_1 + r_2}}^{\infty} g(\tau)d\tau
\]

(6.2)

Thus,

\[
\frac{dE(C)}{dS} = h - \frac{(d_1 + d_2)}{\mu(r_1 + r_2)} \left( 1 - G\left( \frac{S}{r_1 + r_2} \right) \right)
\]

(6.3)

To determine the optimal reserve inventory level $\hat{S}$, it is required to solve

\[
\frac{dE(C)}{dS} = 0
\]

(6.4)

Therefore, the optimal reserve inventory level $\hat{S}$ is given as

\[
G\left( \frac{\hat{S}}{r_1 + r_2} \right) = 1 - \frac{(r_1 + r_2)\mu h}{(d_1 + d_2)}
\]

(6.5)

If,

\[(r_1 + r_2)\mu h > (d_1 + d_2)\]

then no reserve inventory need to be held since one unit of idle time of machines $M_2'$ and $M_2''$ are less expensive than carrying $(r_1 + r_2)$ units of product between breakdowns.

### 6.7 MODIFIED EXPRESSION OF THE MODEL

From the expression for the optimal value of $\hat{S}$ given in (6.5), it is understood that it has a restriction that $\frac{(r_1 + r_2)\mu h}{(d_1 + d_2)} > 1$, otherwise the solution is not a feasible one. Hence, a modification in the expression of the Expected Total Cost $E(C)$ can be incorporated as given below:
Differentiating (6.6) both sides w.r.t. $S$ by using the Leibnitz’s rule of Differentiation of Integrals and equating $\frac{dE(C)}{dS} = 0$ for optimality, we get,

$$
G\left(\frac{\hat{S}}{r_1 + r_2}\right) = \frac{(d_1 + d_2)}{(d_1 + d_2) + h\mu (r_1 + r_2)} < 1
$$

(6.7)

Hence, this solution is without any restriction on the values of $d_1, d_2, r_1, r_2, h$ and $\mu$.

### 6.8 INTER-ARRIVAL TIME BETWEEN BREAKDOWNS AS RANDOM VARIABLE

In this model, the inter-arrival times between successive breakdowns is considered as a random variable, satisfying the extended memory less property so called as Setting the Clock Back Property (SCBZ property). It means that a random variable $X$ is said to possess the SCBZ property, if

$$
\frac{S(x + x_0, \theta_1, \theta_2)}{S(x_0, \theta_1)} = S(x, \theta_2)
$$

where $S(x, \theta)$ is the survivor function which is given as $S(x, \theta) = 1 - F(x, \theta)$ and $x_0$ be a truncation point and fixed such that,

- $X \sim f(x, \theta_1)$ when $X \leq x_0$
- $X \sim f(x, \theta_2)$ when $X > x_0$

Additional details of SCBZ property can be found from Raja Rao B et al [32]. This assumption is made here due to the fact that due to ageing of the machine $M_1$, the random variable $X$ denoting the inter-arrival between
successive breakdowns of $M_1$ may undergo a parametric change / change of distribution.

In this model, the random variable $X$ following the exponential distribution with parameter $\theta_1$ before $x_0$ and exponential distribution with parameter $\theta_2$ after $x_0$ are considered. Then, from equation (3.3) of Chapter 3 for the random variable $X$ with one change point, the pdf of $X$ can be given as follows,

$$f(x) = \begin{cases} 
\frac{1}{\theta_1} e^{-\frac{x}{\theta_1}} & \text{if } X \leq x_0 \\
\frac{1}{\theta_2} e^{x_0 \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)} e^{-\frac{x}{\theta_2}} & \text{if } X > x_0
\end{cases}$$

Now, the mean of $X$ is given as

$$E(X) = \int_0^{x_0} x \frac{1}{\theta_1} e^{-\frac{x}{\theta_1}} dx + \int_{x_0}^{\infty} x \frac{1}{\theta_2} e^{x_0 \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)} e^{-\frac{x}{\theta_2}} dx$$

(6.8)

On simplification,

$$E(X) = \theta_1 + (\theta_2 - \theta_1) e^{\frac{x_0}{\theta_1}}$$

(6.9)

Using (6.9) in (6.7), we get,

$$G\left(\frac{\hat{S}}{r_1 + r_2}\right) = \frac{(d_1 + d_2)}{(d_1 + d_2) + h \left( \theta_1 + (\theta_2 - \theta_1) e^{\frac{-x_0}{\theta_1}} \right) (r_1 + r_2)}$$

(6.10)

6.9 REPAIR TIME AS RANDOM VARIABLE

We know that generally the repair time of a machine cannot be constant. So, it can as well be taken as random variable. Now, we discuss here the optimality for the reserve inventory to be maintained by considering the repair time as distribution $G(.)$ as Uniform, Exponential and Proportional to the random variable.
6.9.1 Case (i)

When $G(.)$ follows a uniform distribution over $(0,a)$, the repair time of $M_i$ will be constant and is independent of time.

Hence, the holding cost will occur if, $a < \frac{s}{r_1 + r_2}$.

Whereas if, $a > \frac{s}{r_1 + r_2}$, then using the equation (6.10), the optimal reserve inventory is given as,

$$\hat{s} = \frac{a(d_1 + d_2)(r_1 + r_2)}{(d_1 + d_2) + h\mu(r_1 + r_2)}$$  \hspace{1cm} (6.11)

6.9.2 Case (ii)

When $G(.)$ follows an exponential distribution with parameter $\lambda$, using equation (6.10), the optimal reserve inventory is given as,

$$\hat{s} = \frac{(r_1 + r_2)}{\lambda} \log \left( \frac{(d_1 + d_2) + h\mu(r_1 + r_2)}{h\mu(r_1 + r_2)} \right)$$  \hspace{1cm} (6.12)

6.9.3 Case (iii)

When $g(y)$ is proportional to $y$, say

$$g(y) = \begin{cases} cy, & 0 \leq y \leq \sqrt{c}, \\ 0, & \text{otherwise} \end{cases}$$

then using equation (6.10), the optimal reserve inventory is given as

$$\hat{s} = (r_1 + r_2) \sqrt{\frac{2(d_1 + d_2)}{c((d_1 + d_2) + h\mu(r_1 + r_2))}}$$  \hspace{1cm} (6.13)
The following illustrations are taken up with proper units of an inventory system for this model.

### 6.9.4 NUMERICAL ILLUSTRATIONS

Changes in $\hat{s}$ when the holding cost $h$ changes and keeping the values of other parameters are fixed arbitrarily.

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$r_1$</th>
<th>$r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>3</td>
<td>100</td>
<td>100</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

The optimal reserve inventory size $\hat{s}$ to be maintained is shown as a graph when the repair time follows the three cases mentioned in the sections 6.9.1, 6.9.2 and 6.9.3 while the holding costs varies.

![Figure 6.2: Changes in $\hat{s}$ due to changes in $h$ when $\alpha = \lambda = c = 1.5$](image)

Figure 6.2: Changes in $\hat{s}$ due to changes in $h$ when $\alpha = \lambda = c = 1.5$
Figure 6.3: Changes in $\hat{S}$ due to changes in $h$ when $a = \lambda = c = 1$

Figure 6.4: Changes in $\hat{S}$ due to changes in $h$ when $a = \lambda = c = 0.5$
The following table shows the optimal reserve inventory sizes $\hat{s}$ for various values of $h$ verses case (i), case (ii) and case (iii).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$a = \lambda = c = 1.5$</th>
<th>$a = \lambda = c = 1$</th>
<th>$a = \lambda = c = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case (i)</td>
<td>Case (ii)</td>
<td>Case (iii)</td>
</tr>
<tr>
<td>10</td>
<td>5.455</td>
<td>4.331</td>
<td>4.924</td>
</tr>
</tbody>
</table>
6.9.5 NUMERICAL ILLUSTRATIONS

Changes in $\hat{S}$ when the idle time costs $d_1 = d_2$, change and keeping the values of other parameters fixed arbitrarily.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>20</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>2</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>3</td>
</tr>
<tr>
<td>$h$</td>
<td>20</td>
</tr>
<tr>
<td>$r_1$</td>
<td>3</td>
</tr>
<tr>
<td>$r_2$</td>
<td>2</td>
</tr>
</tbody>
</table>

The optimal reserve inventory size $\hat{s}$ to be maintained is shown as a graph when the repair time follows the three cases mentioned in the sections 6.9.1, 6.9.2 and 6.9.3 while the idle time costs $d_1$ and $d_2$ varies.

![Figure 6.5: Changes in $\hat{S}$ due to changes in $h$ when $a = \lambda = c = 1.5$](image_url)
Figure 6.6: Changes in $\hat{S}$ due to changes in $d_1$ and $d_2$ when $a = \lambda = c = 1$

Figure 6.7: Changes in $\hat{S}$ due to changes in $d_1$ and $d_2$ when $a = \lambda = c = 0.5$
The following table shows the optimal reserve inventory sizes $\hat{s}$ for various values of $d_1, d_2$ versus case (i), case (ii) and case (iii).

<table>
<thead>
<tr>
<th>$d_1, d_2$</th>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>2.000</td>
<td>1.352</td>
<td>3.333</td>
<td>1.667</td>
<td>2.027</td>
<td>4.082</td>
<td>0.833</td>
<td>4.055</td>
<td>5.773</td>
</tr>
<tr>
<td>200</td>
<td>5.000</td>
<td>3.662</td>
<td>4.714</td>
<td>3.333</td>
<td>5.493</td>
<td>5.773</td>
<td>1.667</td>
<td>10.99</td>
<td>8.165</td>
</tr>
<tr>
<td>350</td>
<td>5.833</td>
<td>5.014</td>
<td>5.092</td>
<td>3.889</td>
<td>7.520</td>
<td>6.236</td>
<td>1.944</td>
<td>15.04</td>
<td>8.819</td>
</tr>
</tbody>
</table>

### 6.9.6 NUMERICAL ILLUSTRATIONS

Changes in $\hat{s}$ when the parameter $\theta_1$ changes and keeping the values of other parameters fixed arbitrarily.

- $x_0 = 20$
- $d_1 = 100$
- $d_2 = 100$
- $\theta_2 = 3$
- $h = 20$
- $r_1 = 3$
- $r_2 = 2$

The optimal reserve inventory size $\hat{s}$ to be maintained is shown as a graph when the repair time follows the three cases mentioned in the sections 6.9.1, 6.9.2 and 6.9.3 while the parameter $\theta_1$ varies.
Figure 6.8: Changes in \( \hat{S} \) due to changes in \( \theta_1 \) when \( a = \lambda = c = 1.5 \)

Figure 6.9: Changes in \( \hat{S} \) due to changes in \( \theta_1 \) when \( a = \lambda = c = 1 \)
Figure 6.10: Changes in $\hat{S}$ due to changes in $\theta_1$ when $a = \lambda = c = 0.5$

The following table shows the optimal reserve inventory sizes $\hat{S}$ for various values of $\theta_1$ versus case (i), case (ii) and case (iii).

Table 6.3: Optimal reserve sizes when the parameter $\theta_1$ varies

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$a = \lambda = c = 1.5$</th>
<th>$a = \lambda = c = 1$</th>
<th>$a = \lambda = c = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{S}$</td>
<td>$\hat{S}$</td>
<td>$\hat{S}$</td>
</tr>
<tr>
<td>Case (i)</td>
<td>Case (ii)</td>
<td>Case (iii)</td>
<td>Case (i)</td>
</tr>
<tr>
<td>0.5</td>
<td>6.000</td>
<td>5.365</td>
<td>5.164</td>
</tr>
<tr>
<td>1.0</td>
<td>5.000</td>
<td>3.662</td>
<td>4.714</td>
</tr>
<tr>
<td>1.5</td>
<td>4.286</td>
<td>2.824</td>
<td>4.364</td>
</tr>
<tr>
<td>2.0</td>
<td>3.750</td>
<td>2.310</td>
<td>4.082</td>
</tr>
<tr>
<td>2.5</td>
<td>3.333</td>
<td>1.959</td>
<td>3.849</td>
</tr>
</tbody>
</table>
6.9.7 NUMERICAL ILLUSTRATIONS

Changes in $\hat{S}$ when the parameter $\theta_2$ changes and keeping the values of other parameters fixed arbitrarily.

| $x_0 = 20$ | $d_1 = 100$, | $d_2 = 100$ | $\theta_1 = 2$ | $\theta_2 = 3$ |

The optimal reserve inventory size $\hat{S}$ to be maintained is shown as a graph when the repair time follows the three cases mentioned in the sections 6.9.1, 6.9.2 and 6.9.3 while the parameter $\theta_2$ varies.

![Figure 6.11: Changes in $\hat{S}$ due to changes in $\theta_2$ when $a = \lambda = c = 1.5$](image)

Figure 6.11: Changes in $\hat{S}$ due to changes in $\theta_2$ when $a = \lambda = c = 1.5$
Figure 6.12: Changes in $\hat{S}$ due to changes in $\theta_2$ when $a = \lambda = c = 1$

Figure 6.13: Changes in $\hat{S}$ due to changes in $\theta_2$ when $a = \lambda = c = 0.5$
The following table shows the optimal reserve inventory sizes $\hat{s}$ for various values of $\theta_2$ verses case (i), case (ii) and case (iii).

Table 6.4: Optimal reserve sizes when the parameter $\theta_2$ varies

<table>
<thead>
<tr>
<th>$\theta_2$</th>
<th>$a = \lambda = c = 1.5$</th>
<th>$a = \lambda = c = 1$</th>
<th>$a = \lambda = c = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{s}$</td>
<td>$\hat{s}$</td>
<td>$\hat{s}$</td>
</tr>
<tr>
<td>Case (i)</td>
<td>2.5</td>
<td>2.5</td>
<td>1.25</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>3.75</td>
<td>3.466</td>
<td>6.931</td>
</tr>
<tr>
<td>Case (iii)</td>
<td>4.082</td>
<td>5.0</td>
<td>7.071</td>
</tr>
<tr>
<td>Case (i)</td>
<td>3.0</td>
<td>2.5</td>
<td>1.25</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>3.75</td>
<td>3.466</td>
<td>6.931</td>
</tr>
<tr>
<td>Case (iii)</td>
<td>4.082</td>
<td>5.0</td>
<td>7.071</td>
</tr>
<tr>
<td>Case (i)</td>
<td>4.5</td>
<td>2.5</td>
<td>1.25</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>3.75</td>
<td>3.466</td>
<td>6.931</td>
</tr>
<tr>
<td>Case (iii)</td>
<td>4.082</td>
<td>5.0</td>
<td>7.071</td>
</tr>
</tbody>
</table>

6.9.8 NUMERICAL ILLUSTRATIONS

Changes in $\hat{s}$ when the consumption rates $r_1 = r_2$, change and keeping the values of other parameters fixed arbitrarily.

| $x_0 = 20$ | $d_1 = 100$ | $d_2 = 100$ | $\theta_1 = 2$ | $h = 20$ | $r_1 = 3$ | $r_2 = 2$ |

The optimal reserve inventory size $\hat{s}$ to be maintained is shown as a graph when the repair time follows the three cases mentioned in the sections 6.9.1, 6.9.2 and 6.9.3 while the consumption rates $r_1$ and $r_2$ varies.
Figure 6.14: Changes in $\hat{S}$ due to changes in $r_1$ and $r_2$ when $a = \lambda = c = 1.5$

Figure 6.15: Changes in $\hat{S}$ due to changes in $r_1$ and $r_2$ when $a = \lambda = c = 1$
Figure 6.16: Changes in $\hat{S}$ due to changes in $r_1$ and $r_2$ when $a = \lambda = c = 0.5$

The following table shows the optimal reserve inventory $\hat{S}$ for various values of $r_1$, $r_2$ versus case (i), case (ii) and case (iii).

Table 6.5  Optimal reserve sizes for various consumption rates $r_1$ and $r_2$

<table>
<thead>
<tr>
<th>$r_1, r_2$</th>
<th>$a = \lambda = c = 1.5$</th>
<th>$a = \lambda = c = 1$</th>
<th>$a = \lambda = c = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{S}$</td>
<td>$\hat{S}$</td>
<td>$\hat{S}$</td>
</tr>
<tr>
<td>Case (i)</td>
<td>Case (ii)</td>
<td>Case (iii)</td>
<td>Case (i)</td>
</tr>
<tr>
<td>2</td>
<td>3.333</td>
<td>1.211</td>
<td>3.433</td>
</tr>
<tr>
<td>3</td>
<td>4.091</td>
<td>1.816</td>
<td>4.671</td>
</tr>
<tr>
<td>4</td>
<td>4.615</td>
<td>2.422</td>
<td>5.729</td>
</tr>
<tr>
<td>5</td>
<td>5.000</td>
<td>3.027</td>
<td>6.667</td>
</tr>
<tr>
<td>6</td>
<td>5.294</td>
<td>3.633</td>
<td>7.515</td>
</tr>
</tbody>
</table>

98
6.10 CONCLUSION

From the figures, graphs and tables of the above examples, the following conclusion can be drawn.

When the holding cost $h$ increases the optimum reserve inventory $\hat{S}$ decreases irrespective of the repair time distribution $G(\cdot)$ follows whether uniform / exponential / proportional to the random variable or not. This is quite reasonable. It is understandable that when the idle cost $d_1$, $d_2$ increase the optimum reserve inventory $\hat{S}$ increases. That is smaller inventory is recommended when the idle cost of machines is more. This is very sensible result.

It is observed from figures 6.7, 6.8 & 6.9 and from table 6.3, the value of $\theta_1$ increases which is the parameter of the repair time distribution of $M_1$, then the optimum reserve inventory $\hat{S}$ decreases which means that the average time to repair the machine $M_1$ decreases.

Similarly, from figures 6.10, 6.11 & 6.12 and from the table 6.4, the value of $\theta_2$ increases which is the parameter of the repair time distribution of $M_1$ after the truncation point $x_0$, the optimum reserve inventory $\hat{S}$ is constant which means the average time to repair the machine $M_1$ is constant.

Moreover, when the consumption rates $r_1, r_2$ increase, then the optimum reserve inventory $\hat{S}$ increases, that is a larger inventory is suggested while there is an increase in the consumption rate and the table 6.5 shows what amount of the optimal reserve inventory is to be maintained for various consumption rates.

6.11 APPENDIX: GENERALIZATION OF THE MODEL FROM ONE OUTPUT TO $n$ INPUT MACHINES

The concept of finding optimal reserve inventory from one output to two input machines can be extended to $n$ machines under the same assumptions made above.
This generalization can be depicted as follows:

Figure 6.17: Reserve Inventory from one output machine to \( n \) input machines

The idle time of machines \( M_i \) for \( i = 1 \) to \( n \) during a breakdown of machine \( M_1 \) will be,

\[
t = \begin{cases} 
0 & \text{if } \tau \leq \frac{S}{r_1 + r_2 + \cdots + r_n} \\
\tau - \frac{S}{r_1 + r_2 + \cdots + r_n} & \text{if } \tau > \frac{S}{r_1 + r_2 + \cdots + r_n}
\end{cases}
\]

Consequently, the expected cost of idle time per breakdown is

\[
(d_1 + d_2 + \cdots + d_n) \int_{\frac{S}{r_1 + r_2 + \cdots + r_n}}^{\infty} \left( \tau - \frac{S}{r_1 + r_2 + \cdots + r_n} \right) g(\tau) d\tau
\]

Thus, the expected cost per unit time becomes,

\[
E(C) = hS + \left( \frac{d_1 + d_2 + \cdots + d_n}{\mu} \right) \int_{\frac{S}{r_1 + r_2 + \cdots + r_n}}^{\infty} \left( \tau - \frac{S}{r_1 + r_2 + \cdots + r_n} \right) g(\tau) d\tau \tag{6.14}
\]

It follows that,

\[
\frac{dE(C)}{dS} = h - \left( \frac{d_1 + d_2 + \cdots + d_n}{\mu (r_1 + r_2 + \cdots + r_n)} \right) \int_{\frac{S}{r_1 + r_2 + \cdots + r_n}}^{\infty} g(\tau) d\tau \tag{6.15}
\]
Therefore, the optimal reserve inventory level $\hat{S}$ is given as

\[
G \left( \frac{S}{r_1 + r_2 + \cdots + r_n} \right) = 1 - \frac{(r_1 + r_2 + \cdots + r_n)\mu h}{(d_1 + d_2 + \cdots + d_n)}
\]  

(6.16)

If

\[
(r_1 + r_2 + \cdots + r_n)\mu h > (d_1 + d_2 + \cdots + d_n)
\]

then, no reserve inventory need be held since one unit of idle time of machines $M_i$ for $i = 1 \text{ to } n$ are less expensive than carrying $(r_1 + r_2 + \cdots + r_n)$ units of product between breakdowns.

### 6.12 MODIFIED EXPRESSION FOR THE GENERALIZED MODEL

The expression for the optimal value of $\hat{S}$ given in (6.8), is feasible only if $\frac{(r_1+r_2+\cdots+r_n)\mu h}{(d_1+d_2+\cdots+d_n)} > 1$, otherwise the solution is not a feasible one. Hence, an insignificant modification in the expression of the Expected Total Cost $E(C)$ can be incorporated as given below:

\[
E(C) = h(r_1 + r_2 + \cdots + r_n) \int_0^{\frac{S}{r_1+r_2+\cdots+r_n}} \left( \frac{S}{r_1 + r_2 + \cdots + r_n} - \tau \right) g(\tau) d\tau
\]

\[
+ \frac{(d_1 + d_2 + \cdots + d_n)}{\mu} \int_{\frac{S}{r_1+r_2+\cdots+r_n}}^{\infty} \left( \tau - \frac{S}{r_1 + r_2 + \cdots + r_n} \right) g(\tau) d\tau
\]

(6.17)

Now applying the Leibnitz’s rule of Differentiation of Integrals on (6.17) and equating $\frac{dE(c)}{ds} = 0$, the optimality is given by,

\[
G \left( \frac{S}{r_1 + r_2 + \cdots + r_n} \right) = \frac{(d_1 + d_2 + \cdots + d_n)}{(d_1 + d_2 + \cdots + d_n) + h\mu(r_1 + r_2 + \cdots + r_n)}
\]

(6.18)