CHAPTER III

SIMPLE - PLANE STRESS PROBLEM

1. RESUME OF PREVIOUS WORK

The problem of finite expansion of a circular hole in a thin infinite plate has been discussed repeatedly in the literature. Taylor (1948) obtained the solution of the problem by using Tresca's Yield condition and the flow rule associated with Mises's yield condition for a rigid perfectly plastic material. Hill (1949) extended Taylor's work for an elastic perfectly plastic material. Prager (1955) solved the problem for a rigid strain hardening material which satisfies Tresca's initial yield condition and the associated flow rule for a certain finite value of the pressure. Alexander and Ford (1954) obtained the solution of the same problem by using Mises's yield condition and the associated Prandtl-Reuss relations. Hodge and Santhanarayanan (1958) dealt with the same problem treated by Prager and obtained the solution for all values of the internal pressure.

In this chapter the solutions for stresses in plane stress problems have been obtained with the help of deformation theory. The results are in closed form for the cases when elastic strains are neglected in Ramberg-Osgood material.

2. BASIC EQUATIONS

Using polar coordinates, the stress equilibrium for an angular section of the plate is

\[
\frac{d}{dr} \left( hr \sigma_r \right) = \frac{h_r}{r} \left( \sigma_\theta - \sigma_r \right)
\]

(3.1)
where \( h_r \) is the thickness of the plate at a radius \( r \).

The strains are given by

\[
\epsilon_r = \frac{du}{d\gamma}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{d\omega}{dz}
\]

Eliminating \( u \) between first two equations we get

\[
\frac{d\epsilon_\theta}{d\gamma} = \frac{\epsilon_r - \epsilon_\theta}{r} \quad \text{(3.3)}
\]

The stress strain relations in the simple J2 deformation theory of plasticity for plane stress problems are

\[
\epsilon_r = \frac{1}{E_0} \left( \sigma_r - \frac{1}{2} \sigma_\theta \right)
\]

\[
\epsilon_\theta = \frac{1}{E_0} \left( \sigma_\theta - \frac{1}{2} \sigma_r \right) \quad \text{(3.4)}
\]

\[
\epsilon_z = -\frac{1}{E_0} \left( \frac{\sigma_r + \sigma_\theta}{2} \right)
\]

The dependence of the secant modulus \( E_0 \) on the effective stress

\[
\bar{\sigma} = \left( \sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2 \right)^{\frac{1}{2}} \quad \text{(3.5)}
\]

may be taken as

\[
\frac{1}{E_0} = \frac{1}{E} \left[ 1 + \frac{3}{7} \left( \frac{\sigma_0}{\bar{\sigma}} \right)^{n-1} \right] \quad \text{(3.6)}
\]

Eliminating the space variables in the above equations by introducing the same nondimensional quantities as were used by Budiansky and Mangasarian (1960),

\[
\xi = (\alpha/\gamma)^2, \quad \Delta_r = \sigma_r/\sigma_0, \quad \Delta_\theta = \sigma_\theta/\sigma_0
\]

\[
\bar{\sigma} = \frac{\sigma}{\sigma_0} = (\Delta_r^2 - \Delta_r \Delta_\theta + \Delta_\theta^2)^{1/2} \quad \text{(3.7)}
\]
where \( a \) is a characteristic dimension of the plate and \( \sigma_0 \) is a given constant boundary traction, we obtain

\[
\frac{1}{h_\gamma} \frac{dh_\gamma}{ds} = - \left[ \frac{\Delta \gamma - \frac{d \gamma}{ds}}{\Delta \gamma} + 2 \epsilon \frac{d A \gamma}{ds} \right] / 2 \epsilon. \quad (3.8)
\]

\[
- 2 \epsilon \frac{d \epsilon_\gamma}{ds} = \epsilon_\gamma - \epsilon_\phi \quad (3.9)
\]

\[
\epsilon_\gamma = \frac{\sigma_0}{E_\phi} \left( \Delta \gamma - \frac{1}{2} \Delta \phi \right)
\]

\[
\epsilon_\phi = \frac{\sigma_0}{E_\phi} \left( \Delta \phi - \frac{1}{2} \Delta \gamma \right) \quad (3.10)
\]

\[
\epsilon_\sigma = - \frac{\sigma_0}{E_\phi} \left( \frac{\Delta \phi + \Delta \gamma}{2} \right)
\]

\[
\frac{1}{E_\phi} = \frac{1}{E} \left[ 1 + \frac{3}{T} \left( \frac{\sigma_0}{E_\phi} \right)^{n-1} \right] \cdot \Delta^{n-1} \quad (3.11)
\]

From equations (3.8), (3.9) and (3.10) and using \( \frac{d \epsilon_\gamma}{ds} = \frac{1}{h_\gamma} \frac{dh_\gamma}{ds} \), we obtain after suitable manipulation

\[
\frac{\sigma_0}{E_\phi} \left[ \frac{d A \gamma}{ds} - \Delta \gamma \frac{d \epsilon_\gamma}{ds} \left( \frac{\sigma_0}{E_\phi} \right)^{n-1} \right] + \frac{\sigma_0}{E_\phi} \left[ \frac{1}{E_\phi} \left( 2 \epsilon \frac{d \gamma}{ds} - \Delta \gamma \right) \right] = 0. \quad (3.12)
\]

With the tentative assumption that \( \Delta \) is a monotonic function of \( \gamma \) as will be confirmed later from the final results, we obtain by division of equation (3.12) by \( d \Delta / ds \) and cancellation of the common factor \( \gamma \)

\[
\frac{3}{E_\phi} \left[ \frac{d A \gamma}{d \Delta} - \Delta \gamma \frac{d \epsilon_\gamma}{d \Delta} \left( \frac{\sigma_0}{E_\phi} \right)^{n-1} \right] + \frac{d \gamma}{ds} \frac{1}{E_\phi} \left( 2 \epsilon \frac{d \gamma}{ds} - \Delta \gamma \right) = 0. \quad (3.13)
\]

3. SOLUTION NEGLECTING ELASTIC STRAINS IN RALBERG-OGGOOD MATERIAL

The analytical work for solutions of specific problems is further simplified by introducing a non-dimensional loading.
factor
\[ \lambda = \frac{\sigma_0}{\xi} \]  \hspace{1cm} (3.14)

and the transformation and parametric representations (Nadai, 1950)
\[ \frac{\lambda_0 + \lambda_\gamma}{2} = \xi = \bar{\sigma} \sin \phi \]  \hspace{1cm} (3.15)
\[ \frac{\lambda_0 - \lambda_\gamma}{2} = \eta = \frac{\bar{\sigma}}{\sqrt{3}} \cos \phi \]

The governing equation (3.13) then reduces to
\[ \left[ 4\lambda - \frac{3(\eta - \bar{\eta}) \sigma_0}{E} \right] \left( \lambda + \lambda_0 \right) \frac{dz}{d\sigma} \]
\[ + (\eta - \bar{\eta}) \lambda^{-1} \left[ \frac{3\sigma_0}{E} \left( \lambda + \lambda_0 \right) + (\lambda + 3\bar{\eta}) \lambda \right] \]
\[ = 0 \] \hspace{1cm} (3.16)

where \[ \lambda = \frac{7}{3 \lambda_0^2} \]  \hspace{1cm} (3.17)
and \[ \lambda = \left( \lambda_0^2 + 3\eta \right)^2 \]  \hspace{1cm} (3.18)

If the applied stress \( \sigma_0 \) be very small \( \lambda \) tends to zero for large values of \( \eta \) and \( A \) tends to infinity, therefore, the equation (3.16) reduces to
\[ \frac{d \xi}{d \sigma} = 0 \]  \hspace{1cm} (3.19)

On the other hand if the applied stress \( \sigma_0 \) be greater than the critical stress \( \sigma_c \), then \( A \) becomes negligibly small for high values of \( \eta \). Even for lower values of \( \eta \), \( A \) tends to zero for high values of \( \sigma_0 \), i.e., we can neglect elastic strains compared to plastic strains. In this case the solution becomes a power law solution while the equation (3.16) becomes
Using the parametric representation shown in the equation (3.15) and integrating we get

\[ \tilde{A} = \frac{c}{(\sin \phi)^{\alpha}} \]  

(3.21)

Where \( c \) is an integration constant.

Substituting this value of \( \tilde{A} \) and those of \( A_r, A_0 \) calculated from (3.15) in terms of \( \phi \) in compatibility equation (3.9) we get after some algebraic manipulations

\[ -\sqrt{3} \tilde{A} (A + \tilde{A}^{n-1}) \cos \phi + 2 \tilde{A} (A + \tilde{A}^{n-1}) \frac{d}{ds} \left[ \frac{\tilde{A}}{2} (4 \omega \phi + \sqrt{3} \cos \phi) \right] + 2 \tilde{A} \frac{d}{ds} \left[ \frac{d}{ds} (4 \omega \phi + \sqrt{3} \cos \phi) \right] (n-1) \tilde{A}^{-2} \frac{d \tilde{A}}{ds} = 0. \]  

(3.22)

When \( A \) is negligibly small i.e. for power law solution the above equation reduces to

\[ \frac{d \phi}{s} = -\frac{d \phi}{\tilde{S} \sin \phi \cos \phi} \]  

(3.23)

which can be easily integrated to give

\[ S = (a/\gamma)^2 = \frac{D}{l \omega \phi} \]  

(3.24)

where \( D \) is an integration constant.

From equation (3.24) we get a relation between \( S \) and \( \phi \).
and equation (3.21) gives a relation between $\theta$ and $\phi$; ultimately we get $\theta$ in terms of $\phi$. Now from equation (3.15), (3.21) and (3.24) we may express the non-dimensional stresses as

$$A_\tau = \frac{1-n}{2n} \left( \frac{\sqrt{3} \Delta - \xi}{\sqrt{3} \Delta} \right)$$

$$A_\phi = \frac{1-n}{2n} \left( \frac{\sqrt{3} \Delta + \xi}{\sqrt{3} \Delta} \right)$$

(3.25)

Further putting the values of $A_\tau$, $A_\phi$ from equation (3.25) into the equation (3.8) and simplifying the resulting differential equation, we get the value of $\lambda_\tau$ after integration as

$$\lambda_\tau = F \left( \frac{\Delta^2 + \xi^2}{2n} \right)$$

(3.26)

where $F$ is an arbitrary constant.

**EXAMPLE 1**: The case of an infinite plate with a circular hole under biaxial tension at infinity.

The boundary conditions are

$$\sigma_\tau = \sigma_\phi = \sigma_\infty \text{ at } \tau = \infty$$

(3.27 a)

$$\sigma_\tau = 0 \text{ at } \tau = a$$

(3.27 b)

The initial condition on the stress plane corresponding to the stress state (3.27 a) at infinity of the physical plane may be represented by

$$A_\tau = A_\phi = 1, \quad \lambda = 1 \text{ at } \xi = 0$$

(3.28)
according to the definition given by equations (3.7) in which
gets the value \( \varepsilon_{\infty} \). Thus at \( \phi = 0 \) we have from (3.15)
\[
\zeta = 1, \quad \eta = 0.
\]  
(3.29)

Similarly, from condition (3.27 b) at the hole boundary
we get
\[
A_r = 0, \quad A_\theta = \beta = k, \quad \zeta = \eta = \frac{k}{2}
\]  
(3.30)
where \( k = \sigma_{\infty}/\sigma_0 \) is the stress concentration factor.

When the factor \( A \) has a very high value then the degenerated
equation (3.19) holds, and the complete solution is given by
\[
\zeta = 1, \quad \eta = \left(\frac{\beta^2 - 1}{3}\right)^{\frac{1}{2}}
\]  
(3.31)
Consequently, the condition (3.30) yields \( k = 2 \). The result is
the same for an elastic material because the stress strain relation
in both cases are linear.

When \( A \) is very small, equation (3.20) holds. Hence the
initial condition is \( \beta = 1 \) at \( \phi = \pi/2 \), corresponding to the conditions at infinity in the physical plane. Therefore,
the integration constant \( C = 1 \).

(3.32)
The condition (3.27 b) at the hole corresponds to \( \beta = k \) at \( \phi = \pi/6 \).
Substituting this condition in equation (3.21) and putting \( C = 1 \), we get the stress concentration factor
\[
k = 2^\frac{1}{k}
\]  
(3.33)
For a perfectly plastic material when \( n = \infty \), we have \( k = 1 \)
and for a linearly strain hardening material when \( \eta = 1 \), we have \( k = 2 \). These results are the same as obtained by Budiansky and Mangasarian (1960) neglecting the effects of change in thickness.

Now applying the boundary condition \( \phi = \pi / 6 \) at the hole boundary i.e. \( \delta = 1 \), we calculate the constant of integration

\[
D = \frac{1}{\sqrt{3}}
\]

from equation (3.24)

On substituting the values of \( c \) and \( D \) in expressions (3.25) and (3.26) we get, \( A_\gamma \), \( A_\theta \), \( h_\gamma \) as

\[
A_\gamma = (1 - q) (1 + 3q^2)^{\frac{1 - \eta}{2\eta}}
\]

\[
A_\theta = (1 + q) (1 + 3q^2)^{\frac{1 - \eta}{2\eta}}
\]

\[
h_\gamma = F \left( q^2 + \frac{1}{3} \right)^{\frac{\eta - 1}{2\eta}}
\]

Figures 3.1, 3.2, 3.3 show the variations of \( A_\gamma \) and \( A_\theta \) with \( \delta \) and of \( h_\gamma / h_\alpha \) with \( \delta \) for the values of \( \eta \) equal to 3, 9, 19.

EXAMPLE 2: The case of a finite circular plate bounded by a rigid wall and having a concentric circular hole under an uniform radial pressure.

The boundary conditions are

\[
\sigma_\gamma = - \sigma_\alpha \quad \text{at} \quad \gamma = \alpha
\]

(3.37 a)

\[
\epsilon_\gamma = 0 \quad \text{at} \quad \gamma = \beta
\]

(3.37 b)

If \( \sigma_\theta \) be the normal stress at the boundary \( \gamma = \frac{b}{2} \), the condition (3.37 b) requires
\((A_y)_b = \frac{\sigma_b}{\alpha} = \mu\)
\((A_\theta)_b = 2 \cdot \frac{\sigma_\theta}{\alpha} = 2\mu\) \hspace{1cm} (3.38)

and \((\delta)_b = \sqrt{3}\mu\).

Hence the conditions (3.37 a) and (3.37 b) may be written as

\(\Delta Y = -1, \quad A_\theta = k, \quad \delta = \sqrt{1+k+k^2}\)

\(\phi = \sin^{-1}\left\{ \frac{1}{2} \frac{k-1}{\sqrt{1+k+k^2}} \right\} = \cos^{-1}\left\{ \frac{\sqrt{3}}{2} \frac{k+1}{\sqrt{1+k+k^2}} \right\}\) \hspace{1cm} (3.39)

For \(\gamma = a\) i.e. \(\xi = 1\), where \(k = \sigma_{\theta a}/\sigma_a\).

and

\(\Delta Y = \mu, \quad A_\theta = 2\mu, \quad \delta = \sqrt{3}\mu\) \hspace{1cm} (3.30)

\(\phi = \frac{\pi}{3}\)

for \(\gamma = b\) i.e. \(\xi = (a/b)^2\)

Using these boundary values we may evaluate the constants \(c\) and \(D\) as

\(c = \sqrt{1+k+k^2} \left\{ \frac{1}{2} \frac{k-1}{\sqrt{1+k+k^2}} \right\} \frac{1}{\delta} = \sqrt{3}/\mu \cdot \left( \frac{\sqrt{3}}{2} \right)\) \hspace{1cm} (3.41)

\(D = \sqrt{3}.(a/b)^2\) \hspace{1cm} (3.42)

The value of \(D\) at \(\gamma = a\) i.e. \(\xi = 1\) is

\(D = \Delta a \phi = \frac{k-1}{\sqrt{3}.(k+1)}\) \hspace{1cm} (3.43)

So from (3.42) and (3.43) we get

\(k = \frac{b^2 + 3a^2}{b^2 - 3a^2}\)\hspace{1cm} (3.44)
Hence for a particular $b$, $k$ is known from (3.44) and $c$ is known from (3.41).

The values of $A_\tau$, $A_\theta$, $A_r$ in terms of $\zeta$ then becomes:

\[
A_\tau = \frac{c}{\sqrt{3}} \cdot \frac{1-n}{2n} \cdot (D^2+\xi^2) \cdot (\sqrt{3}D - \zeta) \tag{3.45}
\]
\[
A_\theta = \frac{c}{\sqrt{3}} \cdot \frac{1-n}{2n} \cdot (D^2+\xi^2) \cdot (\sqrt{3}D + \zeta)
\]

and
\[
A_r = \frac{c}{\sqrt{3}} \cdot \frac{n-1}{2n} \tag{3.46}
\]

Figures 3.4, 3.5 and 3.6 show the variation of $A_\tau$, $A_\theta$ and $A_\gamma/h_a$ with $\zeta$ for the values of $\eta$ equal to 3, 9, 19 in the case of $b = 2a$.

**EXAMPLE 3:** The case of finite plate with a circular hole at the centre subjected to pressure at the hole.

The boundary conditions are:

\[
\sigma_\tau = -\sigma_a \quad \text{at} \quad \tau = a \tag{3.47 a}
\]
\[
\sigma_\tau = 0 \quad \text{at} \quad \tau = b \tag{3.47 b}
\]

The initial condition on the stress plane corresponding to the stress state (3.47 a) at the hole boundary of physical plane may be represented by

\[
A_\tau = -1, \quad A_\theta = k, \quad A_r = \left(1 + k + k^2\right)^{\frac{1}{2}}
\]

and

\[
\phi = \sin^{-1}\left(\frac{1}{2} \cdot \frac{k-1}{\sqrt{1+k+k^2}}\right) = \cos^{-1}\left(\frac{k+1}{\sqrt{1+k+k^2}}\right)
\]

at $\zeta = 1$, where $k = \sigma_\theta / \sigma_a$.
according to the definitions given by equations (3.7) in which
\( \sigma_0 \) gets the value \( \sigma_a \).

Similarly, from condition (3.47 b) at the outer boundary we get

\[ A_r = 0, \quad A_0 = \frac{A}{a}, \quad \phi = \frac{x}{b} \quad \text{at} \quad \xi = \left( a/b \right)^2 \]  (3.49)

where \( \lambda' = \sigma_b / \sigma_a \).

Using the above boundary conditions we may evaluate the constants \( c \) and \( k \) as

\[ c = \left( \frac{k-1}{2} \right) \left( 1 + k + k^2 \right)^{\frac{n-1}{2n}} = \lambda' \left( \frac{1}{n} \right) \]  (3.50)

\[ d = \frac{1}{\sqrt{3}} \left( \frac{a}{b} \right)^2 \]  (3.51)

Further, at \( r = a \)

\[ k = \frac{k-1}{\sqrt{3} (k+1)} \]  (3.52)

Hence \( c \) can be calculated in terms of \( a \) and \( b \) from equation (3.50); and \( A_r \), \( A_0 \), and \( k' / k_a \) can be evaluated in terms of \( \xi \) from (3.45) and (3.46).

Figures 3.7, 3.8 and 3.9 show the variations of \( A_r \), \( A_0 \) and \( k' / k_a \) with \( \xi \) for the values of \( n \) equal to 3, 9, 19 in the case of \( a = 2 \).

In case of an infinite plate i.e. if \( b \) becomes infinity we get from equation (3.52)

\[ k = 1 \]
and hence from (3.50)
\[ \mu = 0. \]

Thus \( \sigma_0 \) at infinity is zero. Since \( \sigma_0 \) also is zero at this boundary, the case becomes the same as an infinite plate with an uniform pressure at the hole boundary.

For \( \eta = 1 \),
\[ \lambda_\tau = \frac{a^2}{r^2 - a^2} \left( 1 - \frac{4a^2}{r^2} \right) \]
\[ \lambda_\theta = \frac{a^2}{r^2 - a^2} \left( 1 + \frac{4a^2}{r^2} \right) \]
which is exactly same as in the case of an elastic solution.

**EXAMPLE 4:** The case of a finite plate with a hole at the centre subjected to uniform radial traction at the outer boundary.

The boundary conditions are
\[ \sigma_\tau = 0 \quad \text{at} \quad \tau = a \]  \hspace{1cm} (3.53 a)
\[ \sigma_\tau = \sigma_0 \quad \text{at} \quad \tau = b \]  \hspace{1cm} (3.53 b)

The conditions (3.53 a) and (3.53 b) may be written as
\[ \lambda_\tau = 0, \quad \lambda_\theta = \frac{3}{4} \tau = k, \quad \phi = \frac{3}{6} \quad \text{at} \quad \xi = 1 \]  \hspace{1cm} (3.54)
where \( k = \frac{\phi_0}{\sigma_0} \)
and
\[ \lambda_\tau = 1, \quad \lambda_\theta = \lambda \quad \text{at} \quad \tau = (1 - \mu + \mu^2)^{-1} \]
\[ \phi = \lambda \omega^{-1} \left\{ \frac{1}{2} \cdot \frac{\lambda + 1}{\sqrt{1 - \mu + \mu^2}} \right\} = \omega^{-1} \left\{ \frac{\sqrt{3}}{2} \cdot \frac{\lambda - 1}{\sqrt{1 - \mu + \mu^2}} \right\} \]  \hspace{1cm} (3.55)
With the above conditions the values of \( C \) and \( D \) become

\[
C = k \left( \frac{1}{\beta} \right)^\frac{1}{\eta} = \left( \frac{\mu + 1}{\mu} \right) \left( 1 - \mu + \mu^2 \right) \frac{\eta - 1}{2\eta} \tag{3.56}
\]

\[
D = \frac{1}{\sqrt{3}} \tag{3.57}
\]

Again at \( r = b \),

\[
D = \frac{\mu + 1}{\sqrt{3} \cdot (\mu - 1)} (a/b)^2 \tag{3.58}
\]

Therefore, \( \mu = \frac{\beta^2 + a^2}{\beta^2 - a^2} \) (3.58)

Hence \( C \) can be calculated in terms of \( a \) and \( b \) from equation (3.56), and \( \lambda_r \), \( \lambda_\theta \), and \( \kappa / \lambda_\alpha \) can be evaluated in terms of \( \beta \) from (3.45) and (3.46).

Figures 3.10, 3.11 and 3.12 show the variations of \( \lambda_r \), \( \lambda_\theta \) and \( \kappa / \lambda_\alpha \) with \( \beta \) for the various values of \( \eta \) in the case of \( \beta = 2a \).

When the plate becomes infinite i.e. \( b \) tends to infinity in that case \( \mu \) becomes equal to 1 and hence from (3.56) we get,

\[
\kappa = \frac{1}{\eta}
\]

which tallies with the result of example 1.

4. **SOLUTIONS WITH ELASTIC STRAINS IN RAMBERG-OSGOOD MATERIAL**

For values of \( A \) other than zero we shall have to
find the stresses numerically, for which the following procedure may be adopted.

We get from equations (3.8), (3.11), (3.15) and third equation of (3.10)

\[
\left[ 2 \pi \Delta \sigma_0 (A + \xi^{-1}) \frac{\Delta \sigma \phi}{(\cos \phi - \frac{\cos \varphi}{\sqrt{3}})} + \frac{2 \pi \Delta \sigma_0 (n-1) \Delta \sigma \phi}{(\cos \phi - \frac{\cos \varphi}{\sqrt{3}})^2} \right] \frac{d \Delta \sigma \phi}{d s} \\
+ \left[ 2 \pi \Delta \sigma_0 (A + \xi^{-1}) \frac{\Delta \sigma \phi}{(\cos \phi - \frac{\cos \varphi}{\sqrt{3}})^2} - \frac{\Delta \sigma \phi}{\frac{\cos \varphi}{\sqrt{3}}} \right] \frac{d \phi}{d s}
\]

\[= A E \left[ \frac{2 \pi d \Delta \sigma \phi}{d s} + \frac{2 \pi \frac{\cos \phi}{\sqrt{3} (\cos \phi - \frac{\cos \varphi}{\sqrt{3}})}} \right] (3.59)\]

Also from equations (3.9), (3.10), (3.11) and (3.15) we get

\[
\left[ \frac{\Delta \sigma \phi + \sqrt{3} \cos \phi}{(\Delta \sigma \phi - \frac{\cos \varphi}{\sqrt{3}})} \right] \frac{d \Delta \sigma \phi}{d s} \\
- \frac{2 (A + \xi^{-1}) \Delta \sigma \phi}{\sqrt{3} (\Delta \sigma \phi - \frac{\cos \varphi}{\sqrt{3}})} \frac{d \phi}{d s}
\]

\[= - \frac{\sqrt{3} (A + \xi^{-1}) \frac{d \Delta \sigma \phi}{d s}}{(\Delta \sigma \phi - \frac{\cos \varphi}{\sqrt{3}})} (3.60)\]

where \(A = \frac{r}{3 \xi^{-1}}, \ \xi = \frac{\sigma_0}{\sigma_1} \) and \(\xi = \frac{\Delta \sigma \phi}{\Delta \sigma \phi - \frac{\cos \varphi}{\sqrt{3}}} \).
Now substituting the value of $\frac{d\phi}{d\xi}$ in terms of $\frac{d\alpha}{d\xi}$ from (3.59) into equation (3.60) we get an equation of $\frac{d\alpha}{d\xi}$ in terms of $\phi$, $\alpha$ and $\gamma$. Since at the boundaries we know $\phi$, $\alpha$ and $\gamma$, we can find $\frac{d\alpha}{d\xi}$ at the hole boundary and then $\frac{d\phi}{d\xi}$ at that boundary from (3.59) and then proceeding as in the article rotating solid disc of Chapter IV we can find the stresses for particular values of $\frac{\sigma_0}{\sigma_1}$ and $E$. 
FIG. 3-1—RADIAL STRESS DISTRIBUTION IN AN INFINITE PLATE WITH A CIRCULAR HOLE AT THE CENTRE SUBJECTED TO EQUAL BIAXIAL TENSION AT INFINITY.
**FIG 3.2** CIRCUMFERENTIAL STRESS DISTRIBUTION IN AN INFINITE PLATE WITH A CIRCULAR HOLE AT THE CENTRE SUBJECTED TO EQUAL BIAXIAL TENSION AT INFINITY.

**FIG 3.3** THICKNESS DISTRIBUTION AN INFINITE PLATE WITH A CIRCULAR HOLE AT THE CENTRE SUBJECTED EQUAL BIAXIAL TENSION AT INFINITY.
Fig. 3-5 Circumferential stress distribution in a circular plate bound by a rigid wall and uniform radial pressure at the concentric circular hole when $b = 2a$. 

$\sigma = \left( \frac{a}{r} \right)^2$
VARIATION OF THICKNESS IN A FINITE CIRCULAR PLATE BOUNDED BY A RIGID WALL AND UNIFORM RADIAL PRESSURE AT THE CONCENTRIC CIRCULAR HOLE WHEN \( b = 2a \)

Fig: 3.6
RADIAL STRESS DISTRIBUTION IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO PRESSURE AT THE HOLE WHEN $b=2a$  

Fig 3.7
CIRCUMFERENTIAL STRESS DISTRIBUTION IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO PRESSURE AT THE HOLE WHEN $b=2a$. Fig: 3.8
VARIATION OF THICKNESS IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO PRESSURE AT THE HOLE WHEN b=2a

Fig: 3.9
RADIAL STRESS DISTRIBUTION IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO UNIFORM RADIAL TRACTION AT THE OUTER BOUNDARY WHEN $b=2a$

$S = (a/r)^2$

Fig. 3.10
CIRCUMFERENTIAL STRESS DISTRIBUTION IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO UNIFORM RADIAL TRACTION AT THE OUTER BOUNDARY WHEN $\theta = 2a$

Fig: 3.11

VARIATION OF THICKNESS IN A CIRCULAR PLATE WITH A HOLE AT THE CENTRE SUBJECTED TO UNIFORM RADIAL TRACTION AT THE OUTER BOUNDARY WHEN $\theta = 2a$

Fig: 3.12