Preface

This thesis embodies the work done by the author under the guidance of Dr. A. Lourdusamy.

Consider the following question: Suppose $2^n$ pebbles are arbitrarily placed on the vertices of an $n$-cube. Does there exist a method that allows us to make a sequence of moves, each move taking two pebbles off one vertex and placing one pebble on an adjacent vertex, in such a way that we can end up with a pebble on any desired vertex? This question is answered in the affirmative by Chung [3]. Pebbling was first introduced into the literature by Chung [3].

Given a graph $G$, distribute $k$ pebbles (indistinguishable markers) on its vertices in some configuration $C$. Specifically, a configuration on a graph $G$ is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$ representing an arrangement of pebbles on $G$. For our purposes, we will always assume that $G$ is connected. A pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. The pebbling number of a connected graph $G$ is the smallest number $f(G)$ such that, however $f(G)$ pebbles are distributed on the vertices of $G$, we can move a pebble to any root vertex by a sequence of pebbling moves [3]. Implicit in this
definition is the fact that if after moving to vertex \( v \) one desires to move to another root vertex, the pebbles reset to their original initial configuration.

Chung [3] defined the **two-pebbling property** and Wang [37] extended Chung’s definition to the **odd two-pebbling property**. Given a distribution of pebbles on \( G \), let \( p \) be the number of pebbles in that distribution, let \( q \) be the number of occupied vertices (vertices with at least one pebble), and let \( r \) be the number of vertices with the odd number of pebbles. We say that \( G \) satisfies the two-pebbling property (respectively, the odd two-pebbling property), if it is possible to move two pebbles to any specified target vertex whenever \( p \) and \( q \) satisfy the inequality \( p + q > 2f(G) \) (respectively, whenever \( p \) and \( r \) satisfy \( p + r > 2f(G) \)). Graham conjectured that for any connected graphs \( G \) and \( H \), \( f(G \times H) \leq f(G)f(H) \) and this conjecture has generated a great deal of interest to study pebbling [3].

The **\( t \)-pebbling number** of a vertex \( v \) in a graph \( G \) is the smallest number \( f_t(v, G) \) with the property that from every placement of \( f_t(v, G) \) pebbles on \( G \), it is possible to move \( t \) pebbles to \( v \) by a sequence of pebbling moves. The \( t \)-pebbling number of the graph \( G \), denoted by \( f_t(G) \), is the maximum \( f_t(v, G) \) over all the vertices of \( G \) [9].

The **cover pebbling number** \( \gamma(G) \), of a connected graph \( G \) [5] is the minimum number of pebbles such that however the pebbles are initially placed on the vertices of the graph we can eventually put a pebble on every vertex simultaneously. One application in [5] for \( \gamma(G) \) is based on a military application where troops must be distributed simultaneously.

In our thesis, in Chapter 1, we collect the basic definitions which are essential for the subsequent Chapters. For graph theoretic terminology, we
refer to Bondy and Murty [2], and Harary [7].

Jahangir graph $J_{n,m}$ for $m \geq 3$ [33] is a graph on $nm+1$ vertices, that is, a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$. In Chapter 2, we study the pebbling number of the Jahangir graph $J_{n,m}$ [22, 23].

Hulbert [4] defines the concept of pebbling number through the concept of bad pebbling distribution. We now generalize this in the setting of $t$-pebbling. If $D_t$ is a distribution of pebbles on the vertices of $G$ and there is some choice of a vertex $v$ ($v$ is any specified root vertex or target vertex) such that it is impossible to move $t$ pebbles to $v$ then we say that $D_t$ is a bad $t$-pebbling distribution. We denote by $D_t(v)$ the number of pebbles on the vertex $v$ in $D_t$ and let $|D_t|$ be the total number of pebbles in $D_t$, that is, $|D_t| = \sum_{v \in V(G)} D_t(v)$. We define the $t$-pebbling number of a graph $G$, $f_t(G)$, to be one more than the maximum $k$ such that there exists a bad $t$-pebbling distribution $D_t$ of size $k$. In Chapter 3, we compute the $t$-pebbling number of Jahangir graphs [24] and we give alternate proofs for the $t$-pebbling numbers of even and odd cycles. Lourdusamy [17] has defined the concept of $2t$-pebbling property and has proved that all even cycles satisfy the $2t$-pebbling property. A graph $G$ satisfies the $2t$-pebbling property if it is possible to move $2t$ pebbles to any specified target vertex of $G$ starting from every configuration in which $p \geq 2f_t(G) - q + 1$ or equivalently $p + q > 2f_t(G)$ for all $t$ where $p$ is the number of pebbles on $G$ and $q$ is the number of vertices with at least one pebble [17]. In Chapter 3, we give an alternate proof for the result that all even cycles satisfy the $2t$-pebbling property. We also prove that all odd cycles satisfy the $2t$-pebbling property. In [17], Lourdusamy has generalized
Graham’s pebbling conjecture into $f_t(G \times H) \leq f(G)f_t(H)$ where $G$ and $H$ are connected graphs. We call this the $t$-pebbling conjecture. He proved that it is true for a graph which is the direct product of a path with an even cycle. We show that the $t$-pebbling conjecture is true for a graph which is the direct product of a path with a cycle.

In [17], A.Lourdusamy has defined the odd $2t$-pebbling property. We say that a vertex $v$ in a graph $G$ satisfies the odd $2t$-pebbling property if, for any arrangement of pebbles with at least $2f_t(v, G) - r + 1$ pebbles, where $r$ is the number of vertices with an odd number of pebbles in the arrangement, it is possible to put $2t$ pebbles on the vertex $v$ using pebbling moves. If every vertex satisfies the odd $2t$-pebbling property then we say that the graph $G$ satisfies the odd $2t$-pebbling property. In Chapter 4, we show that a tree satisfies the odd $2t$-pebbling property. We also prove that the product of trees satisfies the $t$-pebbling conjecture.

A fan graph denoted by $F_n$ is a path $P_{n-1}$ plus an extra vertex connected to all vertices of the path $P_{n-1}$. In Chapter 5, we have computed the $t$-pebbling number of fan graphs. We have shown that fan graphs satisfy the $2t$-pebbling property. Also in Chapter 5, we have proved that the $t$-pebbling conjecture is true for a path by a fan graph, for a complete graph by a fan graph, for an $m$-star ($m > 1$) by a fan graph, for a complete bipartite graph by a fan graph, for a complete $r$-partite graph by a fan graph, for a graph with the $2t$-pebbling property, for a fan graph by a fan graph, for a fan graph by a path, for a fan graph by an $m$-star ($m > 1$), for a fan graph by a complete bipartite graph, for a fan graph by a complete $r$-partite graph, and for a wheel graph by a wheel graph [28].
Recall that a set of vertices of $K$ in $G$ is a covering if every edge of $G$ has at least one end in $K$. Lourdusamy and Punitha Tharani [18, 19] introduced the concept of covering cover pebbling number. The covering cover pebbling number, denoted by $\sigma(G)$, of a connected graph $G$, is the smallest number of pebbles such that, however the pebbles are initially placed on the vertices of the graph, after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of $G$. In Chapter 6, we compute the covering cover pebbling number for cycles. And we also determine the covering cover pebbling number for certain classes of unicyclic graphs [25, 26, 27].

For publication position of the thesis, please refer to [22, 23, 24, 25, 26, 27, 28].