CHAPTER VI

BENDING OF BEAMS

1. RESUME OF PREVIOUS WORK

In 'Introduction to Plasticity' by Hoffman and Sachs (1955) the solution of the case of bending of a beam of narrow rectangular cross-section is obtained with the help of flow theory. In 'Plasticity for Mechanical Engineers' by Johnson and Mellor (1962) the same problem is solved where the stress-strain law is non-linear. In this chapter a deformation theory solution of the problem has been obtained for Ramberg-Osgood (1943) material. The results are seen to tally exactly with the previous results for the limiting cases of perfectly plastic material and perfectly elastic material.

Stress distributions and the displacements in a wide curved bar subjected to pure bending have been obtained by Shaffer and House (1955, 1957) with the help of flow theory when the material is incompressible. Eason (1960) considered the same problem with the help of Tresca's yield condition and associated flow rule when the material is compressible. A deformation theory solution of this problem also is given here for incompressible material.

Several flow theory solutions of short beams with considerable effects of shear stress have been obtained by various authors viz. M.R. Horne (1951), E.T. Onat and R.T. Shield (1954), C.F.A. Leth (1954). However no such solution exists in
deformation theory. Here a numerical solution of this problem using Ramberg-Osgood stress-strain relation is given.

2. A POSSIBLE TRIVIAL SOLUTION FOR BENDING OF A PRISMATIC BEAM OF NARROW RECTANGULAR CROSS SECTION.

Let a long prismatic bar having a rectangular cross-section of width \( b \) and depth \( h \) be subjected to two equal and opposite end moments \( M \). The coordinate axes are chosen in such a way that the \( x \)-axis is in the direction in which strain is prevented (normal to the paper in figure 6.1), and the \( yz \)-plane coincides with the central plane of the beam. The \( y \)-axis is taken positive towards the convex side of the beam, which is bent under terminal couples \( \hat{M} \). Excluding two short portion near the ends, it can be expected that lines parallel to the edges change into concentric circular arcs. Assuming that the width of the beam is very small in comparison with its depth, it can be concluded that elongated prismatic elements, of width \( b \) and depth \( dy \) are deformed as if they would be isolated from the rest of the beam and undergo an axial strain, i.e.

\[
\epsilon_x = \frac{y}{R} \quad (6.1)
\]

where \( R \) is the radius of curvature of the central line of the beam. i.e. we are considering the radius of curvature very large compared to the thickness so that the induced transverse stresses in the \( y \)-direction can be neglected.

For an incompressible material

\[
\epsilon_x = \epsilon_y = - \frac{\epsilon_x}{2} = - \frac{y}{2R} \quad (6.2)
\]
Since \( \sigma_E = \sigma_Y = 0 \), only the stress \( \sigma_X \) remains. So the stress strain relation in simple \( J_2 \) deformation theory becomes:

\[
\epsilon_X = \frac{1}{E} \left[ 1 + \left( \frac{\sigma_X}{\sigma_e} \right)^{\gamma-1} \right] \sigma_X \quad (6.3)
\]

Now assuming the elastic strain is small compared to plastic strain we get from equation (6.3):

\[
\epsilon_X = \frac{1}{E} \cdot \frac{\sigma_X^\gamma}{\sigma_e^{\gamma-1}} \quad (6.4)
\]

Substituting the value of \( \epsilon_X \) from (6.1) in (6.4) we get:

\[
\sigma_X = \left( \frac{E \cdot Y}{R} \cdot \sigma_e^{\gamma-1} \right)^{\frac{1}{\gamma}} \quad (6.5)
\]

The radius of curvature \( R \) of the beam axis is calculated from the equilibrium equation which expresses that the sum of the moments about the \( z \)-axis of all \( \sigma_X \) stresses acting on a cross-section is equal to the applied moment \( M \):

\[
M = 2 \int_0^{h/2} \sigma_X \cdot b \cdot y \cdot dy \quad (6.6)
\]

Substituting the value of \( \sigma_X \) from (6.5) in (6.6):

\[
M = 2 \int_0^{h/2} \left( \frac{E}{R} \cdot \sigma_e^{\gamma-1} \right)^{\frac{1}{\gamma}} \cdot b \cdot y^{\frac{\gamma+1}{\gamma}} \cdot dy
\]

\[
= 2b \cdot \left( \frac{E}{R} \cdot \sigma_e^{\gamma-1} \right)^{\frac{1}{\gamma}} \cdot \left( \frac{h}{2} \right)^{\frac{\gamma+1}{\gamma}} \cdot \left( \frac{2\gamma+1}{\gamma} \right) \quad (6.7)
\]
For \( n = 1 \) i.e. for linear strain hardening material, (6.7) takes the form

\[
M = \frac{E b h^3}{12 R}
\]

which is same as in the elastic case.

For \( n = \infty \) i.e. for perfectly plastic material

\[
M = \frac{b \sigma_0 h^2}{4}
\]

which is the same as given by Hoffman and Sachs (1953).

3. BENDING OF WIDE CURVED BARS

In the previous section we have considered the strains so small that the transverse stresses induced by the curvature can be neglected. Also it was supposed that the neutral surface coincides with the central plane of the sheet throughout the distortion. In the present case we are considering that the strains are of any magnitude and also we do not impose any restriction on neutral surface. Here the only restriction is that the strains are negligible in the width direction i.e. we are considering the bending of a wide curved bar under conditions of plane strain when the material is incompressible.

Let \( r, \theta, z \) be the cylindrical coordinates.

Referring to fig. 6.2 the bar is assumed to be bounded by the areas of the circles \( r = a \) and \( r = b \) (\( b > a \)) and the lines \( \theta = \pm \alpha \).
It is further assumed that the only non-zero stresses are $\sigma_\tau$, $\sigma_\theta$, and $\sigma_z$ and that the stresses depend on $\tau$ alone. The equation of equilibrium is then
\[
\frac{d\sigma_\tau}{d\tau} + \frac{\sigma_\tau - \sigma_\theta}{\tau} = 0.
\] (6.8)

If the radial, circumferential and longitudinal components of displacements are called $u$, $\nu$ and $\omega$ respectively, where $\omega$ is zero because we are concerned with a problem of plane strain, then the expressions for strains in the polar coordinates may be written as
\[
\varepsilon_\tau = \frac{3u}{2\tau}, \quad \varepsilon_\theta = \frac{u}{\tau} + \frac{1}{\tau} \frac{3v}{\theta}, \quad \varepsilon_z = \frac{3\omega}{\tau} = \delta.
\] (6.9)

The stress-strain relations in simple $J_2$ deformation theory are
\[
\varepsilon_\tau = \frac{1}{E_\theta} \left( \sigma_\tau - \frac{1}{2} \sigma_\theta - \frac{1}{2} \sigma_z \right)
\]
\[
\varepsilon_\theta = \frac{1}{E_\tau} \left( \sigma_\theta - \frac{1}{2} \sigma_\tau - \frac{1}{2} \sigma_z \right),
\] (6.10)
\[
\varepsilon_z = \frac{1}{E_\sigma} \left( \sigma_z - \frac{1}{2} \sigma_\tau - \frac{1}{2} \sigma_\theta \right)
\]

Since $\varepsilon_z = 0$, we get $\sigma_z = (\sigma_\tau + \sigma_\theta)/2$. Therefore the above relations become
\[
\varepsilon_\tau = \frac{3}{4E_\theta} \left( \sigma_\tau - \sigma_\theta \right)
\]
\[
\varepsilon_\theta = \frac{3}{4E_\tau} \left( \sigma_\theta - \sigma_\tau \right),
\] (6.11)
where the secant modulus $E_s$ depends on the effective stress:

$$
\bar{\sigma} = \frac{\sqrt{3}}{2} (\sigma_0 - \sigma_r)
$$

and is given by

$$
\frac{1}{E_s} = \frac{1}{E} \left[ 1 + \frac{3}{7} \left( \frac{\sigma_0}{\sigma_r} \right)^{n-1} \right]
$$

The conditions to be satisfied by the stresses are

$$
\sigma_r = 0 \text{ at } r = a \text{ and } r = b
$$

$$
\int_a^b r \sigma_\theta \, dr = M
$$

Shaffer and House (1957) showed that for a wide curved bar with a zero shear stress, radial and circumferential components of displacements respectively become

$$
u = -\left( A/r + B \gamma \right) + k_1 \omega \theta
$$

$$
u = 2B \gamma \theta - k_1 \omega \theta
$$

where $A$, $B$, and $k_1$ are constants.

An examination of the above obtained results show that the curved bar will remain cylindrical and that planes will remain plane throughout the loading history.

With the help of (6.9) and (6.16) we get

$$
\epsilon_r = A/r^2 - B
$$

$$
\epsilon_\theta = -A/r^2 + B
$$
From (6.17) a relation between $\xi_\tau$ and $\xi_\theta$ is obtained as

$$\frac{d\xi_\theta}{d\tau} = \frac{\xi_\tau - \xi_\theta}{\tau} + \frac{2s}{\tau} \quad (6.18)$$

Now introducing the non-dimensional quantities

$$s = \frac{r}{a}, \quad \lambda_\tau = \frac{\xi_\tau}{M/a^2}, \quad \lambda_\theta = \frac{\xi_\theta}{M/a^2} \quad (6.19)$$

$$s = \frac{v}{M/a^2} = \frac{\sqrt{3}}{2} (\lambda_\theta - \lambda_\tau)$$

the equations (6.8), (6.11), (6.13), (6.14), (6.15) and (6.18) become

$$\frac{d\lambda_\tau}{ds} = \frac{\lambda_\theta - \lambda_\tau}{s} \quad (6.20)$$

$$\xi_\tau = \frac{3}{4} \frac{M/a^2}{E_\alpha} (\lambda_\tau - \lambda_\theta) \quad (5.21)$$

$$\xi_\theta = \frac{3}{4} \frac{M/a^2}{E_\alpha} (\lambda_\theta - \lambda_\tau)$$

$$\frac{1}{E_\alpha} = \frac{1}{E} \left[ 1 + \frac{3}{2} \frac{M}{a^2 \sigma_\tau} \left( \lambda_\tau \right)^{n-1} \right] \quad (6.22)$$

$$\lambda_\tau = 0 \quad \text{at} \quad s = 1 \quad \text{and} \quad s = 2 \quad (6.23)$$

$$\int_0^2 s \lambda_\theta \cdot ds = 1 \quad (6.24)$$

$$\frac{d\xi_\theta}{ds} = \frac{\xi_\tau - \xi_\theta}{s} \quad (6.25)$$
Substituting the value of $\xi$, $\xi_0$ from (6.21) in (6.25) and using (6.22) we get after certain simplifications,

$$
\frac{dA_0}{ds} = \frac{1}{2} \left[ \left( A + (A_0 - A_r)^{n-1} \right) (A_r - A_0) + \frac{4}{3} E A B \right] - \frac{dA_r}{ds} \tag{6.26}
$$

where

$$
A = \frac{7 \cdot (2/\sqrt{3})^{n-1}}{3 \cdot (M/a^2, \sigma_1)^{n-1}}
$$

For a particular value of $M/a^2, \sigma_1$ the stress distribution in the bar may be calculated from equations (6.20) and (6.26). Since $A_r = 0$ at $s = 1$ we can find $dA_r/ds$ at $s = 1$ if $A_0$ is known at $s = 1$. We assume some value for $A_0$ at $s = 1$ and find $dA_r/ds$ at that boundary. Then from (6.26) for a particular value of $B$ we find $dA_0/ds$ at $s = 1$. After that following some formula as given in appendix we find $A_r$ at $s = 2$. If it does not coincide with the given boundary conditions (6.23) we go back to the original position and assume some other value for $A_0$ and proceed as before until $A_r$ at $s = 1$ coincides with the given boundary condition (6.23). As soon as $A_r$ satisfies that condition we find whether the absolute value of the integral in (6.24) is equal to 1.

If it is not we assume some other value for $B$ and start from the beginning and go on doing like this until both the conditions (6.23) and (6.24) satisfy. When both the conditions are satisfied we know that is our solution.
Figure 6.3 and 6.4 show the variation of $\lambda$ with respect to $\varepsilon$, i.e., $(\varepsilon/a)$ for the cases $n=3, 5$ and 19 when $\kappa/(a^2\sigma_1) = 1$.

4. BENDING OF A CANTILEVER LOADED AT THE END

Let a cantilever having a narrow rectangular cross-section of unit width be bent by a force $P$ applied at the end (fig. 6.5). The upper and lower edges are free from load. Shearing traction having a resultant $T$ is distributed along the end $x = 0$.

In this case the only non-vanishing stresses are $\sigma_x$ and $T_{xy}$.

The equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0 \quad (6.28)$$

$$\frac{\partial T_{xy}}{\partial x} = 0$$

The compatibility equation is

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^2 T_{xy}}{\partial x \partial y} \quad (6.29)$$

The stress-strain relations in simple $T_x$ deformation theory are

$$\varepsilon_x = \frac{1}{3\nu} \left[ 1 + \left( \frac{T_x}{k^2} \right)^{n-1} \right] \sigma_x$$

$$\varepsilon_y = \frac{1}{3\nu} \left[ 1 + \left( \frac{T_y}{k^2} \right)^{n-1} \right] \left( -\frac{1}{2} \sigma_y \right) \quad (6.30)$$

$$\gamma_{xy} = \frac{1}{4} \left[ 1 + \left( \frac{T_{xy}}{k^2} \right)^{n-1} \right] T_{xy}$$
where \( J_2 = \frac{1}{3} (\sigma^2 x + 3 \sigma y^2) \), \( \sigma \) being the yield stress for the perfectly plastic case, and \( G \) the elastic rigidity.

If the corresponding stresses are defined in terms of a dimensionless stress function \( \phi \) according to the definitions

\[
\sigma_x = \kappa \frac{\partial \phi}{\partial y}
\]

\[
\sigma_y = -\kappa \frac{\partial \phi}{\partial x}
\]

then the first equation of (6.28) is automatically satisfied and from second we get

\[
\frac{\partial^2 \phi}{\partial x^2} = 0
\]

i.e., \( \phi = \kappa \phi(y) \)

where \( \phi(y) \) is any function of \( y \).

The boundary conditions are

\[
\phi = 0 \quad \text{at} \quad \kappa = 0
\]

and

\[
\phi = 0 \quad \text{at} \quad y = \pm c
\]

Also at \( y = 0 \) and \( \kappa = \lambda \) (length of the cantilever) \( \phi \) (6.35) is maximum.

Now with the help of equations (6.28), (6.29), (6.30), (6.31) we get after certain simplification,
If we neglect elastic strains then the above equation takes the form

\[
\begin{align*}
\Phi_{yy} & \left[ 3^{n-1} + (\Phi_y^2 + 3\Phi_x^2) \left( \Phi_y^2 + 3\Phi_x^2 \right) + 2(n-1)\Phi_y^4 \right] \\
+ \Phi_{xy} & \left[ 12(n-1)(\Phi_y^2 + 3\Phi_x^2) \cdot \Phi_y \cdot \Phi_x \right] \\
+ \Phi_{yy} & \left[ (\Phi_y^2 + 3\Phi_x^2) \left( 6(n-1)(\Phi_y^2 + 3\Phi_x^2) \cdot \Phi_y \\
- 2(n-1)(n-2)\Phi_y^3 \right) \\
+ 72(n-1)(n-2)\Phi_y \cdot \Phi_x^2 \right] \\
+ \Phi_x \cdot \Phi_{xy} \cdot \Phi_{yy} & \left[ (\Phi_y^2 + 3\Phi_x^2) \left( 18(n-1)(\Phi_y^2 + 3\Phi_x^2) \\
+ 36(n-1)(n-2)\Phi_y^3 \right) \right] \\
= 0 \\
\end{align*}
\]

\( (6.36) \)
The partial differential equation is solved numerically by superimposing a finite mesh upon the cross-section of the cantilever and replacing the derivatives by the usual centered differences as has been done by Greenberg, Dorn and Wetherell (1962) for the case of a torsion of a square cylinder.

Let \( \phi(i,j) \) represent the value of \( \phi \) at \( x = ih, y = jk \) where \( h \) is the distance between mesh lines in both the \( x \) and \( y \) directions. Then the derivatives are replaced by the following differences.

\[
\begin{align*}
\phi_x(i,j) &= A_1 = \frac{\phi(i+1,j) - \phi(i-1,j)}{2h} \\
\phi_y(i,j) &= A_2 = \frac{\phi(i,j+1) - \phi(i,j-1)}{2h} \\
\phi_{xy}(i,j) &= A_{12} = \frac{\phi(i+1,j+1) - \phi(i-1,j-1) + \phi(i-1,j+1) - \phi(i+1,j-1)}{4h^2} \\
\phi_{yy}(i,j) &= A_{22} = \frac{\phi(i,j+1) - 2\phi(i,j) + \phi(i,j-1)}{h^2} \\
\phi_{xyy}(i,j) &= A_{122} = \frac{\phi(i+1,j+1) + \phi(i+1,j-1) + 2\phi(i,j) - 2\phi(i+1,j) + \phi(i+1,j) - \phi(i,j+1) - \phi(i,j-1)}{h^3} \\
\phi_{yy}(i,j) &= A_{222} = \frac{\phi(i,j+2) - 3\phi(i,j+1) + 3\phi(i,j) - \phi(i,j-1)}{h^3}
\end{align*}
\]

The difference equation obtained from (6.37) replacing the derivatives by the above differences may be solved for \( \phi(i,j) \) as given by

\[
\phi(i,j) = F \left[ e^{-(m-1)(h+k+s)} \right] 
\]

(6.39)
where

\[ F = \frac{1}{c^3 + 6 (n-1) A_2 A_2 + 24 (n-1) A_1 A_2} \]

\[ E = \left\{ 3 \varphi (c', j+1) + \varphi (c', j-1) - \varphi (c', j+2) \right\} \left\{ c^2 + 2(n-1) A_2 A_2 \right\} \]

\[ + 12 (n-1) A_1 A_2 c \left\{ \varphi (c', j+1) + 2 \varphi (c', j+1) - \varphi (c', j+1) - \varphi (c', j-1) \right\} \]

\[ H = \left[ 6 c A_2 + 4 (n-2) (A_2)^2 \right] \left( A_2^2 \right)^2 \]

\[ R = \left[ 9 c A_2 - 12 (n-2) (A_2)^2 + 72 (n-2) A_2 (A_1)^2 \right] \left( A_2^2 \right) \]

\[ S = \left[ 9 c + 36 (n-2) (A_2)^2 \right] A_1 A_2 A_2 \]

and \( c = (A_2)^2 + 3 (A_1)^2 \)

To find the solution we assume some values of \( \varphi \) and compute \( \varphi (c', j) \) with those assumed values and since on the right hand side of (6.39) there is also \( \varphi (c', j) \), so we go on iterating at the same point until latest computed \( \varphi (c', j) \) becomes equal to the previous \( \varphi (c', j) \). Then we proceed to the next point and perform the same operation as before. Now since \( \varphi = \lambda \cdot \delta (y) \), it can be assumed that the values of \( \varphi \) on the line \( x = m \) will be \( m \) times the values of \( \varphi \) on \( x = k \). So that if we calculate \( \varphi \) on one line, the values of \( \varphi \) on other lines can automatically be found out. Also since \( \varphi \) goes on increasing on the \( y \)-axis starting from zero at \( y = e \) and becomes
maximum at \( y = c \) and then starts decreasing in the same proportion and becomes zero at \( y = -c \). i.e. since the values of \( \varphi \) are symmetrical about \( y = 0 \), we find the values of \( \varphi \) on one line starting from \( y = -c \) upto \( y = c \). Initially we assume some values of \( \varphi \) on three consecutive lines and find the values of \( \varphi(y, \xi) \) on the middle line in terms of its neighbouring points using the most recently computed values at those neighbouring points. After a sufficient number of iterations on this particular line, the values of \( \varphi(y, \xi) \) are assumed to converge to the solution of the difference equation. The iteration has been done at some distance from the free end to avoid local end effects.

Figure 6.6 shows the variation of \( \sigma_\alpha \) with respect to \( \frac{M}{\rho} \) for the case \( n = 2, 9 \).
DIMENSIONS IN BENDING OF PRISMATIC BAR OF NARROW RECTANGULAR CROSS-SECTION

Fig: 6.1
GEOMETRY OF THE WIDE CURVED BAR

Fig: 6.2
BAR BENDING WITH END MOMENTS WHEN $M/(a^2 \sigma) = 1$

Fig: 6.3

RADIAL STRESS DISTRIBUTION IN A WIDE CURVED BAR BENDING WITH END MOMENTS WHEN $M/(a^2 \sigma) = 1$

Fig: 6.3
CIRCUMFERENTIAL STRESS DISTRIBUTION IN A WIDE CURVED BAR BENDING WITH END MOMENTS WHEN $M/(\sigma_1)=1$

Fig: 6.4
CANTILEVER LOADED AT THE END
Fig: 6.5

STRESS DISTRIBUTION ACROSS THE PLATE OF A RECTANGULAR BEAM
Fig: 6.6