Chapter 3

Fuzzy ideal theory

3.1 Introduction

The fundamental concept of a fuzzy set was introduced by Zadeh [1] in 1965. Chang [26] defined the notion of fuzzy topology in 1968. Subsequently, various aspects of general topology were extended to fuzzy topology and also many new ideas were formed [40]. To explore some deeper ideas, Lowen [27] gave an alternative definition of fuzzy topology. Now, there are some properties, which are in general, local in nature but in some cases they also reveal as the properties of the whole space. Naturally, investigation on these types of properties are always a point of attraction in general topology as well as in fuzzy topology. Lot of work has done by many authors in general topology. The notion of ideal in general topology shows a direction of study on such local behaviors. Kuratowski [44], Vaidyanathaswamy [45,46], and several other authors have carried out a lot of work in this field [47,48,49,50,51,52]. Recently, Jankovic and Hamlett [48], made

an extensive work and reviewed some works in this field. Now, the notion of fuzzy point and fuzzy q-neighbourhood systems for fuzzy points opens the scope for such analysis in fuzzy topology also.

In this chapter, we extend some of those ideas of general topology in fuzzy topological spaces. In section 2, we define fuzzy ideal on any non-empty set and introduce the notion of fuzzy local function corresponding to a fuzzy topological space. We extend some characterization theorems on local functions of general topological spaces to fuzzy topological spaces. Some notable departure from general topological results are also found. We have succeeded in generating new finer fuzzy topologies by fuzzy ideals from old fuzzy topological spaces. In section 3, we discuss the basic structure of new fuzzy topologies and it is established that the new fuzzy topologies cannot be applied further to find more finer fuzzy topologies by the same fuzzy ideal. We also form a basis for such new finer fuzzy topologies. Finally, in section 4, we define quasi-cover of a fuzzy set and introduce the notion of compatibility between a fuzzy ideal and a fuzzy topological space and obtain some equivalent conditions. Some other interesting results are also found and it is proved that under some compatible conditions the bases for new fuzzy topologies generated by fuzzy ideals are itselfs the new fuzzy topologies.

3.2 Fuzzy ideal, fuzzy local function and generated fuzzy topology

Definition 3.2.1 A non-empty collection of fuzzy sets $I$ of a set $X$ is called a fuzzy ideal on $X$ if and only if

(i) $\mu \in I$ and $\nu \subseteq \mu \Rightarrow \nu \in I$ [heredity],

(ii) $\mu \in I$ and $\nu \in I \Rightarrow \mu \cup \nu \in I$ [finite additivity].
Definition 3.2.2 Let \((X, \tau)\) be a fuzzy topological space and \(I\) be a fuzzy ideal on \(X\). Let \(A\) be any fuzzy set in \(X\). Then the fuzzy local function \(A^*(I, \tau)\) of \(A\) is the union of all fuzzy points \((x)_\lambda\) such that if \(\mu \in N\{(x)_\lambda\}\) and \(I \in I\) then there is at least one \(y \in X\) for which \(\mu(y) + A(y) - 1 > I(y)\).

Therefore, any \((x)_\lambda \notin A^*(I, \tau)\) implies there is at least one \(\mu \in N\{(x)_\lambda\}\) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I(y)\) for some \(I \in I\).

We will occasionally write \(A^*\) or \(A^*(I)\) in place of \(A^*(I, \tau)\) and it will cause no ambiguity.

Example 3.2.1 The simplest fuzzy ideals on \(X\) are \(\{O_X\}\) and \(\mathcal{F}(X)\) (the set of all fuzzy sets in \(X\)). Obviously, for any fuzzy set \(A\) in \(X\),
\[
I = \{O_X\} \Leftrightarrow A^*(I, \tau) = clA.
\]
Also,
\[
I = \mathcal{F}(X) \Leftrightarrow A^*(I, \tau) = O_X.
\]

The following theorem contains some basic properties of fuzzy local functions which are extended from the general topological ones [48]. Some differences are also observed, which are basically due to the inner feature of fuzzy systems.

Theorem 3.2.1 Let \((X, \tau)\) be a fuzzy topological space and \(I, J\) be two fuzzy ideals on \(X\). Then for any two fuzzy sets \(A, B\) in \(X\),
\[
(i) \ A \subseteq B \Rightarrow A^*(I, \tau) \subseteq B^*(I, \tau),
\]
\[
(ii) \ I \subseteq J \Rightarrow A^*(J, \tau) \subseteq A^*(I, \tau),
\]
\[
(iii) \ A^* = cl(A^*) \subseteq clA,
\]
\[
(iv) \ (A^*)^* \subseteq A^*,
\]
\[
(v) \ (A \cup B)^* = A^* \cup B^*,
\]
\[
(vi) \ I \in I \Rightarrow (A \cup I)^* = A^*.
\]
Proof. (i). Since $A \subseteq B$ implies $A(x) \leq B(x)$ for every $x \in X$. Therefore, by Definition 3.2.2, $(x)_{\lambda} \in A^*$ implies $(x)_{\lambda} \in B^*$. This completes the proof of (i).

(ii). Clearly, $I \subseteq J$ implies $A^*(J) \subseteq A^*(I)$, as there may be other fuzzy sets which belongs to $I$ so that for a fuzzy point $(x)_{\lambda}$, $(x)_{\lambda} \in A^*(I)$ but $(x)_{\lambda}$ may not be contained in $A^*(J)$.

(iii). Since $\{0_X\} \subseteq I$ for any fuzzy ideal $I$ on $X$, therefore, by (ii) and Example 3.2.1, $A^*(I) \subseteq A^*(\{0_X\}) = cl(A^*)$, for any fuzzy set $A$ in $X$. Suppose, $(x)_{\lambda} \in cl(A^*)$. So there is at least one $y \in X$ such that $\mu(y) + A^*(y) > 1$ for each q-nbd $\mu$ of $(x)_{\lambda}$. Therefore, $A^*(y) \neq 0$ for such an $y \in X$. Let, $t = A^*(y)$. Clearly, $(y)_{\lambda} \in A^*$. Also $t + \mu(y) > 1$ implies $\mu$ is q-nbd of $(y)_{\lambda}$. Now, $(y)_{\lambda} \in A^*$ implies there is at least one $x' \in X$ such that $\nu(x') + A^*(x') - 1 > I(x')$ for each $\nu \in N(\{(y)_{\lambda}\})$ and $I \in T$. This is also true for $\mu$. So, there is at least one $x'' \in X$ such that $\mu(x'') + A^*(x'') - 1 > I(x'')$, for each $I \in T$. Since, $\mu$ is an arbitrary q-nbd of $(x)_{\lambda}$, therefore $(x)_{\lambda} \in A^*$. Hence, $A^* = cl(A^*) \subseteq clA$.

(iv). By (iii), we have $(A^*)^* = cl((A^*)^*) \subseteq cl(A^*) = A^*$.

(v). Suppose, $(x)_{\lambda} \notin A^* \cup B^*$, i.e., $\lambda \geq (A^* \cup B^*)(x) = \max\{A^*(x), B^*(x)\}$. So, $(x)_{\lambda}$ is not contained in both $A^*$ and $B^*$. This implies there is at least one q-nbd $\mu_1$ of $(x)_{\lambda}$ such that for every $y \in X$, $\mu_1(y) + A^*(y) - 1 \leq I_1(y)$, for some $I_1 \in T$ and similarly there is at least one q-nbd $\mu_2$ of $(x)_{\lambda}$ such that for every $y \in X$, $\mu_2(y) + B^*(y) - 1 \leq I_2(y)$, for some $I_2 \in T$. Let $\mu = \mu_1 \cap \mu_2$. So $\mu$ is also a q-nbd of $(x)_{\lambda}$ and $\mu(y) + A \cup B^*(y) - 1 \leq I_1 \cup I_2(y)$, for every $y \in X$. Therefore, by finite additivity of fuzzy ideal, as $I_1 \cup I_2 \in T$, $(x)_{\lambda} \notin (A \cup B)^*$. Hence $(A \cup B)^* \subseteq A^* \cup B^*$. Clearly, both $A$ and $B \subseteq A \cup B$ implies $A^* \cup B^* \subseteq (A \cup B)^*$ and this completes the proof of (v).

(vi). It is clear that $I \in T$ implies $I^* = 0_X$ so that by (v), $(A \cup I)^* =
Now to generate new fuzzy topologies we are looking for some fuzzy closure operators. So, we first consider the notion of a fuzzy closure operator in fuzzy topology. In the following definition and also in many situations, by $\mathcal{F}(X)$, we will mean the set of all fuzzy sets in $X$.

**Definition 3.2.3 (27)** A fuzzy closure operator $\psi : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ is defined by,

(i) $\psi(0_X) = 0_X$,

(ii) $A \in \mathcal{F}(X) \Rightarrow A \subseteq \psi(A)$,

(iii) $A, B \in \mathcal{F}(X) \Rightarrow \psi(A \cup B) = \psi(A) \cup \psi(B)$,

(iv) $A \in \mathcal{F}(X) \Rightarrow \psi(\psi(A)) = \psi(A)$.

Obviously, $\{ A : \psi(A) = A \}$ constitutes a collection of fuzzy closed sets for a fuzzy topology on $X$.

**Theorem 3.2.2** Let $d : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ be a function such that,

(a) $d(0_X) = 0_X$,

(b) $d(A \cup B) = d(A) \cup d(B)$,

(c) $d(d(A)) \subseteq d(A)$,

where $A, B$ are any fuzzy sets in $X$.

Then $\psi : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ defined by $\psi(A) = A \cup d(A)$ is a fuzzy closure operator.

Clearly, $d$ does not necessarily coincide with the fuzzy derived set operator in the generated fuzzy topology.

**Theorem 3.2.3** The operation $* : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ satisfies all the required condition for $d$. 

28
Proof. Since, \((0_x)^* = 0_x\), \((A \cup B)^* = A^* \cup B^*\) and \((A^*)^* \subseteq A^*\). Therefore, the proof is completed. □

As a consequence of the above theorem, we are now in a position to define a closure operator on a fuzzy topological space.

Let \((X, \tau)\) be a fuzzy topological space and \(\mathcal{I}\) be a fuzzy ideal on \(X\). Let us define \(d^\wedge(A) = A \cup A^*\) for any fuzzy set \(A\) in \(X\). Then by Theorem 3.2.2, it is clear that \(d^\wedge\) satisfies all the conditions of a fuzzy closure operator. Let \(\tau^\wedge(\mathcal{I})\) be the fuzzy topology generated by \(d^\wedge\), i.e.,

\[
\tau^\wedge(\mathcal{I}) = \{A : d^\wedge(A^c) = A^c\}.
\]

Now,

\[
\mathcal{I} = \{0_x\} \Rightarrow d^\wedge(A) = (A \cup A^*) = A \cup cl(A) = d(A),
\]

for every \(A \in \mathcal{F}(X)\).

So, \(\tau^\wedge(\{0_x\}) = \tau\). Again, \(\mathcal{I} = \mathcal{F}(X) \Rightarrow d^\wedge(A) = A\), because \(A^* = 0_x\), for every \(A \in \mathcal{F}(X)\). Therefore, \(\tau^\wedge(\mathcal{F}(X))\) is the fuzzy discrete topology on \(X\). Since, \(\{0_x\}\) and \(\mathcal{F}(X)\) are the two extreme fuzzy ideals on \(X\); therefore, for any fuzzy ideal \(\mathcal{I}\) on \(X\) we have \(\{0_x\} \subseteq \mathcal{I} \subseteq \mathcal{F}(X)\). Hence, we can conclude by Theorem 3.2.1(ii) that

\[
\tau^\wedge(\{0_x\}) \subseteq \tau^\wedge(\mathcal{I}) \subseteq \tau^\wedge(\mathcal{F}(X)),
\]

i.e.,

\[
\tau \subseteq \tau^\wedge(\mathcal{I}) \subseteq \text{fuzzy discrete topology},
\]

for any fuzzy ideal \(\mathcal{I}\) on \(X\).

In particular, we have, for any two fuzzy ideals \(\mathcal{I}\) and \(\mathcal{J}\) on \(X\),

\[
\mathcal{I} \subseteq \mathcal{J} \Rightarrow \tau^\wedge(\mathcal{I}) \subseteq \tau^\wedge(\mathcal{J}).
\]
Next theorem generalizes the above results for any two fuzzy topologies on a set $X$.

**Theorem 3.2.4** Let $\tau_1, \tau_2$ be two fuzzy topologies on $X$. Then for any fuzzy ideal $\mathcal{I}$ on $X$, $\tau_1 \subseteq \tau_2$ implies

(i) $A^*(\tau_2, \mathcal{I}) \subseteq A^*(\tau_1, \mathcal{I})$, for every $A \in \mathcal{F}(X)$, and

(ii) $\tau_1^*(\mathcal{I}) \subseteq \tau_2^*(\mathcal{I})$.

**Proof.** Since, every $\tau_1$ q-nbd of any fuzzy point $(x)_\lambda$ is also a $\tau_2$ q-nbd of $(x)_\lambda$. Therefore, $A^*(\tau_2, \mathcal{I}) \subseteq A^*(\tau_1, \mathcal{I})$ as there may be other $\tau_2$ q-nbd of $(x)_\lambda$ where the condition for $(x)_\lambda \in A^*(\tau_2, \mathcal{I})$ may not hold true, although $(x)_\lambda \in A^*(\tau_1, \mathcal{I})$.

Clearly, $\tau_1^*(\mathcal{I}) \subseteq \tau_2^*(\mathcal{I})$ as $A^*(\tau_2, \mathcal{I}) \subseteq A^*(\tau_1, \mathcal{I})$. $\square$

Actually, new fuzzy topologies restricts a fuzzy point to be an accumulation point of a fuzzy set where they may be an accumulation point of a fuzzy set on old fuzzy topologies. The following theorem shows how the fuzzy derived sets in new generated fuzzy topologies differ from the old ones.

**Theorem 3.2.5** $A^d \subseteq A^d^*$ and $A^{d^*} \subseteq A^*$ for all fuzzy set $A$ in $X$, where $A^d$ denotes the fuzzy derived set of $A$ in $\tau^*$ fuzzy topology.

**Proof.** Since, $\tau \subseteq \tau^*$. Therefore, $(x)_\lambda \in A^d$ implies every q-nbd of $(x)_\lambda$ in fuzzy $\tau^*$-topology is quasi-coincident with $A \Rightarrow$ every q-nbd of $(x)_\lambda$ in fuzzy $\tau$-topology is quasi-coincident with $A \Rightarrow (x)_\lambda \in A^d$, so that $A^d \subseteq A^d^*$.

Again, for any fuzzy point $(x)_\lambda \in A^d^*$ implies $(x)_\lambda \in d^*(A) = A \cup A^*$, i.e., $(x)_\lambda \in A$ or $(x)_\lambda \in A^*$ or both. Now, if $(x)_\lambda \in A$, then for any q-nbd $\mu$ of $(x)_\lambda$ in fuzzy $\tau^*$-topology, there exists $y \in X$ such that $x \neq y$ and $A(y) + \mu(y) > 1$. This implies $(x)_\lambda$ is a fuzzy accumulation point of the fuzzy set $A'$ such that

$$A'(p) = \begin{cases} A(p) & \text{if } p \neq x, \\ \lambda' & \text{if } p = x \text{ and } \lambda' < \lambda. \end{cases}$$
Obviously, $A' \subseteq A$, so that $(A')^* \subseteq A^*$ and also $(x)_\lambda \not\in A'$. Therefore, $(x)_\lambda \in (A')^*$, because $(x)_\lambda \in d'(A') = A' \cup (A')^*$. So, $(x)_\lambda \in A^*$. Hence, $(x)_\lambda \in A^d \Rightarrow (x)_\lambda \in A^* \Rightarrow A^d \subseteq A^*$. □

**Definition 3.2.4** A fuzzy set $\mu$ in $X$ is called fuzzy closed and discrete if and only if $\mu^d = 0_X$.

**Theorem 3.2.6** Let $(X, \tau)$ be a fuzzy topological space with $I$ be a fuzzy ideal on $X$. Then,

(i) $I \in I \Rightarrow I$ is closed and discrete in $(X, \tau^*)$.

(ii) $A^* = cl(A - I)$ for every $I \in I$ and for any fuzzy set $A$ in $X$, (where $A - I$ is a fuzzy set as defined in Definition 2.2.11).

**Proof.** (i). Since, $I \in I \Rightarrow I^* = 0_X$. Therefore, by Theorem 3.2.5, $I^d = 0_X$.

(ii). Since, $(x)_\lambda \in A^*$ implies there is at least one $z \in X$ such that $\mu(z) + A(z) - 1 > I(z)$ for each $\mu \in N\{(x)_\lambda\}$ and $I \in I$; i.e., for each $\mu \in N\{(x)_\lambda\}$, there is at least one $z \in X$ such that, $\mu(z) + (A - I)(z) > 1$, i.e., $(x)_\lambda \in cl(A - I)$, for every $I \in I$. Therefore, $A^* \subseteq cl(A - I)$. Again, any $(x)_\lambda \in cl(A - I)$ implies for each $\mu \in N\{(x)_\lambda\}$, there is at least one $y \in X$ such that $\mu(y) + (A - I)(y) > 1$, i.e., $\mu(y) + A(y) - 1 > I(y)$. Since, this is true for any $I \in I$. Therefore, $(x)_\lambda \in A^*$. This completes the proof. □

The above theorem characterizes an useful fact about the construction of different fuzzy ideals on different fuzzy topologies. It is found that any fuzzy ideal on any fuzzy topology cannot generate new finer fuzzy topologies. The following examples show some cases where the two fuzzy topologies $\tau$ and $\tau^*$ on $X$ are equal.

**Example 3.2.2** (a) If $I$ be a fuzzy ideal on $X$ such that $A^d \subseteq cl(A - I)$ for
every $I \in \mathcal{I}$ and for any fuzzy set $A$ in $X$, then it is clear that $A^d \subseteq A'$ so that $\text{cl}A = \text{cl}'(A)$. Therefore, $\tau = \tau^*$. 

(b) Again, if $\mathcal{I}$ be such that $A^d = (A - I)^d$ for every $I \in \mathcal{I}$, then obviously $\tau = \tau^*$. 

(c) Also, $A^d = A'$, for a fuzzy ideal $\mathcal{I}$ on $X$ implies $\tau = \tau^*$. 

3.3 Basic structure of generated fuzzy topology 

Let $(X, \tau)$ be a fuzzy topological space and $\mathcal{I}$ be a fuzzy ideal on $X$. Let $A$ be a q-nbd of a fuzzy point $(x)_\lambda$ in fuzzy $\tau^*$-topology. Then, there exists $\mu \in \tau^*$ such that $\lambda + \mu(x) > 1$ and $\mu \subseteq A$. 

Now, $\mu \in \tau^* \iff \mu^\sharp$ is $\tau^*$-closed $\iff \text{cl}'(\mu^\sharp) = \mu^\sharp \iff (\mu^\sharp)^* \subseteq \mu^\sharp \iff \mu \subseteq \{(\mu^\sharp)^*\}^\circ$. 

Therefore, $\lambda + \mu(x) > 1 \Rightarrow \lambda + \{(\mu^\sharp)^*\}^\circ(x) > 1 \Rightarrow \lambda + 1 - (\mu^\sharp)^*(x) > 1 \Rightarrow \lambda > (\mu^\sharp)^*(x) \Rightarrow (x)_\lambda \not\supseteq (\mu^\sharp)^*$. 

This implies, there exists at least one q-nbd $\nu_1$ of $(x)_\lambda$ (in $\tau$) such that for every $y \in X$, $\nu_1(y) + \mu^\sharp(y) - 1 \leq I_1(y)$ for some $I_1 \in \mathcal{I}$, i.e., $\nu_1(y) - I_1(y) \leq \mu(y)$ for every $y \in X$. Therefore, as $\nu_1$ is a q-nbd of $(x)_\lambda$ (in $\tau$), there is a $\nu \in \tau$ such that $(x)_\lambda \nu$ and $\nu \subseteq \nu_1$. So, by heredity property of fuzzy ideal we have a $I \in \mathcal{I}$ for which $(x)_\lambda (\nu - I)$ and $(\nu - I) \subseteq \mu$, where $\nu - I$ is a fuzzy set as defined in Definition 2.2.11. Hence, for $\mu \in \tau^*$, we have a $\nu \in \tau$ and $I \in \mathcal{I}$ such that $(\nu - I) \subseteq \mu$. 

Let us denote $\beta(I, \tau) = \{\nu - I : \nu \in \tau, I \in \mathcal{I}\}$. Then we have the following theorem. 

**Theorem 3.3.1** $\beta$ forms a basis for the generated fuzzy topology $\tau^*$ of the fuzzy topological space $(X, \tau)$ with fuzzy ideal $\mathcal{I}$ on $X$. 

**Proof.** Clear from the above construction. □
Now, we site an example which is very important in the sense that it helps us to find some deeper characteristics of the new fuzzy topologies. It also justifies our above construction.

Example 3.3.1 Let $T$ be the fuzzy indiscrete topology on $X$, i.e., $T = \{0, 1\}$. So, $1_X$ is the only q-nbd of every fuzzy point $(x)_\lambda$. Now, $(x)_\lambda \in A^*$ for a fuzzy set $A \Leftrightarrow$ for each $I \in \mathcal{I}$, there is at least one $y \in X$ such that $1 + A(y) - 1 > I(y)$.

This implies, for each $I \in \mathcal{I}$, $A(y) > I(y)$ for at least one $y \in X$. So $A \notin \mathcal{I}$. Therefore, $A^* = 1_X$ if $A \notin \mathcal{I}$ and $A^* = 0_X$ if $A \in \mathcal{I}$. This implies that we have, $cl^*(A) = A \cup A^* = 1_X$, if $A \notin \mathcal{I}$ and $cl^*(A) = A$, if $A \in \mathcal{I}$, for any fuzzy set $A$ in $X$. Hence, $T^*(\mathcal{I}) = \{\mu : \mu \in \mathcal{I}\}$.

Let $\tau \vee T^*(\mathcal{I})$ be the supremum fuzzy topology of $\tau$ and $T^*(\mathcal{I})$, i.e., the smallest fuzzy topology generated by $\tau \cup T^*(\mathcal{I})$. Then we have the following theorem.

Theorem 3.3.2 $\tau^*(\mathcal{I}) = \tau \vee T^*(\mathcal{I})$.

Proof. Follows from the fact that $\beta$ forms a basis for $\tau^*$. □

Lemma 3.3.1 For any two fuzzy ideal $\mathcal{I}$ and $\mathcal{J}$ on $X$, $I \vee J = \{I \cup J : I \in \mathcal{I}, J \in \mathcal{J}\}$ and $I \cap J$ are fuzzy ideals on $X$.

Proof. Follows from the definition of fuzzy ideal. □

Theorem 3.3.3 Let $(X, \tau)$ be a fuzzy topological space and $\mathcal{I}, \mathcal{J}$ be two fuzzy ideals on $X$. Then, for any fuzzy set $A$ in $X$,

(i) $A^*(\mathcal{I} \cap \mathcal{J}) = A^*(\mathcal{I}) \cup A^*(\mathcal{J})$.

(ii) $A^*(\mathcal{I} \vee \mathcal{J}, \tau) = A^*(\mathcal{I}, \tau^*(\mathcal{J})) \cap A^*(\mathcal{J}, \tau^*(\mathcal{I}))$.
Proof. (i). Let \((x)_\lambda \not\in A^*(\mathcal{I}) \cup A^*(\mathcal{J})\). Then, \((x)_\lambda \not\in A^*(\mathcal{I})\) and \((x)_\lambda \not\in A^*(\mathcal{J})\). Now \((x)_\lambda \not\in A^*(\mathcal{I})\) implies there is at least one q-nbd \(\mu\) of \((x)_\lambda\) (in \(\tau\)) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I(y)\), for some \(I \in \mathcal{I}\). Again, \((x)_\lambda \not\in A^*(\mathcal{J})\) implies there is at least one q-nbd \(\nu\) of \((x)_\lambda\) (in \(\tau\)) such that for every \(y \in X\), \(\nu(y) + A(y) - 1 \leq J(y)\), for some \(J \in \mathcal{J}\). Therefore, we have \(\mu \cap \nu + A(y) - 1 \leq I \cap J(y)\), for every \(y \in X\). Since, \(\mu \cap \nu\) is also a q-nbd of \((x)_\lambda\) (in \(\tau\)) and \(I \cap J \in \mathcal{I} \cap \mathcal{J}\), therefore \((x)_\lambda \not\in A^*(\mathcal{I} \cap \mathcal{J})\), so that \(A^*(\mathcal{I} \cap \mathcal{J}) \subseteq A^*(\mathcal{I}) \cup A^*(\mathcal{J})\). Also \(\mathcal{I} \cap \mathcal{J}\) is included in both \(\mathcal{I}\) and \(\mathcal{J}\), so by Theorem 3.2.1(ii), reverse inclusion is obvious, which completes the proof of (i).

(ii). Since \((x)_\lambda \not\in A^*(\mathcal{I} \vee \mathcal{J}, \tau)\) implies there is at least one q-nbd \(\mu\) of \((x)_\lambda\) (in \(\tau\)) such that, for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I'(y)\), for some \(I' \in \mathcal{I} \vee \mathcal{J}\). Therefore, by heredity of fuzzy ideals and considering the structure of fuzzy open sets in generated fuzzy topology, we can find \(\nu\) or \(\nu'\), the q-nbds of \((x)_\lambda\) in \(\tau^*(\mathcal{I})\) or \(\tau^*(\mathcal{J})\) respectively, such that for every \(y \in X\), \(\nu(y) + A(y) - 1 \leq J(y)\) or \(\nu'(y) + A(y) - 1 \leq I(y)\) for some \(J \in \mathcal{J}\) or \(I \in \mathcal{I}\). This implies \((x)_\lambda \not\in A^*(\mathcal{J}, \tau^*(\mathcal{I}))\) or \((x)_\lambda \not\in A^*(\mathcal{I}, \tau^*(\mathcal{J}))\). Thus we have, \(A^*(\mathcal{I}, \tau^*(\mathcal{J})) \cap A^*(\mathcal{J}, \tau^*(\mathcal{I})) \subseteq A^*(\mathcal{I} \vee \mathcal{J}, \tau)\).

Conversely, let \((x)_\lambda \not\in A^*(\mathcal{I}, \tau^*(\mathcal{J}))\). This implies there is at least one q-nbd \(\mu\) in \(\tau^*(\mathcal{J})\) of \((x)_\lambda\) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I_1(y)\), for some \(I_1 \in \mathcal{I}\). Since \(\mu\) is a \(\tau^*(\mathcal{J})\) q-nbd of \((x)_\lambda\), by heredity of fuzzy ideals we have a q-nbd \(\nu\) of \((x)_\lambda\) (in \(\tau\)) such that for every \(y \in X\), \(\nu(y) + A(y) - 1 \leq I \cup J(y)\), for some \(I \in \mathcal{I}\) and \(J \in \mathcal{J}\), i.e., \((x)_\lambda \not\in A^*(\mathcal{I} \vee \mathcal{J}, \tau)\). Thus, \(A^*(\mathcal{I} \vee \mathcal{J}, \tau) \subseteq A^*(\mathcal{I}, \tau^*(\mathcal{J}))\).

Similarly, \(A^*(\mathcal{I} \vee \mathcal{J}, \tau) \subseteq A^*(\mathcal{J}, \tau^*(\mathcal{I}))\) and this completes the proof. □

An important result directly follows from the above theorem that \(\tau^*(\mathcal{I})\) and \([\tau^*(\mathcal{I})]^*(\mathcal{I})\) (in short \(\tau^{**}\)) are equal for any fuzzy ideal \(\mathcal{I}\) on \(X\), i.e., we cannot generate new finer fuzzy topologies by applying the same fuzzy ideal on a fuzzy topology \(\tau\) on \(X\).
Corollary 3.3.1 Let \((X, \tau)\) be a fuzzy topological space and \(I\) be a fuzzy ideal on \(X\). Then, \(A^*(I, \tau) = A^*(I, \tau^*)\) and \(\tau^*(I) = [\tau^*(I)]^*(I)\).

Proof. Putting \(I = J\) in Theorem 3.3.3(ii), we have, \(A^*(I, \tau) = A^*(I, \tau^*)\), which implies \(\tau^*(I) = [\tau^*(I)]^*(I)\). □

Corollary 3.3.2 Let \((X, \tau)\) be a fuzzy topological space and \(I, J\) be two fuzzy ideals on \(X\). Then,

\begin{align*}
(i) \quad & \tau^*(I \cup J) = [\tau^*(J)]^*(I) = [\tau^*(I)]^*(J), \\
(ii) \quad & \tau^*(I \cup J) = \tau^*(I) \vee \tau^*(J), \\
(iii) \quad & \tau^*(I \cap J) = \tau^*(I) \cap \tau^*(J).
\end{align*}

Proof. (i). By Theorem 3.3.3(ii), the result follows.

(ii). By (i), we have, \(\tau^*(I \cup J) = [\tau^*(J)]^*(I) = [\tau^*(I)]^*(J)\) (by Theorem 3.3.2). Since, \(\tau \subseteq \tau^*\) for any fuzzy ideal on \(X\). Therefore, \(\tau^*(I \cup J) = \tau^*(J) \vee \tau \vee T^*(I) = \tau^*(I) \vee \tau^*(J)\).

(iii). Since \(I \cap J\) is included in both \(I\) and \(J\), therefore \(\tau^*(I \cap J)\) is included in both \(\tau^*(I)\) and \(\tau^*(J)\). Now \(\mu\) is a fuzzy open set in \(\tau^*(I) \cap \tau^*(J)\), implies \(\mu^c\) is fuzzy closed set in both \(\tau^*(I)\) and \(\tau^*(J)\). That means \((\mu^c)^*(I) \subseteq \mu^c\) and \((\mu^c)^*(J) \subseteq \mu^c\). So, \((\mu^c)^*(I) \cup (\mu^c)^*(J) \subseteq \mu^c\). Therefore, by Theorem 3.3.3(i), \((\mu^c)^*(I \cap J) \subseteq \mu^c\). Hence, \(\mu \in \tau^*(I \cap J)\). This completes the proof. □

3.4 Compatibility of fuzzy ideals with fuzzy topology

In this section we discuss about the compatibility of fuzzy ideals with the fuzzy topologies. It shows many interesting fuzzy topological features. Many equivalent conditions are found with the compatibility conditions.
Definition 3.4.1 For a fuzzy topological space $(X, \tau)$ with fuzzy ideal $I$, $\tau$ is said to be compatible with $I$, denoted by $\tau \sim I$, if for every fuzzy set $A$ in $X$; if for all fuzzy point $(x)_\lambda \in A$, there exists a q-nbd $\mu$ of $(x)_\lambda$ (in $\tau$) such that $\mu(y) + A(y) - 1 \leq I(y)$ hold for every $y \in X$ and for some $I \in I$, then $A \in I$.

Definition 3.4.2 Let $\{B_\alpha, \alpha \in \Lambda\}$ be an indexed family of fuzzy sets in $X$ such that $B_\alpha \cap A$ for each $\alpha \in \Lambda$, where $A$ is a fuzzy set in $X$. Then $\{B_\alpha, \alpha \in \Lambda\}$ is said to be a quasi-cover of $A$ if and only if, $A(y) + \bigcup_{\alpha \in \Lambda} B_\alpha(y) \geq 1$ for every $y \in X$.

Further, if each $B_\alpha$ is fuzzy open set, then this quasi-cover will be called a fuzzy quasi-open cover of the fuzzy set $A$ in $X$.

Therefore, in either case $A^c \subseteq \bigcup_{\alpha \in \Lambda} B_\alpha$.

Theorem 3.4.1 For a fuzzy topological space $(X, \tau)$ with fuzzy ideal $I$ on $X$, the following are equivalent:

(i) $\tau \sim I$.

(ii) If for every fuzzy set $A$ in $X$ has a fuzzy quasi-open cover $\{B_\alpha, \alpha \in \Lambda\}$ such that for each $\alpha$, $A(y) + B_\alpha(y) - 1 \leq I(y)$, for some $I \in I$ and for every $y \in X$, then $A \in I$.

(iii) For every fuzzy set $A$ in $X$, $A \cap A^* = 0_X$ implies $A \in I$.

(iv) For every fuzzy set $A$ in $X$, $\bar{A} \in I$, where $\bar{A} = \bigcup_{x \in A}$ such that $(x)_\lambda \in A$ but $(x)_\lambda \notin A^*$.

(v) For every fuzzy $\tau^*$-closed set $A$, $\bar{A} \in I$.

(vi) For every fuzzy set $A$ in $X$, if $A$ contains no non-empty fuzzy subset $B$ with $B \subseteq B^*$, then $A \in I$ (here non-empty means $B \neq 0_X$).
Proof. We prove most of the equivalent conditions which ultimately prove all the equivalence.

(i) \(\Rightarrow\) (ii): Let \(\{B_\alpha, \alpha \in \Lambda\}\) be a fuzzy quasi-open cover of a fuzzy set \(A\) in \(X\) such that for each \(\alpha \in \Lambda\), \(A(y) + B_\alpha(y) - 1 \leq I(y)\) for some \(I \in \mathcal{I}\) and for every \(y \in X\). Therefore, as \(\{B_\alpha, \alpha \in \Lambda\}\) is a fuzzy quasi-open cover of \(A\), for each \((x)_\lambda \subseteq A\), there exists at least one \(B_{\alpha_0}\) such that \((x)_\lambda \subseteq B_{\alpha_0}\) and for every \(y \in X\), \(A(y) + B_{\alpha_0}(y) - 1 \leq I(y)\), for some \(I \in \mathcal{I}\). Obviously, \(B_{\alpha_0}\) is a q-nbd of \((x)_\lambda\) (in \(\tau\)). Therefore, as \(\tau \sim \mathcal{I}\), \(A \in \mathcal{I}\).

(ii) \(\Rightarrow\) (i): Clear from the fact that a collection of fuzzy open sets which contains at least one open q-nbd of each fuzzy point of \(A\), constitutes a fuzzy quasi-open cover of \(A\).

(ii) \(\Rightarrow\) (iii): Let \(A \cap A^* = O_X\), i.e., \(\min \{A(y), A^*(y)\} = 0\), for every \(y \in X\). So, a fuzzy point \((x)_\lambda \subseteq A\) implies \((x)_\lambda \not\subseteq A^*\). That means, there is a q-nbd \(\mu\) of \((x)_\lambda\) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I(y)\), for some \(I \in \mathcal{I}\), if \((x)_\lambda \subseteq A\). since \(\mu\) is a q-nbd of \((x)_\lambda\), therefore, there is a fuzzy open set \(\nu\) (in \(\tau\)) such that \((x)_\lambda \nu \subseteq \mu\) and so the collection of such \(\nu\)'s for each \((x)_\lambda \subseteq A\), constitutes a fuzzy quasi-open cover of \(A\). Therefore by condition (ii), \(A \in \mathcal{I}\).

(iii) \(\Rightarrow\) (i): Let for every fuzzy point \((x)_\lambda \subseteq A\), there is a q-nbd \(\mu\) of \((x)_\lambda\) (in \(\tau\)) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I(y)\), for some \(I \in \mathcal{I}\). That means \((x)_\lambda \not\subseteq A^*\). Now, there are two cases: either \(A^*(x) = 0\) or, \(A^*(x) \neq 0\) but \(\lambda > A^*(x)\). Let, if possible, \((x)_\lambda \subseteq A\) be such that \(\lambda > A^*(x)\) \(\neq 0\). Let \(\lambda' = A^*(x)\). Then the fuzzy point \((x)_{\lambda'} \subseteq A^*\) and also \((x)_{\lambda'} \subseteq A\). This implies for each q-nbd \(\nu\) of \((x)_{\lambda'}\), and for each \(I \in \mathcal{I}\), there is at least one \(y \in X\) such that \(\nu(y) + A(y) - 1 > I(y)\). Since \((x)_{\lambda'} \subseteq A\), this contradicts the assumption for every fuzzy point of \(A\). So, \(A^*(x) = 0\). That means, \((x)_\lambda \subseteq A\) implies \((x)_\lambda \not\subseteq A^*\). Now this is true for every fuzzy set \(A\) in \(X\). So, for every fuzzy set \(A\) in \(X\), \(A \cap A^* = O_X\). Hence, by condition
(iii), we have $A \in I$, which implies $r \sim I$.

(iii) $\Rightarrow$ (iv): Let the fuzzy point $(x)_{\lambda} \in \tilde{A}$. That means $(x)_{\lambda} \subseteq A$ but $(x)_{\lambda} \not\subseteq A^*$. So, there is a q-nbd $\mu$ of $(x)_{\lambda}$ such that for every $y \in X$, $\mu(y) + A(y) - 1 \leq I(y)$, for some $I \in I$. Since $A \subseteq A$, so for every $y \in X$, $\mu(y) + \tilde{A}(y) - 1 \leq I(y)$, for some $I \in I$. Therefore, $(x)_{\lambda} \not\subseteq \tilde{A}^*$, so that either $\tilde{A}^*(x) = 0$ or $\tilde{A}^*(x) \neq 0$ but $\lambda > \tilde{A}^*(x)$. Let $(x)_{\lambda_i}$ be a fuzzy point such that $\lambda_1 \leq \tilde{A}^*(x) < \lambda$, i.e., $(x)_{\lambda_i} \in \tilde{A}^*$. So for each q-nbd $\nu$ of $(x)_{\lambda_i}$ and for each $I \in I$, there is at least one $y \in X$ such that $\nu(y) + \tilde{A}(y) - 1 > I(y)$. Since $A \subseteq A$, for each q-nbd $\nu$ of $(x)_{\lambda_i}$ and for each $I \in I$, there is at least one $y \in X$ such that $\nu(y) + A(y) - 1 > I(y)$. This implies $(x)_{\lambda_i} \subseteq A^*$. But as $\lambda_1 < \lambda$, $(x)_{\lambda} \subseteq \tilde{A}$ implies $(x)_{\lambda_i} \subseteq \tilde{A}$, and therefore $(x)_{\lambda_i} \not\subseteq A^*$. This is a contradiction. Hence $\tilde{A}^*(x) = 0$, so that $(x)_{\lambda} \subseteq \tilde{A}$ implies $(x)_{\lambda} \not\subseteq \tilde{A}^*$ with $\tilde{A}^*(x) = 0$. Thus we have $\tilde{A} \cap \tilde{A}^* = 0_X$, for every fuzzy set $A$ in $X$. Hence, by condition (iii), $\tilde{A} \in I$.

(iv) $\Rightarrow$ (v): Straightforward.

(iv) $\Rightarrow$ (vi): Let $A$ be any fuzzy set in $X$ such that it contains no non-empty (i.e., not $0_X$) fuzzy subset $B$ with $B \subseteq B^*$. Clearly, for every fuzzy set $A$ in $X$, $A = \tilde{A} \cup (A \cap A^*)$. Therefore, $A^* = (\tilde{A} \cup (A \cap A^*))^* = \tilde{A}^* \cup (A \cap A^*)^*$ (by Theorem 3.2.1(v)). Now by condition (iv), $\tilde{A} \in I$ so that $\tilde{A}^* = 0_X$. Hence, $(A \cap A^*)^* = A^*$. But $(A \cap A^*) \subseteq A^*$ so that $(A \cap A^*) \subseteq (A \cap A^*)^*$. This contradicts the hypothesis about every fuzzy set $A$ in $X$ that, it contains no non-empty fuzzy subset $B$ with $B \subseteq B^*$. Therefore, $(A \cap A^*) = 0_X$ so that $A = \tilde{A}$ and hence by condition (iv), $A \in I$.

(vi) $\Rightarrow$ (iv): Since, for every fuzzy set $A$ in $X$, $\tilde{A} \cap \tilde{A}^* = 0_X$. Therefore, by condition (vi), as $\tilde{A}$ contains no non empty fuzzy subset $B$ with $B \subseteq B^*$, $\tilde{A} \in I$. 38
(v) ⇒ (i): Let \( A \) be any fuzzy set in \( X \). Let for every fuzzy point \( (x)_\lambda \in A \), there is a q-nbd \( \mu \) of \( (x)_\lambda \) (in \( r \)) such that for every \( y \in X \), \( \mu(y) + A(y) - 1 \leq I(y) \), for some \( I \in \mathcal{I} \). This implies \( (x)_\lambda \not\in A^* \). Let \( B = A \cup A^* \), then \( B^* = (A \cup A^*)^* = A^* \cup (A^*)^* = A^* \). So, \( cl^*(B) = B \cup B^* = B \). That means, \( B \) is fuzzy \( r^* \)-closed set. Therefore, by condition (v), \( \tilde{B} \in \mathcal{I} \). Again, any fuzzy point \( (y)_t \in \tilde{B} \) implies \( (y)_t \not\in B^* \). So, as \( B = A \cup A^* \), \( (y)_t \not\in A \). Now, by hypothesis about \( A \), we have for every \( (x)_\lambda \in A \), \( (x)_\lambda \not\in A^* \). So, \( \tilde{B} = A \). Hence \( A \in \mathcal{I} \), i.e., \( r \sim \mathcal{I} \). □

**Theorem 3.4.2** Let \((X, \tau)\) be a fuzzy topological space with \( \mathcal{I} \) be any fuzzy ideal on \( X \). Then the following are equivalent and implied by \( r \sim \mathcal{I} \):

(i) For every fuzzy set \( A \) in \( X \), \( Ap|A^* = \emptyset_X \) implies \( A^* = \emptyset_X \).

(ii) For every fuzzy set \( A \) in \( X \), \( A^* = \emptyset_X \) (where \( \tilde{A} \) is as defined in Theorem 3.4.1(iv)).

(iii) For every fuzzy set \( A \) in \( X \), \( (A \cap A^*)^* = A^* \).

**Proof.** Clear from Theorem 3.4.1. □

**Theorem 3.4.3** Let \((X, \tau)\) be a fuzzy topological space with \( \mathcal{I} \) be any fuzzy ideal on \( X \). Let \( r \sim \mathcal{I} \). Then a fuzzy set in \( X \) is closed with respect to \( r^* \) if and only if it is the union of a fuzzy set which is closed with respect to \( r \) and a fuzzy set in \( \mathcal{I} \).

**Proof.** Let \( A \) be a fuzzy set in \( X \) such that it is fuzzy \( r^* \)-closed. That means \( A^* \subseteq A \) and we have, \( A = \tilde{A} \cup A^* \). Since \( r \) is compatible with \( \mathcal{I} \), therefore \( \tilde{A} \in \mathcal{I} \). Also, \( A^* \) is always \( r \)-closed (by Theorem 3.2.1(iii)). This completes the necessary part of the proof.

Conversely, let \( A \) be any fuzzy set in \( X \) such that \( A = B \cup I \), where \( clB = B \) and \( I \in \mathcal{I} \). Therefore, by Theorem 3.2.1, \( A^* = B^* \cup I^* = B^* \subseteq clB = B \subseteq A \). That
means $A^* \subseteq B \subseteq A$. So, we have, $cl^*(A) = A \cup A^* = A$ and this implies $A$ is fuzzy $\tau^*$-closed set. □

An important consequence of Theorem 3.4.3 is the following corollary.

**Corollary 3.4.1** The fuzzy topology $\tau$ is compatible with the fuzzy ideal $I$ on $X$ implies $\beta(I, \tau)$, a basis for $\tau^*$ is itself a fuzzy topology and also $\beta = \tau^*$. ..

**Proof.** Clear from the previous theorem. □

Thus we have generalised the notion of ideal to fuzzy ideal in fuzzy systems. Now, the idea of quasi-coincident relation enables us to extend the notion of local functions in general topology to fuzzy local functions in fuzzy topology. Our fuzzy local functions reflects the true fuzzy feature through the fuzzy quasi-neighborhood systems. Clearly, many results are generalised in fuzzy topology with these two notions. Now we are able to find new finer fuzzy topologies generated by fuzzy ideals from old ones. Naturally, it will reflect more and more fuzzy topological behaviours like different compactness criterion, connectednes, semiregularizations, etc. It is also interesting to find that how much we can extend new fuzzy topologies with a fuzzy ideal in hand. We have found that we may generate but the same fuzzy topology with repeated application of same fuzzy ideal. This is actually natural consistency of the theory. It is also interesting to observe the compatibility of fuzzy ideals with different fuzzy topologies which actually shows the inner connection between fuzzy ideals and fuzzy topologies. And it turns out ultimately that the bases for new fuzzy topologies are the new fuzzy topologies if the fuzzy ideals are compatible with the old fuzzy topologies. All the above results are now opens a new direction in studying fuzzy topology. Our next chapter will show some cases.