The modeling of a system is one of the very important disciplines to analyze the characteristics of the system. This is because once the mathematical formulation is developed; the different aspects of the system can be critically examined with the help of standard mathematical techniques. In our present study, we develop a model for the buffer estimation required at the user's site in video broadcasting. For a user, the cost of the system is one of the most important factors and hence the service charges. In spite of the fact that a service may be very important and useful, its cost factor may be too high to encourage its utilization by the most of the desirous persons. The storage space in user's system for the video-on-demand services, one of the very important parameters, is directly related to the system cost. More the storage space, the more will be the cost of the system. Therefore, from a user's point of view, it becomes necessary that the buffer storage needed be minimum possible. However, at the same time, the services should not be disrupted. The jitter delay is an inherent parameter of the video-on-demand systems, and further, this parameter is very much random. Fortunately, there is a well-developed and established theory for random variables, and their characteristics. In chapter 3, we proposed a mathematical model for the storage space. In this chapter, we model the jitter delay as random variable and accordingly the model has been developed.

In this model, the buffer size after averaging over the random jitter delays is given by the following relation (3.5)
subject to
\[ \sum_{i=1}^{n} \tau_i = T \quad \text{and} \quad \tau_i > (m-1)\Delta_{2i} \quad \text{for} \quad i=1,2,\ldots,n \]

where $\beta_i$ is in $[\Delta_{2i-1}, \Delta_{2i}]$ and $0 < \Delta_{2i-1} < \Delta_{2i}$, $i = 1,2,\ldots,n$.

We maintain $\tau_i > (m-1)\Delta_{2i}$ so that the buffer storage is non-negative. The negative buffer storage does not have any meaning in present study.

Function $f(\beta_1, \beta_2, \ldots, \beta_n)$ can be written as $f_{\beta_1}(\beta_1)f_{\beta_2}(\beta_2)\ldots f_{\beta_n}(\beta_n)$ as $\beta_i$ are independent random variables.

Since $\beta_i$s are independent random variables, we consider different models of the random variable to develop the formulation for the buffer estimation. For general description, we consider that the different jitter delays assume values over different intervals. For example, the jitter delay, $\beta_i$ takes values over interval $[\Delta_{2i-1}, \Delta_{2i}]$. Since the jitter delays can assume any values over the given interval, it is imperative that the random variables to be considered must be of continuous types. We will consider three different distributions of random variables, namely, Uniform, Beta and Normal distributions. Since the actual nature of the jitter delay is not known, the uniform distribution is a natural choice. The Beta distribution provides more flexibility over the
uniform distribution. The sum of many distributions irrespective their nature leads to asymptotically normal distribution and hence the normal distribution is also considered.

5.1 Uniform Distribution

In this section, we model the jitter delay as uniform random variable $X$ for which the probability density function is given by

$$f_X(x) = \begin{cases} \frac{1}{(\Delta_2 - \Delta_1)} & \text{for } \Delta_1 \leq x \leq \Delta_2 \\ 0 & \text{otherwise} \end{cases}$$

(5.2)

It may be noted that the integrand in equation (5.1) is inverse sum of the terms of the form

$$\frac{1}{\tau_i - (m-1)\beta}$$

In this expression, $m$ and $\tau_i$ are fixed (constant) quantities, and $\beta$ is a random variable. This expression, being a function of a random variable, is also a random variable. We define this expression as a function $g_{\beta}(x)$ of the random variable $\beta$, denoting it by $Y_i$,

$$Y_i = g_{\beta}(x) = \frac{1}{\tau_i - (m-1)x} \quad \text{for } \Delta_{i-1} \leq x \leq \Delta_i$$

(5.3)
Since the random variable $\beta_i$ takes values over the interval $[\Delta_{2i-1}, \Delta_{2i}]$, the random variable $Y_i$ will take values over the interval $\left[ \frac{1}{\tau_i - (m-1)\Delta_{2i-1}}, \frac{1}{\tau_i - (m-1)\Delta_{2i}} \right]$.

For a given value $y$ of $Y_i$, let $x^*$ denote the solution of equation (5.3). The equation in (5.3) has a unique solution which is given by

$$x^* = \frac{y \tau_i - 1}{(m-1)y}$$

We also have

$$\left| g'_{\beta_i}(x^*) \right| = (m-1)y^2$$

The density function of $Y_i$, denoted by $f_{Y_i}(y)$, is given by

$$f_{Y_i}(y) = \frac{f_{\beta_i}(x^*)}{\left| g'_{\beta_i}(x^*) \right|}$$

Substituting the values of $x^*$ and $\left| g'_{\beta_i}(x^*) \right|$ from (5.4) in (5.5), we have

$$f_{Y_i}(y) = \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})y^2}$$

Thus the probability density function of $Y_i$ is given by

$$f_{Y_i}(y) = \begin{cases} \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})y^2} & \text{for } \frac{1}{\tau_i - (m-1)\Delta_{2i-1}} \leq y \leq \frac{1}{\tau_i - (m-1)\Delta_{2i}} \\ 0 & \text{otherwise} \end{cases}$$

\[ (5.6) \]
In order to estimate the density function of the random variable \( Y = Y_1 + Y_2 + \ldots + Y_n \), we can apply either the convolution operation multiple times or the characteristic function method. We follow the second approach because of simplicity. In this approach, we first find out the characteristic functions of \( Y_i, \ i = 1, 2, \ldots, n \), multiply all the characteristic functions, and then apply the inversion of the characteristic function to the product.

The characteristic function of \( Y_i \), denoted by \( \Phi_i(\omega) \), is given by

\[
\Phi_i(\omega) = \int_{-\infty}^{\infty} f_{Y_i}(y) e^{i\omega y} dy
\]  

(5.7)

Substituting the value of \( f_{Y_i}(y) \) from (5.6) in (5.7), and noting that \( Y_i \) assumes values over the interval \( \left[ \frac{1}{\tau_i - (m-1)\Delta_{2i-1}}, \frac{1}{\tau_i - (m-1)\Delta_{2i}} \right] \) only, we have

\[
\Phi_i(\omega) = \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})} \frac{1}{y^2} e^{i\omega y} dy
\]  

(5.8)

where \( \theta_{2i-1} = \frac{1}{\tau_i - (m-1)\Delta_{2i-1}} \) and \( \theta_{2i} = \frac{1}{\tau_i - (m-1)\Delta_{2i}} \).
We write (5.8) as follows.

\[ \Phi_i(\omega) = K_i I \]  
\[ \text{(5.9)} \]

where \( K_i \) and \( I \) are given by

\[ K_i = \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})} \]

\[ I = \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{1}{y^2} e^{j\omega y} \, dy \]  
\[ \text{(5.10)} \]

After performing integration in (5.10), we have

\[ I = \left( \frac{e^{j\omega \theta_{2i-1}}}{\theta_{2i-1}} - \frac{e^{j\omega \theta_{2i}}}{\theta_{2i}} \right) + (j\omega) \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{e^{j\omega y}}{y} \, dy \]  
\[ \text{(5.11)} \]

We denote

\[ I_1 = \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{e^{j\omega y}}{y} \, dy \]  
\[ \text{(5.12)} \]

We need to find out the value of the integral in (5.12) such that its value is of the form \( C_1 e^{j\omega \theta_{2i-1}} + C_2 e^{j\omega \theta_{2i}} \), where \( C_1 \) and \( C_2 \) are to be determined.
Since \(0 < \theta_{2i-1} < \theta_{2i}\), we have

\[
0 < \frac{\theta_{2i-1}}{\theta_{2i}} \quad \text{or} \quad 0 < 1 - \frac{\theta_{2i-1}}{\theta_{2i}} \quad \text{or} \quad 0 < \frac{|\theta_{2i-1} - \theta_{2i}|}{\theta_{2i}} < 1 \quad (5.13)
\]

To find out the value of \(I_1\), we need to use the condition given in (5.13) for convergence.

Putting \(y = \theta_{2i} + \theta_{2i} x\) in (5.12), we have

\[
I_1 = \int_{\theta_{2i-1} - \theta_{2i}}^{\theta_{2i}} \frac{e^{j\omega\theta_{2i}(1+x)}}{1+x} \, dx = e^{j\omega\theta_{2i}} \int_{\theta_{2i-1} - \theta_{2i}}^{\theta_{2i}} \frac{e^{j\omega\theta_{2i}x}}{1+x} \, dx
\]

Or,

\[
I_1 = e^{j\omega\theta_{2i}} \int_{\theta_{2i-1} - \theta_{2i}}^{\theta_{2i}} e^{j\omega\theta_{2i}x} \sum_{r=0}^{\infty} (-1)^r x^r \, dx
\]

Or,

\[
I_1 = e^{j\omega\theta_{2i}} \sum_{r=0}^{\infty} (-1)^r \int_{\theta_{2i-1} - \theta_{2i}}^{\theta_{2i}} e^{j\omega\theta_{2i}x} x^r \, dx \quad (5.14)
\]

Putting \(v = j\omega\theta_{2i} x\) in (5.14), we have

\[
I_1 = e^{j\omega\theta_{2i}} \sum_{r=0}^{\infty} (-1)^r \int_{0}^{\infty} e^{v} \left( \frac{v}{j\omega\theta_{2i}} \right)^r \frac{1}{(j\omega)\theta_{2i}} \, dv
\]
Or,

\[
I_1 = e^{j\omega_2} \sum_{r=0}^{\infty} (-1)^r \frac{1}{(j\omega_2)^{r+1}} \int_0^1 e^v v^r dv
\]

The integrand in (5.15) is of a form of infinite series, i.e.,

\[
\int_0^1 e^v v^r dv = \sum_{r=0}^{\infty} (-1)^r \frac{1}{(j\omega_2)^{r+1}} \left\{ c_r e^{j\omega_\theta_2} - p_r ((\theta_{2i-1} - \theta_{2i}) j\omega) e^{j\omega_\theta_{2i}} \right\}
\]

where \(c_r\) are constants and \(p_r\) are polynomials in \((\theta_{2i-1,2i}) j\omega\).

For \(r = 0, 1, 2, 3, 4 \& 5\) and putting \((\theta_{2i-1,2i}) j\omega = \eta\), we have from (5.16)

\[
\begin{align*}
p_6(\eta) &= 1 \quad \text{and} \quad c_0 = 1 \\
p_1(\eta) &= \eta - 1 \quad \text{and} \quad c_1 = -1 \\
p_2(\eta) &= \eta^2 - 2\eta + 2 \quad \text{and} \quad c_2 = 2 \\
p_3(\eta) &= \eta^3 - 3\eta^2 + 6\eta - 6 \quad \text{and} \quad c_3 = -6 \\
p_4(\eta) &= \eta^4 - 4\eta^3 + 12\eta^2 - 24\eta + 24 \quad \text{and} \quad c_4 = 24 \\
p_5(\eta) &= \eta^5 - 5\eta^4 + 20\eta^3 - 60\eta^2 + 120\eta - 120 \quad \text{and} \quad c_5 = -120
\end{align*}
\]
We write $I$ given in (5.10), using (5.15) and (5.16),

$$I = \left( \frac{e^{j\omega \theta_{2i-1}} - e^{j\omega \theta_{2i}}}{\theta_{2i-1} - \theta_{2i}} \right) + (j\omega) \left[ \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega \theta_{2i})^{r+1}} e^{j\omega \theta_{2i}} - \sum_{r=0}^{\infty} (-1)^r \frac{1}{(j\omega \theta_{2i})^{r+1}} p_r \left((\theta_{2i-1} - \theta_{2i})j\omega\right)e^{j\omega \theta_{2i-1}} \right]$$

(5.18)

Finally, $\Phi_i(\omega)$ defined in (5.9), using (5.18), can be written as

$$\Phi_i(\omega) = K_i \left( \frac{e^{j\omega \theta_{2i-1}} - e^{j\omega \theta_{2i}}}{\theta_{2i-1} - \theta_{2i}} \right) + (j\omega) \left[ \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega \theta_{2i})^{r+1}} e^{j\omega \theta_{2i}} - \sum_{r=0}^{\infty} (-1)^r \frac{1}{(j\omega \theta_{2i})^{r+1}} p_r \left((\theta_{2i-1} - \theta_{2i})j\omega\right)e^{j\omega \theta_{2i-1}} \right]$$

Or,

$$\Phi_i(\omega) = K_i \left( -\frac{1}{\theta_{2i}} + j\omega \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega \theta_{2i})^{r+1}} \right) e^{j\omega \theta_{2i}} +$$

$$K_i \left( \frac{1}{\theta_{2i-1}} - j\omega \sum_{r=0}^{\infty} (-1)^r \frac{p_r}{(j\omega \theta_{2i})^{r+1}} \left((\theta_{2i-1} - \theta_{2i})j\omega\right) \right) e^{j\omega \theta_{2i-1}}$$

We write $\Phi_i(\omega)$ in the following form

$$\Phi_i(\omega) = K_i \left(A_{2i-1}e^{j\omega \theta_{2i-1}} + A_2 e^{j\omega \theta_{2i}} \right)$$

(5.19)
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(5.20)

\[ A_{2i-1} = \frac{1}{\theta_{2i-1}} - j\omega \sum_{r=0}^{\infty} (-1)^r \frac{p_r ((\theta_{2i-1} - \theta_{2i}) j\omega)}{(j\omega \theta_{2i})^{r+1}} \]

\[ A_{2i} = \left( -\frac{1}{\theta_{2i}} + j\omega \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega \theta_{2i})^{r+1}} \right) \]

The expressions of \( A_{2i-1} \) and \( A_{2i} \) are in the form of infinite series. For computation purposes, we need to have finite terms only. Therefore, we can write \( A_{2i-1} \) and \( A_{2i} \) in following form

\[
\frac{1}{(j\omega)^t} \left( \kappa_0 (j\omega)^t + \kappa_1 (j\omega)^{t-1} + \kappa_2 (j\omega)^{t-2} + \ldots + \kappa_t \right)
\]

where \( \kappa_t \) is the coefficient of \( (j\omega)^t \). The value of \( t \), a positive integer, depends on how many terms are taken for approximating the values of \( A_{2i-1} \) and \( A_{2i} \) in their infinite series. The expression in (5.21) as such represents \( A_{2i-1} \), and for representing \( A_{2i} \), \( \kappa_0 \) is necessarily zero.

The characteristic function of \( Y \), denoted by \( \Phi_Y(\omega) \), is given by

\[ \Phi_Y(\omega) = \Phi_1(\omega) \Phi_2(\omega) \ldots \Phi_n(\omega) \]  \hspace{2cm} (5.22)

Substituting the values of \( \Phi_i(\omega) \), \( i=1,2,\ldots,n \), from (5.19) in (5.22), we have

\[ \Phi_Y(\omega) = \prod_{i=1}^{n} K_i \left\{ A_{2i-1} e^{j\omega \theta_{2i-1}} + A_{2i} e^{j\omega \theta_{2i}} \right\} \]  \hspace{2cm} (5.23)
We write $\Phi_Y(\omega)$ in the following form

$$\Phi_Y(\omega) = K \Psi$$  \hspace{1cm} (5.24)

where $K$ and $\Psi$ are defined

$$K = \prod_{i=1}^{n} K_i$$  \hspace{1cm} (5.25)

$$\Psi = \prod_{i=1}^{n} \left( A_{2i} e^{j\omega \theta_{2i}} + A_{2i-1} e^{j\omega \theta_{2i-1}} \right)$$  \hspace{1cm} (5.26)

Using the result (4.11) of $P$-to-$S$ theorem of Chapter 4, we can write $\Psi$ given in (5.26) in the following form

$$\Psi = \sum_{k=1}^{2^n} \left( A_{k_1} A_{k_2} ... A_{k_n} e^{j\omega (\theta_{k_1} + \theta_{k_2} + ... + \theta_{k_n})} \right)$$  \hspace{1cm} (5.27)

where $k_1, k_2, ..., k_n$ are the indices of $\theta$ of $k^{th}$ term in $\Psi$.

The index values, $k_1, k_2, ..., k_n$ of the terms occurring in (5.27) can be found out by using the binary code tree described in section 4.5 of Chapter 4.

Each term in $A_{k_i}$ has negative powers of $(j\omega)$ if $k_i$ is even. When $k_i$ is odd, there is also a constant term besides negative powers of $(j\omega)$. 

Therefore, the product $A_{k_1} A_{k_2} \ldots A_{k_n}$ can be written as

$$A_{k_1} A_{k_2} \ldots A_{k_n} = \frac{1}{(j\omega)^n} \left( d_0 (j\omega)^n + d_1 (j\omega)^{n-1} + \ldots + d_m \right) = \sum_{i=0}^{m} \frac{d_i}{(j\omega)^i}$$

(5.28)

where $d_i$ denotes the coefficient of $(j\omega)^{-i}$. $d_i$ can be found out by using multiple convolution discussed in section 4.7 of Chapter 4.

We now find out the density function of $Y$ from its characteristic function $\Phi_Y(\omega)$. Its density function is given by

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_Y(\omega) e^{-j\omega y} dy$$

(5.29)

We may write $f_Y(y)$ as follows

$$f_Y(y) = \frac{1}{2\pi} \left( \int_{-\infty}^{0} \Phi_Y(\omega) e^{-j\omega y} dy + \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega y} dy \right)$$

Or,

$$f_Y(y) = \frac{1}{2\pi} \left( \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega (y-y)} dy + \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega y} dy \right)$$

(5.30)
The random variable $Y$ assumes positive values only. Therefore, $f_Y(y)$ in (5.31) can be written as

$$f_Y(y) = \frac{1}{2\pi} \int_0^\infty \Phi_Y(\omega)e^{-j\omega y} dy$$

This integral can be found out by using \textit{Inverse Laplace} transform, i.e.,

$$f_Y(y) = \frac{1}{2\pi} L^{-1}\{\Phi_Y(\omega)\}$$  \hspace{1cm} (5.31)

Using (5.24), (5.25) and (5.27), we can write (5.31) in the following form

$$f_Y(y) = L^{-1}\left\{ \frac{K}{2\pi} \sum_{k=1}^{2^n} A_{k_1}A_{k_2}...A_{k_n} e^{j\omega(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n})} \right\}$$  \hspace{1cm} (5.32)

Substituting the values of $A_{k_1}A_{k_2}...A_{k_n}$ from (5.28) in (5.32), we have

$$f_Y(y) = \frac{K}{2\pi} \sum_{k=1}^{2^n} L^{-1}\left( \sum_{l=0}^{nt} d_l e^{j\omega(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n})} \right) \frac{1}{(j\omega)^l}$$
The function $f_y(y)$ can be written in the following form

$$f_y(y) = \frac{K}{2\pi} \sum_{k=1}^{\infty} \left\{ \sum_{l=1}^{n_k} \frac{d_l(y-(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n}))^{-1}}{(-1)^{(l-1)!}} u(y-(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n})) \right\}$$

for $(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n}) \leq y \leq \sum_{i=1}^{n} \theta_{2i}$

(5.33)

where $u(y)$ is the unit step function. It is given by

$$u(y) = \begin{cases} 
1 & \text{if } y \geq 0 \\
0 & \text{if } y < 0 
\end{cases}$$

The random variable $Y$ has minimum and maximum values as $(\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n})$ and

$$\sum_{i=1}^{n} \theta_{2i}$$

respectively.

We now define a function $g_Y(y)$ of $Y$, denoting it $Z$,

$$Z = g_Y(y) = \frac{1}{y} \text{ for } (\theta_{k_1}+\theta_{k_2}+...+\theta_{k_n}) \leq y \leq \sum_{i=1}^{n} \theta_{2i}$$

We want to find out the mean and the variance of $Z$. For variance, we need to find out the second order moment.
The mean of $Z$, denoted by $M_1$, is given by

$$
M_1 = \sum_{i=1}^{n} \theta_{2i} \int_{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \frac{1}{y} f_y(y) dy
$$

Substituting the value of $f_y(y)$ from (5.33) in above integral, we have

$$
M_1 = \frac{K}{2\pi} \sum_{k=1}^{2^n} \left\{ \frac{\sum_{i=1}^{n} \theta_{2i}}{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \int_{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \frac{1}{y} d_0 \text{Dirac} \left( y - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \right) dy + \sum_{l=1}^{nt} \frac{\sum_{i=1}^{n} \theta_{2i}}{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \int_{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \frac{d_l(y - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}))^{l-1}}{(-1)^l (l-1)! y} dy \right\}
$$

(5.34)

After performing integration in (5.34), we have

$$
M_1 = \frac{K}{2\pi} \sum_{k=1}^{2^n} \left\{ \frac{d_0}{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} + \sum_{l=1}^{nt} \frac{d_l}{(-1)^l (l-1)!} (R_{12}^l - R_{11}^l) \right\}
$$

(5.35)
where

\[
R_{12}^l = \frac{\alpha_{2}^{l-1}}{l-1} + \left(\frac{l-1}{1}\right) (-\zeta) \frac{\alpha_{2}^{l-2}}{l-2} + \ldots + \left(\frac{l-1}{1}\right) (-\zeta)^{l-2} \frac{\alpha_{2}}{l-2} + \left(\frac{l-1}{1}\right) (-\zeta)^{l-1} \ln \alpha_{2}
\]

(5.36)

\[
R_{11}^l = \frac{\zeta^{l-1}}{l-1} + (-1) \left(\frac{l-1}{1}\right) \frac{\zeta^{l-2}}{l-2} + \ldots + (-1)^{l-2} \left(\frac{l-1}{1}\right) \zeta^{l-1} + (-1)^{l-1} \left(\frac{l-1}{1}\right) \zeta^{l-1} \ln \zeta
\]

(5.37)

and \( \alpha_{2} = \sum_{i=1}^{n} \theta_{2i}, \ \zeta = (\theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}) \)

The second order moment, denoted by \( M_{2} \), is given by

\[
M_{2} = \sum_{i=1}^{n} \int_{\theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}} \frac{1}{y^{2}} f_{\gamma}(y) dy
\]

Substituting the value of \( f_{\gamma}(y) \) from (5.33) in above integral, we have

\[
M_{2} = \frac{K}{2\pi} \sum_{k=1}^{\infty} \left\{ \sum_{i=1}^{n} \int_{\theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}} \frac{d_{0}}{y^{2}} \ \text{Dirac} \left( y - (\theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}) \right) dy + \sum_{i=1}^{nt} \int_{(-1)^{l}(l-1)! \theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}} \left( y - (\theta_{k_{1}} + \theta_{k_{2}} + \ldots + \theta_{k_{n}}) \right)^{l-1} \frac{d_{l}}{y^{2}} dy \right\}
\]

(5.38)
Denoting \( \alpha_2 = \sum_{i=1}^{n} \theta_{2i} \) and \( \zeta = (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \)

\[
M_2 = \frac{K}{2\pi} \sum_{k=1}^{2^n} \frac{d_0}{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})^2} + \sum_{l=1}^{nt} \frac{d_1}{(-1)^l (l-1)!} (R_{22}^l - R_{21}^l)
\]

(5.39)

where

\[
R_{21}' = \left\{ \frac{\kappa^{l-2}}{l-2} \right\} + \left\{ \frac{(l-1)\kappa^{l-2}}{l-3} \right\} + \ldots + \left\{ \frac{(-1)^{l-3}(l-1)\kappa^{l-2} + (-1)^{l-2}(l-2)\kappa^{l-2} \ln \kappa - (-1)^{l-1}\kappa^{l-2}}{l-3} \right\}
\]

(5.40)

\[
R_{22}' = \left\{ \frac{\alpha_2^l}{l-2} \right\} + \left\{ \frac{(l-1)\alpha_2^l}{l-3} \right\} + \ldots + \left\{ \frac{(-1)^{l-3}(l-1)\alpha_2 + (-1)^{l-2}(l-2)\alpha_2 \ln \alpha - (-1)^{l-1}}{l-3} \right\}
\]

(5.41)

Thus we have the mean and the variance of \( Z \) as \( M_1 \) and \( M_2 - M_1^2 \), respectively. The values of \( M_1 \) and \( (M_2 - M_1^2) \) provide the mean and the variance of the storage space, respectively.
5.2 Beta Distribution

In this section, the jitter delays are modeled as Beta random variables. The Beta distribution function provides greater flexibility than the uniform distribution function.

The Beta density function of a random variable \( \beta_i \) is given by

\[
f_{\beta_i}(x) = \begin{cases} 
\frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)} x^b (1-x)^c & \text{for } 0 < x < 1, \ b > 0, \ c > 0 \\
0 & \text{otherwise}
\end{cases}
\]

We restrict the random variable \( \beta_i \) to assume values over an interval \([\Delta_{2i-1}, \Delta_{2i}]\). In that case, the Beta density function is given by

\[
f_{\beta_i}(x) = \begin{cases} 
\frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)B_i} x^b (1-x)^c & \text{for } \Delta_{2i-1} \leq x \leq \Delta_{2i}, b > 0, \ c > 0 \\
0 & \text{otherwise}
\end{cases}
\]

where \( B_i \), the normalizing factor, is given by

\[
B_i = \left( \frac{\Delta_{2i}^{b+1}}{b+1} - \frac{c \Delta_{2i}^{b+2}}{b+2} + \ldots + (-1)^{c-1} \frac{\Delta_{2i}^{b+c}}{b+c} + (-1)^c \frac{\Delta_{2i}^{b+c+1}}{b+c+1} \right)
\]

\[
- \left( \frac{\Delta_{2i-1}^{b+1}}{b+1} - \frac{c \Delta_{2i-1}^{b+2}}{b+2} + \ldots + (-1)^{c-1} \frac{\Delta_{2i-1}^{b+c}}{b+c} + (-1)^c \frac{\Delta_{2i-1}^{b+c+1}}{b+c+1} \right)
\]

(5.44)
We define a function \( g_{\beta_i}(x) \) of the random variable \( \beta_i \), denoting it by \( Y_i \), as follow

\[
Y_i = g_{\beta_i}(x) = \frac{1}{\tau_i - (m-1)x} \tag{5.45}
\]

For a given value \( y \) of the random variable \( Y_i \), let \( x^* \) denote the solution of equation (5.45). The solution is unique and given by

\[
x^* = \frac{y \tau_i - 1}{(m-1)y} \tag{5.46}
\]

We also have

\[
\left| g'_{\beta_i}(x^*) \right| = (m-1)y^2 \tag{5.47}
\]

The density function of \( Y_i \), denoted by \( f_{Y_i}(y) \), is given by

\[
f_{Y_i}(y) = \frac{f_{\beta_i}(x^*)}{\left| g'_{\beta_i}(x^*) \right|} \tag{5.47}
\]

Substituting the values of \( x^* \) and \( g'_{\beta_i}(x^*) \) from (5.46) in (5.47), we have

\[
f_{Y_i}(y) = \frac{\Gamma(b+c+2)x_1^b(1-x_1)^c}{\Gamma(b+1)\Gamma(c+1)(m-1)y^2B_i}
\]

\[
f_{Y_i}(y) = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)B_i(m-1)^{b+c+1}y^{b+c+2}} \left[ (\tau_i y - 1)^b(m-1)y - (\tau_i y - 1)^c \right]
\]

for \( b > 0, \quad c > 0, \quad \theta_{2-1} \leq y \leq \theta_2 \) \tag{5.48}
where \( \theta_{2i-1} = \frac{1}{\tau_i - (m-1)\Delta_{2i-1}} \) and \( \theta_{2i} = \frac{1}{\tau_i - (m-1)\Delta_{2i}} \)

In order to find out the density function of the random variable \( Y = Y_1 + Y_2 + \ldots + Y_n \), we follow the same approach as in case of Uniform random variables, i.e., find out the characteristic function of each of \( Y_i \), take their product and then apply inverse characteristic function to the product.

The characteristic function of \( Y_i \), denoted by \( \Phi_i(\omega) \), is given by

\[
\Phi_i(\omega) = \int_{-\infty}^{\infty} f_{Y_i}(y) e^{i\omega y} \, dy
\]

Substituting the value of \( f_{Y_i}(y) \) from (5.48) in (5.49), we have

\[
\Phi_i(\omega) = \int_{-\infty}^{\infty} \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)B(m-1)^{b+c+1}} \left[ (\tau_i y - 1)^b (m-1) y - (\tau_i y - 1)^c \right] e^{i\omega y} \, dy
\]

Since \( Y_i \) assumes values over the interval \( \left[ \frac{1}{\tau_i - (m-1)\Delta_{2i-1}}, \frac{1}{\tau_i - (m-1)\Delta_{2i}} \right] \) only, we write (5.50) as follows

\[
\Phi_i(\omega) = \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)B(m-1)^{b+c+1}} \left[ (\tau_i y - 1)^b (m-1) y - (\tau_i y - 1)^c \right] e^{i\omega y} \, dy
\]
For simplicity, we take $b = 1$ and $c = 1$, and denoting

$$\Omega_i = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)(m-1)^3 B_i} = \frac{6}{(m-1)^3 B_i}$$

We write $\Phi_i(\omega)$ as follows

$$\Phi_i(\omega) = \Omega_i \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{(\tau_i y - 1)(m-1)y - (\tau_i y - 1)}{y^4} e^{j\omega y} dy$$

(5.52)

After performing integration in (5.52), we have

$$\Phi_i(\omega) = \Omega_i \left\{ \frac{1}{3} \left( \frac{e^{j\omega \theta_{2i}}}{\theta_{2i}^3} - \frac{e^{j\omega \theta_{2i-1}}}{\theta_{2i-1}^3} \right) + \int_{\theta_{2i-1}}^{\theta_{2i}} \left( \frac{2\tau_i - m + 1 - j\omega/3 + \tau_i (m-1-\tau_i)}{y^3} + \frac{\tau_i (m-1-\tau_i)}{y^2} \right) e^{j\omega y} dy \right\}$$

Or,

$$\Phi_i(\omega) = \Omega_i \left\{ \frac{1}{3} \left( \frac{e^{j\omega \theta_{2i}}}{\theta_{2i}^3} - \frac{e^{j\omega \theta_{2i-1}}}{\theta_{2i-1}^3} \right) + I_1 \right\}$$

(5.53)
where $I_1$ is given by

$$I_1 = \int_{\theta_{2i-1}}^{\theta_{2i}} \left\{ \left( 2\tau_i - m + 1 - \frac{j\omega}{3} \right) \frac{1}{y^3} + \frac{\tau_i (m-1-\tau_i)}{y^2} \right\} e^{j\omega y} \, dy$$

After performing integration in $I_1$, we have

$$I_1 = (2\tau_i - m + 1 - \frac{j\omega}{3}) \left( e^{j\omega \theta_{2i-1}} - e^{j\omega \theta_{2i}} \right) + I_2$$  \hspace{1cm} (5.54)$$

where $I_2$ is given by

$$I_2 = \left( \tau_i (m-1-\tau_i) + (2\tau_i - m + 1) \frac{j\omega}{2} - \frac{(j\omega)^2}{6} \right) \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{e^{j\omega y}}{y^2} \, dy$$

We write $I_2$ in the following form

$$I_2 = \left( \tau_i (m-1-\tau_i) + (2\tau_i - m + 1) \frac{j\omega}{2} - \frac{(j\omega)^2}{6} \right) I_3$$  \hspace{1cm} (5.55)$$

where $I_3$ is given by
\[ I_3 = \int_{\theta_{2i-1}}^{\theta_{2i}} \frac{e^{j\omega y}}{y^2} dy \]

\( I_3 \) is same as \( I \) defined in (5.10). Therefore, we can write its value from (5.18)

\[ I_3 = \left( \frac{e^{j\omega \theta_{2i-1}}}{\theta_{2i-1}} - \frac{e^{j\omega \theta_{2i}}}{\theta_{2i}} \right) + (j\omega) \left\{ \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega \theta_{2i})^{r+1}} e^{j\omega \theta_{2i}} - \right\} \left\{ \sum_{r=0}^{\infty} (-1)^r \frac{1}{(j\omega \theta_{2i})^{r+1}} p_r ((\theta_{2i-1} - \theta_{2i}) j\omega) e^{j\omega \theta_{2i-1}} \right\} \]

(5.56)

\( \Phi_i(\omega) \) defined in (5.53), using (5.54) and (5.55), can be written in the following form

\[ \Phi_i(\omega) = \Omega_i \left\{ \frac{1}{3} \left( \frac{e^{j\omega \theta_{2i}}}{\theta_{2i}^3} - \frac{e^{j\omega \theta_{2i-1}}}{\theta_{2i-1}^3} \right) + (2\tau_i - m + 1 - \frac{j\omega}{3}) \left( \frac{e^{j\omega \theta_{2i-1}}}{2\theta_{2i-1}^2} - e^{j\omega \theta_{2i}} \right) \right\} + \left( \tau_i (m - 1 - \tau_i) + (2\tau_i - m + 1) \frac{j\omega}{2} - \frac{(j\omega)^2}{6} \right) I_3 \]

where \( I_3 \) is defined in (5.56).

After simplification, \( \Phi_i(\omega) \) can be written in the following form
We write $\Phi_i(\omega)$ defined in (5.57) in the following form

$$\Phi_i(\omega) = \Omega_i \left( A_{2i-1} e^{j\omega \theta_{2i-1}} + A_{2i} e^{j\omega \theta_{2i}} \right)$$

where

$$A_{2i-1} = \left( \frac{1}{3\theta_{2i-1}^3} - \frac{2}{3} \frac{1}{2\theta_{2i-1}^2} + \left( \frac{m+1}{\theta_{2i-1}} - \frac{j\omega}{3} \right) \frac{1}{2\theta_{2i-1}^2} \right) +$$

$$\left( \tau_i(m-1-\tau_i) + (2\tau_i - m + 1) \frac{j\omega}{2} - \frac{(j\omega)^2}{6} \right) \left( \frac{1}{\theta_{2i-1}^2} - j\omega \sum_{r=0}^{\infty} (-1)^r \frac{p_r((\theta_{2i-1} - \theta_{2i})j\omega)}{(j\omega \theta_{2i})^{r+1}} \right)$$

(5.59)
\[ A_{2i} = \left( \frac{1}{3\theta_{2i}^3} - \left( 2\tau_i - m + 1 - \frac{j\omega}{3} \right) \frac{1}{2\theta_{2i}^2} \right) + \]

\[ \left( \tau_i(m - 1 - \tau_i) + (2\tau_i - m + 1) \frac{j\omega}{2} - \frac{(j\omega)^2}{6} \right) \left( -\frac{1}{\theta_{2i}} + j\omega \sum_{r=0}^{\infty} (-1)^r \frac{c_r}{(j\omega\theta_{2i})^{r+1}} \right) \]

(5.60)

The expressions of \( A_{2i-1} \) and \( A_{2i} \) are in the form of infinite series. For computational purposes, we need to consider finite terms only. Therefore, we can write these expressions in the following form:

\[ A_{2i-1} = \frac{1}{(j\omega)^t} \left( t_0(j\omega)^{t+2} + t_1(j\omega)^{t+1} + t_2(j\omega)^t + \ldots + t_{t+2} \right) \]

\[ A_{2i} = \frac{1}{(j\omega)^t} \left( \kappa_0(j\omega)^{t+1} + \kappa_1(j\omega)^t + \kappa_2(j\omega)^{t-1} + \ldots + \kappa_{t+1} \right) \]

(5.61)

where \( t_i \) is the coefficient of \( (j\omega)^{2-i} \) in expression of \( A_{2i-1} \) and \( \kappa_i \) is the coefficient of \( (j\omega)^{1-i} \) in expression of \( A_{2i} \). The value of \( t \), a positive integer, depends on how many terms have been considered to approximate the values of \( A_{2i-1} \) and \( A_{2i} \) in their infinite series.

The equation (5.58) along with (5.59) and (5.60) gives characteristic function \( \Phi_i(\omega) \), \( i = 1, 2, \ldots, n \). The product of these characteristic functions will give the characteristic function of \( Y \). Denoting its characteristic function by \( \Phi_Y(\omega) \), we have
\[
\Phi_Y(\omega) = \Phi_1(\omega)\Phi_2(\omega)\ldots\Phi_n(\omega) \tag{5.62}
\]

Substituting the value of \(\Phi_i(\omega), i = 1, 2,\ldots, n\) from (5.59) in (5.63), we have

\[
\Phi_Y(\omega) = \prod_{i=1}^{n} \Omega_i \left( A_{2i-1} e^{j\omega \theta_{2i-1}} + A_{2i} e^{j\omega \theta_{2i}} \right) \tag{5.63}
\]

We write (5.63) in the following form

\[
\Phi_Y(\omega) = \Omega \mathcal{S} \tag{5.64}
\]

where \(\Omega = \prod_{i=1}^{n} \Omega_i\) and

\[
\mathcal{S} = \prod_{i=1}^{n} \left( A_{2i-1} e^{j\omega \theta_{2i-1}} + A_{2i} e^{j\omega \theta_{2i}} \right)
\]

Using the result (4.11) of Chapter 4, \(\mathcal{S}\) can be written as follows

\[
\mathcal{S} = \sum_{k=1}^{2^n} \left( A_{k_1} A_{k_2} \ldots A_{k_n} e^{j\omega (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} \right) \tag{5.65}
\]

The index values, \(k_1, k_2,\ldots, k_n\) can be found out using the binary code tree described in section 4.5 of Chapter 4.
Each $A_k$ can be written in a polynomial form in $(j\omega)$ after taking an appropriate factor $1/(j\omega)^t$ common. The product $A_{k_1} A_{k_2} \ldots A_{k_n}$ can be written as

$$A_{k_1} A_{k_2} \ldots A_{k_n} = \frac{1}{(j\omega)^t} \left( d_0 (j\omega)^{n(t+2)} + d_1 (j\omega)^{n(t+2)-1} + \ldots + d_{n(t+2)} \right)$$

(5.66)

where $d_i$ is the coefficient of $(j\omega)^{2n-i}$. The value of $t$, a positive integer, depends upon how many terms are taken for approximating the values of $A_{2i-1}$ and $A_{2i}$.

We write the product $A_{k_1} A_{k_2} \ldots A_{k_n}$ in (5.66) in the following form

$$A_{k_1} A_{k_2} \ldots A_{k_n} = \mathcal{R}_1 + d_{2n} + \mathcal{R}_2$$

(5.67)

where

\begin{align*}
\mathcal{R}_1 &= \sum_{l=0}^{2n-1} d_l (j\omega)^{2n-l} \\
\mathcal{R}_2 &= \sum_{l=2n+1}^{n(t+2)} d_l (j\omega)^{2n-l}
\end{align*}

(5.68) (5.69)
Using (5.65), (5.67), (5.68) & (5.69), we can write (5.64) in the following form

\[
\Phi_Y(\omega) = \sum_{k=1}^{2^n} \left[ \sum_{l=0}^{2n-1} d_l(j\omega)^{2n-l} + d_{2n} + \sum_{l=2n+1}^{n+2} d_l(j\omega)^{2n-l} \right] e^{j\omega(\theta_1 + \theta_2 + \ldots + \theta_n)}
\]

(5.70)

We now find out the probability density function of \( Y \) from its characteristic function \( \Phi_Y(\omega) \), which is given by

\[
f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_Y(\omega) e^{-j\omega y} \, dy
\]

(5.71)

Now, we can write \( f_Y(y) \) as follow

\[
f_Y(y) = \frac{1}{2\pi} \left( \int_{-\infty}^{0} \Phi_Y(\omega) e^{-j\omega y} \, dy + \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega y} \, dy \right)
\]

Or,

\[
f_Y(y) = \frac{1}{2\pi} \left( \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega y} \, dy + \int_{0}^{\infty} \Phi_Y(\omega) e^{-j\omega y} \, dy \right)
\]

(5.72)
The random variable $Y$ assumes positive values only. Therefore, $f_Y(y)$ in (5.72) can be written as follows

\[ f_Y(y) = \frac{1}{2\pi} \int_0^{\infty} \Phi_Y(\omega) e^{-j\omega y} \, dy \]

This integral can be simplified using Inverse Laplace transform, i.e.,

\[ f_Y(y) = \frac{1}{2\pi} L^{-1}\left\{ \Phi_Y(\omega) \right\} \]  
\[ (5.73) \]

Substituting $\Phi_Y(\omega)$ from (5.70) in (5.73), we have

\[ f_Y(y) = L^{-1} \left\{ \frac{\Omega}{2\pi} \sum_{k=1}^{2^n} \left[ \sum_{l=0}^{2n-1} d_l(j\omega)^{2n-l} \right] \right. \\
\left. + d_{2n} + \sum_{l=2n+1}^{n(t+2)} d_l(j\omega)^{2n-l} \right\} e^{j\omega(t_1+\theta_2+\ldots+\theta_n)} \]
Or,

\[ f_Y(y) = \frac{\Omega}{2\pi} \sum_{k=1}^{2^n} \left( \sum_{l=0}^{2n-1} d_l(j\omega)^{2n-1} \right) \left[ \sum_{l=0}^{n(t+2)} d_l(j\omega)^{2n-l} \right] e^{j\omega(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} \]

After simplification, (5.74) is written as follows

\[ f_Y(y) = \frac{\Omega}{2\pi} \sum_{k=1}^{2^n} \left[ \sum_{l=0}^{2n-1} (-1)^{2n-l} d_l \text{Dirac}^{(2n-1)}(y-(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \right] + d_{2n} \text{Dirac} (y-(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) + \]

\[ \sum_{l=2n+1}^{n(t+2)} (-1)^{2n-l} d_l (y-(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}))^{2n-l-1} \]

\[ (5.75) \]

\textit{Dirac}^{(0)}(x) \text{ represents } i^{th} \text{ derivative of the } \text{Dirac}(x) \text{ function.}

Now we define a function \( g_Y(y) \) of the random variable \( Y \), denoting it by \( Z \),

\[ Z = g_Y(y) = \frac{1}{y} \quad \text{for} \quad (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \leq y \leq \sum_{i=1}^{n} \theta_{2i} \quad (5.76) \]
We now find out the mean and the variance of $Z$. The mean of $Z$, denoted by $M_1$, is given by

$$M_1 = \sum_{i=1}^{n} \theta_{2i} \int f_Y(y) \frac{1}{y} \, dy$$

Substituting the value $f_Y(y)$ from (5.75), we write $M_1$ as follow

$$M_1 = \frac{\Omega}{2\pi} \sum_{k=1}^{2^n} (H_1^k + H_2^k + H_3^k)$$

where

$$H_1^k = \sum_{l=2n+1}^{n(t+2)} \frac{(-1)^{2n-l}}{(2n-l-1)!} d_l \sum_{i=1}^{n} \theta_{2i} \int (x-(\theta_{k_1} + \theta_{k_2} + ... + \theta_{k_n}))^{2n-l-1} \frac{dx}{x}$$

$$H_2^k = \int d_{2n} \text{Dirac} (x-(\theta_{k_1} + \theta_{k_2} + ... + \theta_{k_n})) \frac{dx}{x}$$

(5.78)

(5.79)
\[ H_3^k = \sum_{i=0}^{2n-1} (-1)^{2n-l} d_i \int \text{Dirac} \ (2n-l) \left( x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \right) dx \]

where \( \varepsilon \) is a very small quantity but positive.

After performing integration in \( H_1^k \), we have

\[ H_1^k = \sum_{l=2n+1}^{n(t+2)} \frac{(-1)^{2n-l} d_l R_i^k}{(2n-l-1)!} \]

where \( R_i^k \) is given by

\[
R_i^k = \left\{ \frac{\alpha^{2n-l-1}_2}{2n-l-1} + \frac{2n-l-1}{1} \right\} \left\{ \frac{\alpha^{2n-l-2}_2}{2n-l-2} + \frac{2n-l-2}{2n-l-2} \right\} \alpha_2 + \zeta^{2n-l-1} \ln \alpha_2
\]

\[
\left\{ \frac{\zeta^{2n-l-1}}{2n-l-1} + \frac{(-1)^{2n-l-1}}{1} \right\} \left\{ \frac{\zeta^{2n-l-1}}{2n-l-2} + \frac{2n-l-1}{2n-l-2} \right\} \zeta^{2n-l-1} + \zeta^{2n-l-1} \ln \zeta
\]

where \( \alpha_2 = \sum_{i=1}^{n} \theta_{2i} \) and \( \zeta = (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \)

(5.82)
The integrals $H_k^2$ and $H_k^3$ defined in (5.79) and (5.80), respectively, have two aspects depending upon the condition that the point $(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})$ is excluded or included. If this point is excluded, the contribution of $H_k^2$ and $H_k^3$ would be zero. However, for better accuracy, we include this point wherever such situation arises. $H_k^2$ can be written in the following form

$$H_k^2 = \frac{d_{2n}}{\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}} \quad (5.83)$$

We write $H_k^3$ as follows

$$H_k^3 = \sum_{l=0}^{2n-1} (-1)^{2n-l} \alpha_l \sum_{i=1}^{n} \theta_{2i} \int (\text{Dirac}(2n-l)(x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \frac{1}{x}) dx$$

where $\varepsilon$ is a very small quantity but positive. \hspace{\textwidth}

The integrand in $H_k^3$ denoted by $X_l$ can be written as

$$X_l = \lim_{\varepsilon \to 0} \sum_{i=1}^{n} \theta_{2i} \int (\text{Dirac}(2n-l)(x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \frac{1}{x}) dx$$

Or,

$$X_l = \left(-1\right)^{2n-l} \frac{d}{dx}^{2n-l} \left(\frac{1}{x}\right) \bigg|_{x=(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})}$$

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Thus we have

\[ H^k_3 = \sum_{l=0}^{2n-1} (-1)^{2n-l} d_l \left( \frac{(2n-l)!}{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})^{2n-l+1}} \right) \]  

(5.85)

The second order moment of \( Z \), denoted by \( M_2 \), is given by

\[ M_2 = \frac{\sum_{i=1}^{n} \theta_{2i}}{\int f_Y(y) \frac{1}{y^2} dy} \]

Substituting the value of \( f_Y(y) \) from (5.76), we write \( M_2 \) as follows

\[ M_2 = \frac{\Omega}{2\pi} \sum_{k=1}^{2n} \left( G^k_1 + G^k_2 + G^k_3 \right) \]  

(5.86)

where

\[ G^k_1 = \sum_{l=2n+1}^{n(t+2)} \frac{(-1)^{2n-l}}{(2n-l-1)!} \sum_{i=1}^{n} \theta_{2i} \int (x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}))^{2n-l-1} \frac{dx}{x^2} \]  

(5.87)
\[ G_2^k = \sum_{i=1}^{\theta_{2i}} \int_{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} d_{2n} \text{Dirac}(x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \frac{dx}{x^2} \]

(5.88)

\[ G_3^k = \sum_{l=0}^{n} (-1)^{2n-l} \sum_{i=1}^{\theta_{2i}} \int_{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} \left(\text{Dirac}^{(2n-l)}(x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \right) \frac{1}{x^2} dx \]

(5.89)

After performing integration in (5.87), we have

\[ G_1^k = \sum_{l=2n+1}^{n(t+2)} \frac{(-1)^{2n-l}}{(2n-l-1)!} d_{l}(R'_l - R''_l) \]

(5.90)

where \( R'_l \) and \( R''_l \) are given below

\[
R'_l = \begin{cases} 
\frac{\alpha_{2n-l-2}}{2n-l-2} + & \begin{pmatrix} 2n-l-1 \\ 1 \end{pmatrix} (-\zeta) \frac{\alpha_{2n-l-3}}{2n-l-3} + \ldots + \\
\begin{pmatrix} 2n-l-1 \\ 2n-l-3 \end{pmatrix} (-\zeta)^{2n-l-3} \alpha_2 + & \begin{pmatrix} 2n-l-1 \\ 2n-l-2 \end{pmatrix} (-\zeta)^{2n-l-2} \ln(\alpha_2) - \frac{(-\zeta)^{2n-l-1}}{\alpha_2} 
\end{cases}
\]

(5.91)
\[ R_i^* = \begin{cases} \frac{\alpha_1^{2n-l-2}}{2n-l-2} + \binom{2n-l-1}{1} (-1)^{2n-l-2} \frac{\zeta^{2n-l-2}}{2n-l-3} + \ldots + \\ \left( \frac{2n-l-1}{2n-l-3} \right)^{2n-l-3} \zeta^{2n-l-2} + (-1)^{2n-l-2} \zeta^{2n-l-2} \ln(\zeta) - (-1)^{2n-l-1} \zeta^{2n-l-2} \end{cases} \] (5.92)

where \( \alpha_2 = \sum_{i=1}^{n} \theta_{2i} \) and \( \zeta = (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n}) \)

After performing integration in \( G_2^k \) defined in (5.88), we have

\[ G_2^k = \frac{d_{2n}}{\left( \theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n} \right)^2} \] (5.93)

We denote the integrand in \( G_3^k \) defined in (5.89) by \( X_I \) as follows

\[ X_I = \lim_{\varepsilon \to 0} \sum_{\epsilon} \int_0^1 \left( \text{Dirac} \left( 2n-l \right) (x - (\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})) \right)^{2n-l} \left( \frac{1}{x^2} \right) dx \]

After performing integration, we have

\[ X_I = \left( -1 \right)^{2n-l} \frac{d^{2n-l}}{dx^{2n-l}} \left( \frac{1}{x^2} \right) \Bigg|_{x=(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})} \]
Thus we have

\[ G_3^k = \sum_{l=0}^{2n-1} (-1)^{2n-l} d_l \left( \frac{(2n-l+1)!}{(\theta_{k_1} + \theta_{k_2} + \ldots + \theta_{k_n})^{2n-l+2}} \right) \] (5.94)

\( M_1 \) and \( M_2 \) defined in (5.77) and (5.86) are the first and second order moments of random variable \( Z \), respectively. The values of \( M_1 \) and \((M_2 - M_1^2)\) provide the average value and the variance of the storage space, respectively.
5.3 Normal (Gaussian) Distribution

A random variable $X$ is said to be normal or Gaussian with parameters $\mu$ and $\sigma$ if its density function is of the following form:

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad -\infty < x < \infty$$

(5.95)

In this section, we use the central limit theorem (ref. section 4.4 in Chapter 4) to derive the formulation for mean and variance of the buffer storage using normal distribution.

We assume that the jitter delay $\beta_i$ is a uniform random variable that assumes values over an interval $[\Delta_{2i-1}, \Delta_{2i}]$. Its density function is given by

$$f_{\beta_i}(x) = \begin{cases} \frac{1}{\Delta_{2i} - \Delta_{2i-1}} & \text{for} \quad \Delta_{2i-1} \leq x \leq \Delta_{2i} \\ 0 & \text{otherwise} \end{cases}$$

(5.96)

We define a new variable $Y_i$ as a function $g_{\beta_i}(x)$ of the random variable $\beta_i$ as follows

$$Y_i = g_{\beta_i}(x) = \frac{1}{\tau_i - (m-1)x} \quad \text{for} \quad \Delta_{2i-1} \leq x \leq \Delta_{2i}$$

(5.97)
The mean of $Y_i$, denoted by $\mu_{Y_i}$, is given by

$$\mu_{Y_i} = E\{Y_i\} = \int_{\Delta_{2i-1}}^{\Delta_{2i}} g_i(x) f_{\beta_i}(x) dx$$

(5.98)

$$\mu_{Y_i} = \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})} \log \left( \frac{\tau_i - (m-1)\Delta_{2i-1}}{\tau_i - (m-1)\Delta_{2i}} \right)$$

(5.99)

The second order moment of $Y_i$, denoted by $M_{2Y_i}$ is given by

$$M_{2Y_i} = E\{Y_i^2\} = \int_{\Delta_{2i-1}}^{\Delta_{2i}} (g_i(x))^2 f_{\beta_i}(x) dx$$

(5.100)

$$M_{2Y_i} = \frac{1}{(\tau_i - (m-1)\Delta_{2i})(\tau_i - (m-1)\Delta_{2i-1})}$$

(5.101)

The variance of $Y_i$, denoted by $\sigma_{Y_i}^2$, is given by

$$\sigma_{Y_i}^2 = M_{2Y_i} - \mu_{Y_i}^2$$

(5.102)
We denote \( n \) random variables \( Y_1, Y_2, \ldots, Y_n \) corresponding to jitter delays \( \beta_1, \beta_2, \ldots, \beta_n \), respectively. Further, for simplicity we assume that these jitter delays \( \beta_1, \beta_2, \ldots, \beta_n \) are distributed uniformly over the respective intervals \( [\Delta_1, \Delta_2], [\Delta_3, \Delta_4], \ldots, [\Delta_{2n-1}, \Delta_{2n}] \). It may be noted that since \( \beta_i, i = 1, 2, \ldots, n \), is random variable uniformly distributed over the interval \( [\Delta_{2i-1}, \Delta_{2i}] \), \((\tau_i \text{ and } m \text{ are fixed quantities})\). \( Y_i, i = 1, 2, \ldots, n, \) is also random variable uniformly distributed over the interval \[
\left[ \frac{1}{\tau_i - (m-1)\Delta_{2i-1}}, \frac{1}{\tau_i - (m-1)\Delta_{2i}} \right].
\]
Denote their respective means and variances of \( Y_1, Y_2, \ldots, Y_n \) by \( \mu_{Y_1}, \mu_{Y_2}, \ldots, \mu_{Y_n} \) and \( \sigma_{Y_1}^2, \sigma_{Y_2}^2, \ldots, \sigma_{Y_n}^2 \).

Now we define a new random variable \( Y \) as the sum of \( Y_1, Y_2, \ldots, Y_n \).

\[
Y = Y_1 + Y_2 + \ldots + Y_n
\]  

(5.103)

By applying the central limit theorem, \( Y \) can be approximated by a normal random variable asymptotically.

Therefore, the probability density function of \( Y \) is given by

\[
f_Y(y) = \begin{cases} 
\frac{1}{N\sigma_Y}\sqrt{2\pi} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} & \text{for } \sum_{i=1}^{n} \theta_{2i-1} \leq y \leq \sum_{i=1}^{n} \theta_{2i} \\
0 & \text{otherwise}
\end{cases}
\]  

(5.104)
where

\[
\theta_2 = \frac{1}{\tau_i - (m-1)\Delta_{2i}} \quad \text{and} \quad \theta_{2i-1} = \frac{1}{\tau_i - (m-1)\Delta_{2i-1}}
\]

and \( \mu_Y \) and \( \sigma_Y^2 \) are the mean and variance of \( Y \) and are given by

\[
\mu_Y = \mu_{Y_1} + \mu_{Y_2} + \ldots + \mu_{Y_n}
\]

\[
\mu_Y = \sum_{i=1}^{n} \frac{1}{(m-1)(\Delta_{2i} - \Delta_{2i-1})} \log \left( \frac{\tau_i - (m-1)\Delta_{2i-1}}{\tau_i - (m-1)\Delta_{2i}} \right)
\] (5.105)

\[
\sigma_Y^2 = \sigma_{Y_1}^2 + \sigma_{Y_2}^2 + \ldots + \sigma_{Y_n}^2
\]

\[
\sigma_Y^2 = \sum_{i=1}^{n} \left( \frac{1}{(\tau_i - (m-1)\Delta_{2i})(\tau_i - (m-1)\Delta_{2i-1})} - \mu_{Y_i}^2 \right)
\] (5.106)

and \( N_F \), normalizing factor, is given by

\[
N_F = \frac{\sum_{i=1}^{n} \theta_{2i}}{\sum_{i=1}^{n} \theta_{2i-1}} \int e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} \frac{1}{\sigma_Y \sqrt{2\pi}} \, dy
\]

\[\text{Stochastic Modeling for Buffer Storage}\]
We define a function of $Y$, denoted by $Z$, as $Z = g_Y(y) = 1/y$. The mean of $Z$, denoted by $M_I$, is given by

$$M_I = \int_{\alpha_1}^{\alpha_2} f_Y(y) \frac{1}{y} \, dy$$

Substituting the value $f_Y(y)$ from (5.104) in the above expression, we write $M_I$ as follows

$$M_I = \frac{1}{\sigma_Y \sqrt{2\pi}} \int_{\xi_1}^{\xi_2} \frac{1}{\xi} e^{-\frac{(\xi - \mu_Y)^2}{2\sigma_Y^2}} \, d\xi$$

where $\alpha_1 = \sum_{i=1}^{n} \theta_{2i+1}$ and $\alpha_2 = \sum_{i=1}^{n} \theta_{2i}$

We can write $M_I$ as follows

$$M_I = \frac{1}{\sigma_Y \sqrt{2\pi}} \int_{\xi_1}^{\xi_2} \frac{1}{\xi} \sum_{k=0}^{\infty} \frac{(-1)^k (\xi - \mu_Y)^{2k}}{(2\sigma_Y^2)^k k!} \, d\xi$$

Or,

$$M_I = \frac{1}{\sigma_Y \sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\sigma_Y^2)^k k!} \int_{\xi_1}^{\xi_2} \frac{(\xi - \mu_Y)^{2k}}{\xi} \, d\xi$$

(5.107)
After simplification, we write (5.107) in the following form

$$M_1 = \frac{1}{\sigma_Y \sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\sigma_Y^2)^{2k} k!} (I_{\alpha_2} - I_{\alpha_1})$$

(5.108)

where

$$I_{\alpha_2} = \left( \frac{\alpha_{2k}}{2k} - \binom{2k}{1} \frac{\alpha_{2k-1}}{2k-1} \mu_Y + \cdots + (-1)^{2k-1} \binom{2k}{2k-1} \alpha_{2k} \mu_Y^{2k-1} + \mu_Y^{2k} \ln \alpha_2 \right)$$

(5.109)

$$I_{\alpha_1} = \left( \frac{\alpha_{2k}}{2k} - \binom{2k}{1} \frac{\alpha_{2k-1}}{2k-1} \mu_Y + \cdots + (-1)^{2k-1} \binom{2k}{2k-1} \alpha_{2k} \mu_Y^{2k-1} + \mu_Y^{2k} \ln \alpha_1 \right)$$

(5.110)

The second order moment of $Z$, denoted by $M_2$, is given by

$$M_2 = \frac{\alpha_2}{\alpha_1} \int_{\alpha_1}^{\alpha_2} f_Y(y) \frac{1}{y^2} dy$$

Substituting the value $f_Y(y)$ from (5.104) in this expression, we write $M_2$ as follows

$$M_2 = \frac{1}{\sigma_Y \sqrt{2\pi}} \int_{\alpha_1}^{\alpha_2} \frac{1}{\xi^2} e^{-\frac{(\xi - \mu_Y)^2}{2\sigma_Y^2}} d\xi$$
Or,

\[ M_2 = \frac{1}{\sigma_Y \sqrt{2\pi}} \sum_{k=0}^{\infty} \int \frac{(-1)^k}{(2\sigma_Y^2)^{2k}} \frac{(\xi - \mu_Y)^{2k}}{\xi^2} d\xi \]

\[ M_2 = \frac{1}{\sigma_Y \sqrt{2\pi}} \sum_{k=0}^{\infty} \int \frac{(-1)^k}{(2\sigma_Y^2)^{2k}} \frac{(\xi - \mu_Y)^{2k}}{\xi^2} \alpha_1^2 \frac{d\xi}{\alpha_1} \]

(5.111)

After simplification, we write (5.111) in the following form

\[ M_2 = \frac{1}{\sigma_Y \sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\sigma_Y^2)^{2k} k!} (I'_{\alpha_2} - I'_{\alpha_1}) \]

(5.112)

where

\[ I'_{\alpha_2} = \left( \frac{\alpha_{2k-1}}{2k-1} + \frac{2k}{2k-2} \right) \frac{2^{2k-2}}{2k-2} (-\mu_Y) + \ldots + \left( \frac{2k}{2k-1} \right) \frac{\alpha_{2k} (-\mu_Y)^{2k-2}}{2k-1} \ln \alpha_2 - \left( \frac{(-\mu_Y)^{2k}}{\alpha_2} \right) \]

(5.113)

\[ I'_{\alpha_1} = \left( \frac{\alpha_{2k-1}}{2k-1} + \frac{2k}{2k-2} \right) \frac{2^{2k-2}}{2k-2} (-\mu_Y) + \ldots + \left( \frac{2k}{2k-1} \right) \frac{\alpha_{2k} (-\mu_Y)^{2k-2}}{2k-1} \ln \alpha_1 - \left( \frac{(-\mu_Y)^{2k}}{\alpha_1} \right) \]

(5.114)

Thus we have developed the formulation for first and second order moments \( M_1 \) and \( M_2 \), respectively, of \( Z \), defined in (5.108) along with (5.109) and (5.110), and (5.112)
along with (5.113) and (5.114), respectively. The value of $M_1$ and $(M_2 - M_1^2)$ provide the mean and variance $\sigma_Z^2$ of the storage space, respectively.

Thus we have developed stochastic models for estimating the buffer storage incorporating the jitter delays as Uniform, Beta and Normal random variables.