
Fuzzy ideal theory
Fuzzy local function and generated fuzzy topology

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Abstract

In this paper we introduce the notion of fuzzy ideal in fuzzy set theory. The concept of fuzzy local function is also introduced here by utilizing the q-neighbourhood structure for a fuzzy topological space. These concepts are discussed with a view to find new fuzzy topologies from the original one. The basic structure, especially a basis for such generated fuzzy topologies and several relations between different fuzzy ideals and fuzzy topologies are also studied here. Finally, the notion of compatibility of fuzzy ideals with fuzzy topologies, are introduced and some equivalent conditions concerning this are established here. © 1997 Elsevier Science B.V.

Keywords: Fuzzy topology; Fuzzy ideal; Fuzzy local function

1. Introduction

The fundamental concept of a fuzzy set was introduced by Zadeh [11]. Subsequently, Chang defined the notion of fuzzy topology [3]. An alternative definition of fuzzy topology was given by Lowen [7]. Since then various aspects of general topology were investigated and carried out in fuzzy sense by several authors of this field. The local properties of a space which may also be in certain cases the properties of the whole space, are important field for study in both general and fuzzy topology. In general topology, by introducing the notion of ideal, Kuratowski [6], Vaidyanathaswamy [9, 10] and several other authors carried out such analyses. Recently, there has been an extensive study on the importance of ideal in general topology in the paper of Jankovic and Hamlett [5]. The notion of fuzzy point and q-neighbourhood (hereafter, in short, q-nbd) of a fuzzy point introduces the scope of such analysis in fuzzy topology [2, 8].

Our aim in this paper is to extend those ideas of general topology in fuzzy topological space (hereafter, in short, fts). In Section 3, we define fuzzy ideal for a set and introduce the notion of fuzzy local function corresponding to a fts. We have deduced some characterization theorems for such concepts exactly analogous to general topology and succeeded in finding out the generated new fuzzy topologies for any fts. In Section 4, we discuss...
the basic structure of new fuzzy topology and it is established that the new fuzzy topology cannot be further generated with the same fuzzy ideal. Finally, in Section 5, we define quasi-cover of a fuzzy set and introduce the notion of compatibility of fuzzy ideal with a fts and obtain some results concerning these. Since fuzzy ideal is also a dual fuzzy filter this paper will give another insight on the study of fuzzy filter theory [1, 4], Also, in the framework of this paper there are still many other aspects of fts, namely, compactness, semiregularization, submaximal, etc. which can be investigated further.

2. Preliminaries

Throughout the paper, by \((X, \tau)\) (or simply \(X\)) we mean a fts in Chang's [3] sense. A fuzzy point in \(X\) with support \(x \in X\) and value \(\lambda (0 < \lambda \leq 1)\) is denoted by \(x_\lambda\). A fuzzy point \(x_\lambda\) is said to be contained in a fuzzy set \(A\) in \(X\) if and only if \(\lambda \leq A(x)\) and this will be denoted by \(x_\lambda \in A\) [2].

For a fuzzy set \(A\) in \(X\), cl\((A)\) and \(A^*\) will respectively denote closure and complement of \(A\). The constant fuzzy sets taking values 0 and 1 on \(X\) are denoted by \(0_X\) and \(1_X\) respectively. A fuzzy set \(A\) in a fts \((X, \tau)\) is said to be quasi-coincident with a fuzzy set \(B\), denoted by \(A \equiv B\), if there exists \(x \in X\) such that \(A(x) + B(x) > 1\) [8]. Obviously, for any two fuzzy set \(A\) and \(B\), \(A \equiv B\) will imply \(B \equiv A\).

A fuzzy set \(A\) in a fts \((X, \tau)\) is called a q-nbd of a fuzzy point \(x_\lambda\) if and only if there exists a fuzzy open set \(\mu\) such that \(x_\lambda \in N(x)\). We will denote the set of all q-nbd of \(x_\lambda\) in \((X, \tau)\) by \(N(x)\).

A fuzzy point \(x_\lambda\) is called an accumulation point of a fuzzy set \(A\) in a fts \((X, \tau)\) if and only if (i) any q-nbd of \(x_\lambda\) is quasi-coincident with \(A\), (ii) if \(x_\lambda \in A\), any q-nbd of \(x_\lambda\) and \(A\) are quasi-coincident at some fuzzy point \(y\), such that \(x \neq y\) [2, 8]. It is known [8] that a fuzzy point \(x_\lambda\) is a cl\((A)\) if and only if for every q-nbd \(B\) of \(x_\lambda\), \(B \equiv A\). In the above definition of accumulation point, if only (i) holds then \(x_\lambda\) is an adherence point of \(A\) [8]. Also it is known [8] that \(B = \) derived fuzzy set of \(A\). For definitions and results not explained in this paper [2, 3, 7, 8] may be referred.

3. Fuzzy ideal, fuzzy local function and generated fuzzy topology

**Definition 3.1.** A non-empty collection of fuzzy sets \(\mathcal{J}\) of a set \(X\) is called a fuzzy ideal on \(X\) if and only if

(i) \(\mu \in \mathcal{J}\) and \(\nu \subseteq \mu \Rightarrow \nu \in \mathcal{J}\) [heredity],

(ii) \(\mu \in \mathcal{J}\) and \(\nu \in \mathcal{J} \Rightarrow \mu \cup \nu \in \mathcal{J}\) [finite additivity].

**Definition 3.2.** Let \((X, \tau)\) be a fts and \(\mathcal{J}\) be a fuzzy ideal on \(X\). Let \(A\) be any fuzzy set of \(X\). Then the fuzzy local function \(A^*(\mathcal{J}, \tau)\) of \(A\) is the union of all fuzzy points \(x_\lambda\) such that if \(\mu \in N(x_\lambda)\) and \(I \in \mathcal{J}\) then there is at least one \(y \in X\) for which \(\mu(y) + A(y) - 1 > I(y)\).

Therefore, any \(x_\lambda \notin A^*(\mathcal{J}, \tau)\) (for any \(x_\lambda \notin A\) (any fuzzy set) implies hereafter, \(x_\lambda\) is not contained in the fuzzy set \(A\), i.e., \(\lambda > A(x)\) implies there is at least one \(\mu \in N(x_\lambda)\) such that for every \(y \in X\), \(\mu(y) + A(y) - 1 \leq I(y)\) for some \(I \in \mathcal{J}\).

We will occasionally write \(A^*\) or \(A^*(\mathcal{J})\) for \(A^*(\mathcal{J}, \tau)\) and it will cause no ambiguity.

**Example 3.3.** The simplest fuzzy ideals on \(\{0_X\}\) and \(\mathcal{J}(X)\), the set of all fuzzy sets of \(X\) (hereafter, if necessary, \(\mathcal{J}(X)\) will carry the same meaning). Obviously, \(\mathcal{J} = \{0_X\} \Rightarrow A^*(\mathcal{J}, \tau) = \) cl\((A)\), for any fuzzy set \(A\) of \(X\) and \(\mathcal{J} = \mathcal{J}(X) \Rightarrow A^*(\mathcal{J}, \tau) = 0_X\).

The following theorem contains basic results of fuzzy local function which are known in general topology [5] and also shows the difference between two structures.

**Theorem 3.4.** Let \((X, \tau)\) be a fts and \(\mathcal{J}, \mathcal{J}\) be two fuzzy ideals on \(X\). Then for any fuzzy sets \(A, B\) of \(X\),

(i) \(A \subseteq B \Rightarrow A^*(\mathcal{J}, \tau) \subseteq B^*(\mathcal{J}, \tau)\),

(ii) \(\mathcal{J} \subseteq \mathcal{J} \Rightarrow A^*(\mathcal{J}, \tau) \subseteq A^*(\mathcal{J}, \mathcal{J})\),

(iii) \(A^* = \) cl\((A^*)\) \subseteq cl\((A)\),

(iv) \(A^* \subseteq A^*\),

(v) \((A \cup B)^* \subseteq A^* \cup B^*\),

(vi) \(I \in \mathcal{J} \Rightarrow (A \cup I)^* = A^*\).

**Proof.** (i) Since \(A \subseteq B\) implies \(A(x) \subseteq B(x)\) for every \(x \in X\), therefore, by Definition 3.2, \(x_\lambda \in A^*\)
implies \( x_1 \in B^* \), which completes the proof of (i).

(ii) Clearly, \( \mathcal{J} \subseteq \mathcal{J} \) implies \( A^*(\mathcal{J}) \subseteq A^*(\mathcal{J}) \), as there may be other fuzzy sets which belong to \( \mathcal{J} \) so that for a fuzzy point \( x_1, x_1 \in A^*(\mathcal{J}) \) but \( x_1 \) may not be contained in \( A^*(\mathcal{J}) \).

(iii) Since \( \{0_x\} \subseteq \mathcal{J} \) for any fuzzy ideal \( \mathcal{J} \) on \( X \), therefore by (ii) and Example 3.3, \( A^*(\mathcal{J}) \subseteq A^*(\{0_x\}) = \text{cl}(A) \), for any fuzzy set \( A \). Suppose, \( x_2 \in \text{cl}(A) \). So there is at least one \( y \in X \) such that \( \mu(y) + A^*(y) > 1 \) for each q-nbd \( \mu \) of \( x_2 \). Hence, \( A^*(\mathcal{J}) \neq 0 \). Let \( t = \text{cl}(A) \). Clearly, \( y_i \in A^* \) and \( t + \mu(y) > 1 \) so that \( \mu \) is also a q-nbd of \( y_i \). Now, \( y_i \in A^* \) implies there is at least one \( x' \in X \) such that \( v(x') + A^*(x') - 1 > I(x') \) for each \( v \in N(y_i) \) and \( I \in \mathcal{J} \). This is also true for \( \mu \). So there is at least one \( x'' \in X \) such that \( \mu(x'') + A^*(x'') - 1 > I(x'') \), for each \( I \in \mathcal{J} \). Since \( \mu \) is an arbitrary q-nbd of \( x_2 \), therefore \( x_2 \in A^* \). Hence, \( A^* = \text{cl}(A^*) = \text{cl}(A) \).

(iv) Suppose, \( x_i \notin A^* \cup B^* \), i.e., \( \lambda > (A^* \cup B^*) (x_i) = \max \{A^*(x), B^*(x_i)\} \). So \( x_i \) is not contained in both \( A^* \) and \( B^* \). This implies there is at least one q-nbd \( \mu_i \) of \( x_i \) such that for every \( y \in X \), \( \mu_i(y) + A^*(y) - 1 \leq I_i(y), \) for some \( I_i \in \mathcal{J} \) and similarly, there is at least one q-nbd \( \mu_i \) of \( x_i \) such that, for every \( y \in X \), \( \mu_i(y) + B^*(y) - 1 \leq I_i(y), \) for some \( I_i \in \mathcal{J} \). Let \( \mu = \mu \cap \mu_2 \). So \( \mu \) is also a q-nbd of \( x_i \) and \( \mu(y) + A \cup B - 1 \leq I_1 \cup I_2(y) \) for every \( y \in X \). Therefore, by finite additivity of fuzzy ideal, as \( I_1 \cup I_2 \in \mathcal{J} \), \( x_i \notin (A \cup B)^* \). Hence \( (A \cup B)^* \subseteq A^* \cup B^* \). Clearly, both \( A \) and \( B \subseteq A \cup B \) implies \( A^* \cup B^* \subseteq (A \cup B)^* \) and this completes the proof of (v).

(vi) It is clear that \( I \in \mathcal{J} \) implies \( I^* = 0_x \) so that \( (A \cup I)^* = A^* \cup I^* = A^* \).

**Theorem 3.6.** Let \( d: \mathcal{T}(X) \to \mathcal{T}(X) \) be a function such that

(a) \( d(0_x) = 0_x \),
(b) \( d(A \cup B) = d(A) \cup d(B) \),
(c) \( d(A) \subseteq d(A) \),

where \( A, B \) are any fuzzy sets of \( X \). Then \( \psi: \mathcal{T}(X) \to \mathcal{T}(X) \) defined by \( \psi(A) = A \cup d(A) \) is a fuzzy closure operator. Clearly, \( d \) does not necessarily coincide with fuzzy derived set operator in the generated fuzzy topology.

**Theorem 3.7.** \( \ast: \mathcal{T}(X) \to \mathcal{T}(X) \) satisfies all the required condition for \( d \).

**Proof.** Since \( 0^*_X = 0_x \) and \( (A \cup B)^* = A^* \cup B^* \) and \( (A^*)^* = A^* \), the proof is completed. □

Let \( (X, \tau) \) be a tfs and \( \mathcal{J} \) be a fuzzy ideal on \( X \). Let us define \( \text{cl}(A) = A \cup (A^*)^* \) for any fuzzy set \( A \) of \( X \). Clearly, \( \text{cl}^* \) is a fuzzy closure operator. Let \( \tau^*(\mathcal{J}) \) be the fuzzy topology generated by \( \text{cl}^* \), i.e., \( \tau^*(\mathcal{J}) = \{ A: \text{cl}^*(A) = A \} \). Now, \( \mathcal{J} = \{0_x\} \Rightarrow \text{cl}^*(A) = A \cup A^* = A \cup (A \cap \text{cl}(A)) \), for every \( A \in \mathcal{T}(X) \). So, \( \tau^*(\{0_x\}) = \tau \). Again, \( \mathcal{J} = \mathcal{T}(X) \Rightarrow \text{cl}^*(A) = A \), because \( A^* = 0_x \), for every \( A \in \mathcal{T}(X) \). So, \( \tau^*(\mathcal{J}(X)) \) is the fuzzy discrete topology on \( X \). Since \( \{0_x\} \) and \( \mathcal{T}(X) \) are the extreme fuzzy ideals on \( X \), therefore for any fuzzy ideal \( \mathcal{J} \) on \( X \) we have \( \{0_x\} \subseteq \mathcal{J} \subseteq \mathcal{T}(X) \). So we can conclude by Theorem 3.4(ii), \( \tau^*(\{0_x\}) \subseteq \tau^*(\mathcal{J}) \subseteq \tau^*(\mathcal{T}(X)) \), i.e., \( \tau \subseteq \tau^*(\mathcal{J}) \subseteq \tau^*(\mathcal{T}(X)) \) is fuzzy discrete topology, for any fuzzy ideal \( \mathcal{J} \) on \( X \). In particular, we have, for any two fuzzy ideals \( \mathcal{J} \) and \( \mathcal{J} \) on \( X \), \( \mathcal{J} \subseteq \mathcal{J} \Rightarrow \tau^*(\mathcal{J}) \subseteq \tau^*(\mathcal{J}) \).

**Theorem 3.8.** Let \( \tau_1, \tau_2 \) be two fuzzy topologies on \( X \). Then for any fuzzy ideal \( \mathcal{J} \) on \( X \), \( \tau_1 \subseteq \tau_2 \) implies

(i) \( A^*(\tau_1, \mathcal{J}) \subseteq A^*(\tau_1, \mathcal{J}) \),
(ii) \( \tau^*_1(\mathcal{J}) \subseteq \tau^*_2(\mathcal{J}) \).

**Proof.** Since every \( \tau_1 \) q-nbd of any fuzzy point \( x_1 \) is also a \( \tau_2 \) q-nbd of \( x_1 \). Therefore, \( A^*(\tau_2, \mathcal{J}) \subseteq A^*(\tau_1, \mathcal{J}) \) as there may be other \( \tau_2 \) q-nbd of \( x_1 \) where the condition for \( x_1 \in A^*(\tau_1, \mathcal{J}) \) may not hold true, although \( x_1 \in A^*(\tau_1, \mathcal{J}) \). Clearly, \( \tau^*_1(\mathcal{J}) \subseteq \tau^*_2(\mathcal{J}) \) as \( A^*(\tau_2, \mathcal{J}) \subseteq A^*(\tau_1, \mathcal{J}) \). □
Theorem 3.9. \( A^* \subseteq A^* \) and \( A^* \subseteq A^* \) for all fuzzy set \( A \) of \( X \), where \( A^* \) denotes the fuzzy derived set of \( A \) in \( \tau^* \) fuzzy topology.

Proof. Since, \( \tau \subseteq \tau^* \). Therefore, \( x_1 \in A^* \) implies every q-nbd of \( x_1 \) in fuzzy \( \tau^* \)-topology is quasi-coincident with \( A \Rightarrow \) every q-nbd of \( x_1 \) in fuzzy \( \tau \)-topology is quasi-coincident with \( A \Rightarrow x_1 \in A^* \), so that \( A^* \subseteq A^* \).

Again, for any fuzzy point \( x_1 \in A^* \) implies \( x_1 \in \text{cl}^*(A) = A \cup A^* \), i.e., \( x_1 \in A \) or \( x_1 \in A^* \) or both. Now, if \( x_1 \in A \), then for any q-nbd \( \mu \) of \( x_1 \) in fuzzy \( \tau^* \)-topology, there exists \( y \in X \) such that \( x \neq y \) and \( A(y) + \mu(y) > 1 \). This implies \( x_1 \) is a fuzzy accumulation point of the fuzzy set \( A^* \) such that

\[
A^*(p) = \begin{cases} 
A(p) & \text{if } p \neq x, \\
\lambda' & \text{if } p = x \text{ and } \lambda' < \lambda.
\end{cases}
\]

Obviously, \( A' \subseteq A \), so that \( (A')^* \subseteq A^* \) and also \( x_1 \notin A^* \). Hence, \( x_1 \in (A')^* \), because \( x_1 \in \text{cl}^*(A') = A' \cup (A')^* \). So, \( x_1 \in A^* \). Therefore, \( x_1 \in A^* \Rightarrow x_1 \in A^* \Rightarrow A^* \subseteq A^* \). □

Definition 3.10. A fuzzy set \( \mu \) of \( X \) is called fuzzy closed and discrete if and only if \( \mu^d = 0_X \).

Theorem 3.11. Let \((X, \tau)\) be a fis with \( \mathcal{I} \) a fuzzy ideal on \( X \). Then,

(i) \( I \in \mathcal{I} \Rightarrow I \) is closed and discrete in \((X, \tau^*)\).

(ii) \( A^* = \text{cl}(A - I) \) for every \( I \in \mathcal{I} \) and for any fuzzy set \( A \) of \( X \), where \( A - I \) is the fuzzy set defined by \( (A - I)(x) = \max \{A(x) - I(x), 0\} \), for every \( x \in X \).

Proof. (i) \( I \in \mathcal{I} \Rightarrow I^* = 0_X \). Therefore by Theorem 3.9, \( I^* = 0_X \).

(ii) Clear from the definition of fuzzy local function and closure of a fuzzy set. □

The above theorem characterizes a useful fact about the construction of different fuzzy ideals in relation with the original fuzzy topology and the generated fuzzy topology. The following examples show some cases where the two fuzzy topologies \( \tau \) and \( \tau^* \) on \( X \) are equal.

Example 3.12. (a) If \( \mathcal{I} \) be a fuzzy ideal on \( X \) such that \( A^d \subseteq \text{cl}(A - I) \) for every \( I \in \mathcal{I} \) and for any fuzzy set \( A \) of \( X \), then it is clear that \( A^d \subseteq A^* \) so that \( \text{cl}(A) = \text{cl}^*(A) \). Therefore, \( \tau = \tau^* \).

(b) Again, if \( \mathcal{I} \) be such that \( A^d = (A - I)^d \) for every \( I \in \mathcal{I} \), then obviously \( \tau = \tau^* \).

(c) Also, \( A^d = A^* \), for a fuzzy ideal \( \mathcal{I} \) on \( X \) implies \( \tau = \tau^* \).

4. Basic structure of generated fuzzy topology

Let \((X, \tau)\) be a fis and \( \mathcal{I} \) be a fuzzy ideal on \( X \). Let \( A \) be a q-nbd of a fuzzy point \( x_1 \) in fuzzy \( \tau^* \)-topology. Therefore, there exists \( \mu \in \tau^* \) such that \( \lambda + \mu(x) > 1 \) and \( \mu \subseteq A \).

Now, \( \mu \in \tau^* \iff \mu^d \in \tau^* \text{-closed} \iff \text{cl}^*(\mu^d) = \mu^d \iff (\mu^d)^* \subseteq \mu \iff (\mu^d)^* \subseteq \mu \subseteq (\mu^d)^* \). Therefore, \( \lambda + \mu(x) > 1 \Rightarrow \lambda + ((\mu^d)^*)(x) > 1 \Rightarrow \lambda + (\mu^d)^*(x) > 1 \Rightarrow \lambda > (\mu^d)^*(x) \Rightarrow x_1 \notin (\mu^d)^* \).

This implies there exists at least one q-nbd \( v_1 \) of \( x_1 \) (in \( \tau \)) such that for every \( y \in X, v_1(y) + \mu^d(y) - 1 \leq I_1(y) \) for some \( I_1 \in \mathcal{I} \), i.e., \( v_1(y) - I_1(y) \leq \mu(y) \) for every \( y \in X \). Therefore, as \( v_1 \) is a q-nbd of \( x_1 \) (in \( \tau \)), there is a \( v \in \tau \) such that \( x_1qv \subseteq v_1 \) and by heredity property of fuzzy ideal we have a \( I \in \mathcal{I} \) for which \( x_1q(v - I) \subseteq \mu \), where \( (v - I)(y) = \max \{v(y) - I(y), 0\} \), for every \( y \in X \). Hence, for \( \mu \in \tau^* \), we have a \( v \in \tau \) and \( I \in \mathcal{I} \) such that, \( (v - I) \subseteq \mu \).

Let us denote \( \beta(\mathcal{I}, \tau) = \{v - I: v \in \tau, I \in \mathcal{I}\} \).

Then we have the following theorem.

Theorem 4.1. \( \beta(\mathcal{I}, \tau) \) forms a basis for the generated fuzzy topology \( \tau^* \) of the fis \((X, \tau)\) with fuzzy ideal \( \mathcal{I} \) on \( X \).

Proof. Straight forward. □

We site an example which is very important for the further results that justifies the above construction.

Example 4.2. Let \( T \) be the fuzzy indiscrete topology on \( X \), i.e., \( T = \{0_X, 1_X\} \). So \( 1_X \) is the only q-nbd of every fuzzy point \( x_1 \). Now, \( x_1 \in A^* \) for a fuzzy set \( A \) for each \( I \in \mathcal{I} \), there is at least one \( y \in X \) such that \( 1 + A(y) - 1 > I(y) \).
This implies, for each $I \in \mathcal{J}$, $A(y) > I(y)$ for at least one $y \in X$. So $A \not\in \mathcal{J}$. Therefore, $A^* = 1_X$ if $A \not\in \mathcal{J}$ and $A^* = 0_X$ if $A \in \mathcal{J}$. This implies that we have, $cl^*(A) = A \cup A^* = 1_X$ if $A \not\in \mathcal{J}$ and $cl^*(A) = A$, if $A \in \mathcal{J}$, for any fuzzy set $A$ of $X$. Hence $T^* = \{\mu: \mu^* \in \mathcal{J}\}$.

Let $\tau \vee T^*(\mathcal{J})$ be the supremum fuzzy topology of $\tau$ and $T^*(\mathcal{J})$, i.e., the smallest fuzzy topology generated by $\tau \cup T^*(\mathcal{J})$. Then we have the following theorem.

**Theorem 4.3.** $T^*(\mathcal{J}) = \tau \vee T^*(\mathcal{J})$.

**Proof.** Follows from the fact that $\beta$ forms a basis for $\tau^*$.

**Lemma 4.4.** For any two fuzzy ideal $\mathcal{J}$ and $\mathcal{J}^*$ on $X$, $\mathcal{J} \vee \mathcal{J}^* = \{I \cup J: I \in \mathcal{J}, J \in \mathcal{J}^*\}$ and $\mathcal{J} \cap \mathcal{J}^*$ are fuzzy ideals on $X$.

**Proof.** Clear from definition.

**Theorem 4.5.** Let $(X, \tau)$ be a fts and $\mathcal{J}$ be a fuzzy ideal on $X$. Then, for any fuzzy set $A$ of $X$,

(i) $A^*(\mathcal{J} \vee \mathcal{J}^*) = A^*(\mathcal{J}) \cup A^*(\mathcal{J}^*)$.

(ii) $A^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq A^*(\mathcal{J}, x^*(\mathcal{J})) \cap A^*(\mathcal{J}^*, x^*(\mathcal{J}))$.

**Proof.**

(i) Let $x \notin A^*(\mathcal{J} \cup \mathcal{J}^*)$. Then, $x \notin A^*(\mathcal{J})$ and $x \notin A^*(\mathcal{J}^*)$. Now $x \notin A^*(\mathcal{J})$ implies there is at least one q-nbd $\mu$ of $x_1$ (in $\mathcal{J}$) such that for every $y \in X$, $\mu(y) + A(y) - 1 \leq I(y)$, for some $I \in \mathcal{J}$. Again, $x \notin A^*(\mathcal{J}^*)$ implies there is at least one q-nbd $\nu$ of $x_1$ (in $\mathcal{J}^*$) such that for every $y \in X$, $\nu(y) + A(y) - 1 \leq I(y)$, for some $I \in \mathcal{J}$. Therefore, we have $\mu \vee \nu + A(y) - 1 \leq I(y)$, for every $y \in X$. Since, $\mu \vee \nu$ is also a q-nbd of $x_1$ (in $\mathcal{J}$) and $I \in \mathcal{J}$, therefore $x \notin A^*(\mathcal{J} \cup \mathcal{J}^*)$, so that $A^*(\mathcal{J} \cup \mathcal{J}^*) \subseteq A^*(\mathcal{J}) \cup A^*(\mathcal{J}^*)$. Also $\mathcal{J} \cap \mathcal{J}^*$ is included in both $\mathcal{J}$ and $\mathcal{J}^*$, so by Theorem 3.4(ii), reverse inclusion is obvious, which completes the proof of (i).

(ii) Since $x \notin A^*(\mathcal{J} \cap \mathcal{J}^*)$, there is at least one q-nbd $\mu$ of $x_1$ (in $\mathcal{J}$) such that, for every $y \in X$, $\mu(y) + A(y) - 1 \leq I(y)$, for some $I \in \mathcal{J} \cap \mathcal{J}^*$. Therefore, by heredity of fuzzy ideals and considering the structure of fuzzy open sets in generated fuzzy topology, we can find $v$ or $v'$, the q-nbds of $x_1$ in $\tau^*(\mathcal{J})$ or $\tau^*(\mathcal{J}^*)$ respectively, such that for every $y \in X$, $v(y) + A(y) - 1 \leq I(y)$ or $v'(y) + A(y) - 1 \leq I(y)$ for some $I \in \mathcal{J}$ or $I \in \mathcal{J}^*$. This implies $x \notin A^*(\mathcal{J} \cap \mathcal{J}^*)$ or $x \notin A^*(\mathcal{J}^*, x^*(\mathcal{J}))$. Thus we have, $A^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq A^*(\mathcal{J} \cup \mathcal{J}^*)$.

An important result follows from the above theorem that $\tau^*(\mathcal{J})$ and $[\tau^*(\mathcal{J})]^*([\mathcal{J}])$ (in short $\tau^*$) are equal for any fuzzy ideal on $X$.

**Corollary 4.6.** Let $(X, \tau)$ be a fts and $\mathcal{J}$ be a fuzzy ideal on $X$. Then, $A^*(\mathcal{J}, \tau^*) = A^*(\mathcal{J}^*, \tau^*)$ and $\tau^*(\mathcal{J}) = [\tau^*(\mathcal{J})]^*([\mathcal{J}])$.

**Proof.** Putting $\mathcal{J} = \mathcal{J}$ in Theorem 4.5(ii) we have the required result.

**Corollary 4.7.** Let $(X, \tau)$ be a fts and $\mathcal{J}$ be two fuzzy ideal on $X$. Then,

(i) $\tau^*(\mathcal{J} \cup \mathcal{J}^*) = [\tau^*(\mathcal{J})]^*([\mathcal{J}])$,

(ii) $\tau^*(\mathcal{J} \cap \mathcal{J}^*) = \tau^*(\mathcal{J}) \vee \tau^*(\mathcal{J}^*)$,

(iii) $\tau^*(\mathcal{J} \cap \mathcal{J}^*) = \tau^*(\mathcal{J} \cap \mathcal{J}^*)$.

**Proof.**

(i) By Theorem 4.5(ii) the result follows.

(ii) By (i), we have, $\tau^*(\mathcal{J} \cap \mathcal{J}^*) = [\tau^*(\mathcal{J})]^*([\mathcal{J}]) = \tau^*(\mathcal{J}) \vee \tau^*(\mathcal{J}^*)$ (by Theorem 4.3).

Since, $\tau^* \subseteq \tau^*$ for any fuzzy ideal on $X$. Therefore, $\tau^*(\mathcal{J} \cup \mathcal{J}^*) = \tau^* \vee \tau^*(\mathcal{J}) \vee \tau^*(\mathcal{J}^*) = \tau^*(\mathcal{J}) \vee \tau^*(\mathcal{J}^*)$.

(iii) Since $\mathcal{J} \cap \mathcal{J}^*$ is included in both $\mathcal{J}$ and $\mathcal{J}^*$, $\tau^*(\mathcal{J} \cap \mathcal{J}^*)$ is included in both $\tau^*(\mathcal{J})$ and $\tau^*(\mathcal{J}^*)$. Now $\mu$ is a fuzzy open set in $\tau^*(\mathcal{J} \cap \mathcal{J}^*)$ and $\tau^*(\mathcal{J})$. That means $(\mu^*)^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq \mu$ and $(\mu^*)^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq \mu'$. So, $(\mu^*)^*(\mathcal{J} \cup \mu^*)^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq \mu \cap \mu'$. Therefore, by Theorem 4.3(ii), $(\mu^*)^*(\mathcal{J} \cap \mathcal{J}^*) \subseteq \mu \cap \mu'$. Hence, $\mu \in \tau^*(\mathcal{J} \cap \mathcal{J}^*)$. This completes the proof.
5. Compatibility of fuzzy ideals with fuzzy topology

Definition 5.1. For a fts $(X, \tau)$ with fuzzy ideal $\mathcal{I}$, $\tau$ is said to be compatible with $\mathcal{I}$, denoted by $\tau \sim \mathcal{I}$, if for every fuzzy set $A$ of $X$; if for all fuzzy point $x_1 \in A$, there exists a q-nbd $B_0$ of $x_1$ (in $\tau$) such that $\mu(y) + A(y) - 1 < I(y)$ hold for every $y \in X$ and for some $I \in \mathcal{I}$, then $A \in \mathcal{I}$.

Definition 5.2. Let $\{B_\alpha, \alpha \in A\}$ be an indexed family of fuzzy sets of $X$ such that $B_{\alpha} \in A$ for each $\alpha \in A$, where $A$ is a fuzzy set of $X$. Then $\{B_\alpha, \alpha \in A\}$ is said to be a quasi-cover of $A$ if and only if, $A(y) + \bigcup_{\alpha \in A} B_\alpha(y) \geq 1$ for every $y \in X$.

Further, if each $B_\alpha$ is fuzzy open set, then this quasi-cover will be called a fuzzy quasi-open cover of the fuzzy set $A$ of $X$.

Therefore, in either case $A^* \subseteq \bigcup_{\alpha \in A} B_\alpha$.

Theorem 5.3. For a fts $(X, \tau)$ with fuzzy ideal $\mathcal{I}$ on $X$ the following conditions are equivalent:

(i) $\tau \sim \mathcal{I}$.

(ii) If for every fuzzy set $A$ of $X$ has a fuzzy quasi-open cover $\{B_\alpha, \alpha \in A\}$ such that for each $\alpha$, $A(y) + B_\alpha(y) - 1 < I(y)$, for some $I \in \mathcal{I}$ and for every $y \in X$, then $A \in \mathcal{I}$.

(iii) For every fuzzy set $A$ of $X$, $A \cap A^* = 0_X$ implies $A \in \mathcal{I}$.

(iv) For every fuzzy set $A$ of $X$, $A \in \mathcal{I}$, where $A = \bigcup_{x \in X} x A$ such that $x A \in A$ but $x A \notin A^*$. [Here fuzzy point $x$ is said to be compatible with fuzzy point $x_1$.]

(v) For every fuzzy $\tau$-closed set $A$, $A \in \mathcal{I}$ (If $A$ is defined as in (iv)).

(vi) For every fuzzy set $A$ of $X$, if $A$ contains no non-empty fuzzy subset $B$ with $B \subseteq B^*$, then $A \in \mathcal{I}$ (Here non-empty means $B \neq 0_X$).

Proof. We prove most of the equivalent conditions which ultimately prove all the equivalence.

(i) $\Rightarrow$ (ii): Let $\{B_\alpha, \alpha \in A\}$ be a fuzzy quasi-open cover of a fuzzy set $A$ of $X$ such that for each $\alpha \in A$, $A(y) + B_\alpha(y) - 1 < I(y)$ for some $I \in \mathcal{I}$ and for every $y \in X$. Therefore, as $\{B_\alpha, \alpha \in A\}$ is a fuzzy quasi-open cover of $A$, for each $x_1 \in A$, there exists at least one $B_\alpha$ such that $x_1 \in B_\alpha$ and for every $y \in X$, $A(y) + B_\alpha(y) - 1 < I(y)$, for some $I \in \mathcal{I}$. Obviously, $B_\alpha$ is a q-nbd of $x_1$ (in $\tau$). Therefore, as $\tau \sim \mathcal{I}$, $A \in \mathcal{I}$.

(ii) $\Rightarrow$ (i): Clear from the fact that a collection of fuzzy open sets which contains at least one open q-nbd of each fuzzy point of $A$, constitutes a fuzzy quasi-open cover of $A$.

(ii) $\Rightarrow$ (iii): Let $A \cap A^* = 0_X$, i.e., $\min\{A(y), A^*(y)\} \geq 0$, for every $y \in X$. So, a fuzzy point $x_1 \in A^*$ implies $x_1 \notin A^*$. That means, there is a q-nbd $B_\alpha$ of $x_1$ such that for every $y \in X$, $\mu(y) + A(y) - 1 < I(y)$, for some $I \in \mathcal{I}$, if $x_1 \in A$. Since $\mu$ is a q-nbd of $x_1$, therefore, there is a fuzzy open set $v$ (in $\tau$) such that $x_1 \in v$ and so the collection of such $v$s for each $x_1 \in A$, constitutes a fuzzy quasi-open cover of $A$. Therefore by condition (ii), $A \in \mathcal{I}$.

(iii) $\Rightarrow$ (i): For every fuzzy point $x_1 \in A$, there is a q-nbd $\mu$ of $x_1$ (in $\tau$) such that for every $y \in X$, $\mu(y) + A(y) - 1 < I(y)$, for some $I \in \mathcal{I}$. That means $x_1 \notin A^*$. Now, there are two cases: either $A^*(x) = 0$ or, $A^*(x) \neq 0$ but $\lambda > A^*(x)$. Let, if possible, $x_1 \in A$ be such that $\lambda > A^*(x) \neq 0$. Let $A^*(x) = \lambda$. Then the fuzzy point $x_1 \in A^*$ and also $x_1 \notin A$. This implies for each q-nbd $v$ of $x_1$ and for each $I \in \mathcal{I}$, there is at least one $y \in X$ such that $v(y) + A(y) - 1 > I(y)$. Since $x_1 \in A$, this contradicts the assumption for every fuzzy point of $A$. So, $A^*(x) = 0$. That means, $x_1 \in A$ implies $x_1 \notin A^*$. Now this is true for every fuzzy set $A$ of $X$. So, for every fuzzy set $A$ of $X$, $A \cap A^* = 0_X$. Hence, by condition (iii), we have $A \in \mathcal{I}$, which implies $\tau \sim \mathcal{I}$.

(iii) $\Rightarrow$ (iv): Let the fuzzy point $x_1 \in A^*$. That means $x_1 \in A$ but $x_1 \notin A^*$. So, there is a q-nbd $\mu$ of $x_1$ such that for every $y \in X$, $\mu(y) + A(y) - 1 < I(y)$, for some $I \in \mathcal{I}$. Since $\mu \subseteq A$, so for every $y \in X$, $\mu(y) + A(y) - 1 < I(y)$, for some $I \in \mathcal{I}$. Therefore, $x_1 \notin A^*$. But as $x_1 \notin A^*$, so that both $A^*(x) = 0$ or $A^*(x) \neq 0$ but $\lambda > A^*(x)$. Let $x_1$ be a fuzzy point such that $\lambda_1 \leq A^*(x) \leq \lambda$, i.e., $x_1 \in A^*$. So, for each q-nbd $v$ of $x_1$ and for each $I \in \mathcal{I}$, there is at least one $y \in X$ such that $v(y) + A(y) - 1 > I(y)$. Since $\lambda_1 \leq A^*(x) \leq \lambda$, i.e., $x_1 \in A^*$. So, for each q-nbd $v$ of $x_1$ and for each $I \in \mathcal{I}$, there is at least one $y \in X$ such that $v(y) + A(y) - 1 > I(y)$. This implies $x_1 \notin A^*$. But as $\lambda_1 < \lambda$, $x_1 \in A$ implies $x_1 \notin A^*$. This is a contradiction. Hence $A^*(x) = 0$, so that $x_1 \notin A$ implies $x_1 \notin A^*$. Thus we have $A \cap A^* = 0_X$, for every fuzzy set $A$ of $X$. Hence, by condition (iii), $A \in \mathcal{I}$.

(iv) $\Rightarrow$ (v): Straightforward.

(iv) $\Rightarrow$ (vi): Let $A$ be any fuzzy set of $X$ that contains no non-empty (i.e., not $0_X$) fuzzy subset.
B with \( B = B^* \). Clearly, for every fuzzy set \( A \) of \( X \),
\[ A = \bar{A} \cup (A \cap A^*) \]
Therefore, \( A^* = (\bar{A} \cup (A \cap A^*))^* \) (by Theorem 3.4(iii)). Now by condition (iv), \( \bar{A} \in \mathscr{I} \) so that \( \bar{A} = 0_X \). Hence, \((A \cap A^*)^* = A^* \). But \( A \cap A^* \subseteq A^* \) so that \( A \cap A^* \subseteq (A \cap A^*)^* \). This contradicts the hypothesis about every fuzzy set \( A \) of \( X \) that, it contains no non-empty fuzzy subset \( B \) with \( B \subseteq B^* \). Therefore, \( A \cap A^* = 0_X \) so that \( A = \bar{A} \) and hence by condition (iv), \( A \in \mathscr{I} \).

(vi) \( \Rightarrow \) (iv): Since, for every fuzzy set \( A \) of \( X \), \( A \cap A^* = 0_X \). Therefore, by condition (vi), as \( \bar{A} \) contains no non-empty fuzzy subset \( B \) with \( B \subseteq B^* \), \( \bar{A} \in \mathscr{I} \).

(v) \( \Rightarrow \) (i): Let \( A \) be any fuzzy set of \( X \). Let for every fuzzy point \( x \in A \), there is a q-nbd \( U \) of \( x \) in \( \mathscr{I} \) such that for every \( y \in U \), \( \mu(y) + \mu(A(y)) - 1 < \mu(x) \), for some \( l \in \mathscr{I} \). This implies \( x \notin A^* \). Let \( B = A \cup A^* \), then \( B^* = (A \cup A^*)^* = A^* \cup (A^*)^* = A^* \). So, \( \mathcal{C}^*(B) = B \cup B^* = B \). That means, \( B \) is fuzzy \( \tau^* \)-closed set. Therefore, by condition (v), \( B \in \mathscr{I} \). Again, any fuzzy point \( y \in B \) implies \( y \in B \) but \( y \notin B^* = A^* \). So, as \( B = A \cup A^* \), \( y \notin A \). Now, by hypothesis about \( A \), we have for every \( x \in A \), \( x \notin A^* \). So, \( \bar{B} = A \). Hence \( A \in \mathscr{I} \), i.e., \( \tau \sim \mathscr{I} \).

Theorem 5.4. Let \( (X, \tau) \) be a fts with \( \mathscr{I} \) be any fuzzy ideal on \( X \). Then the following are equivalent and implied by \( \tau \sim \mathscr{I} \):

(i) For every fuzzy set \( A \) of \( X \), \( A \cap A^* = 0_X \) implies \( A^* = 0_X \).

(ii) For every fuzzy set \( A \) of \( X \), \( \bar{A} = 0_X \) (\( \bar{A} \) is defined as in Theorem 5.3(iv)).

(iii) For every fuzzy set \( A \) of \( X \), \( (A \cap A^*)^* = A^* \).

Proof. Clear from Theorem 5.3.

Theorem 5.5. Let \( (X, \tau) \) be a fts with \( \mathscr{I} \) be any fuzzy ideal on \( X \). Let \( \tau \sim \mathscr{I} \) and \( \tau \) is a fuzzy set with \( \tau \) is closed with respect to \( \tau^* \) if and only if it is the union of a fuzzy set which is closed with respect to \( \tau \) and a fuzzy set in \( \mathscr{I} \).

Proof. Let \( A \) be a fuzzy set of \( X \) such that it is fuzzy \( \tau^* \)-closed. That means \( A^* \subseteq A \) and we have, \( A = \bar{A} \cup A^* \). Since \( \tau \) is compatible with \( \mathscr{I} \), therefore \( \bar{A} \in \mathscr{I} \). Also, \( A^* \) is always \( \tau^* \)-closed (by Theorem 3.4(ii)). This completes the necessary part of the proof.

Conversely, let \( A \) be any fuzzy set of \( X \) such that \( A = B \cup I \), where \( \mathcal{C}(B) = B \) and \( I \in \mathscr{I} \). Therefore, by Theorem 3.4, \( A^* = B^* \cup I^* = B^* \subseteq \mathcal{C}(B) = B \subseteq A \). That means \( A^* \subseteq B \subseteq A \). So, we have, \( \mathcal{C}(A) = A \cup A^* = A \) and this implies \( A \) is fuzzy \( \tau^* \)-closed set.

An important consequence of Theorem 5.5 is the following corollary.

Corollary 5.6. The fuzzy topology \( \tau \) is compatible with the fuzzy ideal \( \mathscr{I} \) on \( X \) implies \( \mathscr{I}(\mathscr{F}, \tau) \), a basis for \( \tau^* \) (defined earlier) is itself a fuzzy topology and also \( \beta = \tau^* \).

Proof. Clear from the previous theorem.

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References

On fuzzy Local Functions, Fuzzy Semiregularizations and Fuzzy Mappings

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Abstract:

In this paper some characterization theorems relating fuzzy open sets and generated fuzzy topologies by fuzzy ideals are given. We define a criterion, $\mathcal{I}$-interior for a fuzzy topological space with a given fuzzy ideal. With the help of this criterion, we study fuzzy semiregularization spaces associated with the generated fuzzy topologies, the original fuzzy topologies and also with the fuzzy subspaces. Finally, we discuss some fuzzy mappings between different fuzzy topological spaces and some relations are also established in connection with earlier results.

Keywords:

Fuzzy topology, fuzzy ideal, fuzzy local function, fuzzy semiregularization, fuzzy subspace, fuzzy mapping.

1. Introduction

In [6], we defined fuzzy local function for a fuzzy set of any fuzzy topological space (in short, fts) with the help of a fuzzy ideal. We succeeded in generating new fuzzy topologies from original one. In continuation of that work [6], some characterization theorems relating fuzzy open sets, the generated fuzzy topologies by the fuzzy ideals are given in Section 3. The departure from the general topology is also noticed here [8]. Giles [7] defined the bold intersection for any two fuzzy sets. With the help of this we construct a new fuzzy set for any two fuzzy sets (actually the bold intersection) and establish some results. We also define a criterion for every fts with fuzzy ideal $\mathcal{I}$, namely $\mathcal{I}$-interior, and obtain an important result connecting fuzzy open sets. We give an example in support of this criterion. Now, corresponding to any fts, there is a coarser fts, called fuzzy semiregularization space. These spaces have interesting characteristics that they share some property, called fuzzy semiregular property, with the original fuzzy topology. In [2], the notion of a fuzzy regular open set was introduced and it was shown that the family of fuzzy regular open sets form a base for a fuzzy topology $\tau$, (called
fuzzy semiregularization topology) coarser than original $\tau$ on a set $X$. Petricevic [11],
gave a new definition of fuzzy $\delta$-open and $\delta$-closed sets and showed that the fuzzy
topology generated by fuzzy $\delta$-open sets ($\tau^0$) is precisely the fuzzy semiregularization
topology $\tau$. In Section 4, we have studied fuzzy semiregularization spaces associated
with fts $(X, \tau)$ and generated fts $(X, \tau^0)$ and for a given condition it has been shown
that they are same for those above two fuzzy topologies. In this connection, we have also
studied fuzzy subspaces as generalizations of those results. Finally, we discuss some
fuzzy mapping between two fts defined in [2, 10, 11, 13] and some new results are
obtained in connection with the earlier results.

2. Preliminaries

We mean by a fts $(X, \tau)$ (or, simply $X$), a fts in Chang's sense [5]. A fuzzy point in
$X$ with support $x \in X$ and value $\lambda (0 < \lambda \leq 1)$ is denoted by $x_\lambda$. A fuzzy point $x_\lambda$
is said to be contained in a fuzzy set $A$ of $X$ if and only if $\lambda \leq A(x)$ and this will be
denoted by $x_\lambda \in A$ [12]. $x_0 \notin A$ will mean $\lambda > A(x)$. For any fuzzy set $A$ of $X$, $A^c$, $\text{int} A$ and $\text{cl} A$ will denote complement, interior and closure of $A$ respectively. For
some cases we will explicitly mention the fuzzy topologies $\tau, \sigma$, etc., on different sets and
write $\text{int} A, \text{int}_\sigma A$ and $\text{cl} A, \text{cl}_\mu A$, etc., to avoid notational overlapping. $0_X$ and $1_X$
will denote the constant fuzzy sets with values 0 and 1 on $X$ respectively. A fuzzy set $A$
in fts $X$ is said to be quasi-coincident with a fuzzy set $B$ if there is at least one $x \in X$
such that $A(x) + B(x) > 1$ and this will be denoted by $A \sim B$. A fuzzy set $\mu$ in fts
$(X, \tau)$ is said to be a quasi-neighborhood (in short, q-nbd) of a fuzzy point $x_\lambda$
if and only if there exists a fuzzy open set $\nu$ such that $x_\lambda \in \nu$ [4, 12]. We will denote the set of all q-nbds of $x_\lambda$ in $(X, \tau)$ by $N(x_\lambda)$. For any two fuzzy sets $A$ and $B$ of $X$, $A - B$ is a
fuzzy set defined by $(A - B)(x) = \max\{A(x) - B(x), 0\}$ for all $x \in X$. For restriction of a fuzzy set $\mu$ of $X$ to $Y \subseteq X$, we denote $\mu_Y$ and this means the support of $\mu$
is changed from $X$ to $Y \subseteq X$ and $\mu_Y(x) = \mu(x)$ for every $x \in Y$ [12]. Also $X \backslash Y$
will denote the difference between the sets $X$ and $Y$. A fuzzy set $\mu$ of a fts $(X, \tau)$ is said
to be fuzzy regularly open (respectively, regularly closed) if and only if $\mu = \text{int}(\text{cl} \mu)$
(respectively, $\text{cl}(\text{int} \mu) = \mu$) [2]. A fuzzy set $\mu$ is called fuzzy $\delta$-open if and only if
$\mu = \vee_\mu$, where $\mu$, are fuzzy regular open sets [11]. $\mu$ is called $\delta$-closed if its
complement is fuzzy $\delta$-open. A fts $(X, \tau)$ is called fuzzy regular (respectively, fuzzy
almost regular) if for each fuzzy point $x_\lambda$ and each fuzzy open (respectively, regular open)
set $\mu$ containing $x_\lambda$ there is a fuzzy open set $\nu$ such that $x_\lambda \in \nu \subseteq \text{cl} \nu \subseteq \mu$
[1, 11]. $(X, \tau)$ is called fuzzy semiregular if and only if $\tau = \tau_\delta$. Clearly, fuzzy regularity
implies fuzzy semiregularity and fuzzy almost regularity. Also every fuzzy almost
regular, fuzzy semiregular space is fuzzy regular [1]. For definitions and results not
explained in this paper we refer [6, 10, 11, 12]. We now recall some definitions and
results from [6] which will be used here.
Definition 2.1, [6]. A non-empty collection of fuzzy sets $\mathcal{F}$ of a set $X$ is called a fuzzy ideal on $X$ if and only if (i) $\mu \in \mathcal{F}$ and $\nu \subseteq \mu \Rightarrow \nu \in \mathcal{F}$ [heredity], (ii) $\mu \in \mathcal{F}$ and $\nu \in \mathcal{F} \Rightarrow \mu \cup \nu \in \mathcal{F}$ [finite additivity].

Definition 2.2, [6]. Let $(X, \tau)$ be a fts and $\mathcal{F}$ be a fuzzy ideal on $X$. Let $A$ be any fuzzy set of $X$. Then the fuzzy local function $A'(\mathcal{F}, \tau)$ of $A$ is the union of all fuzzy points $x_i$ such that if $\nu \in N(x_i)$ and $I \in \mathcal{F}$ then there is at least one $y \in X$ for which $\nu(y) + A(y) - I(y) > 0$.

Therefore, any $x_i \in A'(\mathcal{F}, \tau)$ implies that there is at least one $\mu \in N(x_i)$ such that for every $y \in X$, $\mu(y) + A(y) - I(y) > 0$ for some $I \in \mathcal{F}$.

We will write $A'(\mathcal{F})$ or simply $A'$ in place of $A'(\mathcal{F}, \tau)$ and it will cause no ambiguity.

Theorem 2.1, [6]. Let $(X, \tau)$ be a fts and $\mathcal{F}, \mathcal{G}$ be two fuzzy ideals on $X$. Then for any fuzzy sets $A, B$ of $X$,

\begin{enumerate}
  \item $A \subseteq B \Rightarrow A'(\mathcal{F}, \tau) \subseteq B'(\mathcal{F}, \tau)$,
  \item $\mathcal{F} \subseteq \mathcal{G} \Rightarrow A'(\mathcal{G}, \tau) \subseteq A'(\mathcal{F}, \tau)$,
  \item $A' = \text{cl}(A') \subseteq \text{cl}(A)$,
  \item $(A')' \subseteq A'$,
  \item $(A \cup B)' = A' \cup B'$,
  \item $I \in \mathcal{F} \Rightarrow (A \cup I)' = A'$.
\end{enumerate}

Now the operator $\text{cl}^*$ defined by $\text{cl}^*(A) = A \cup A'$ for any fuzzy set $A$ of $(X, \tau)$ is obviously a fuzzy closure operator [6, 9] and the fuzzy topology generated by it is denoted by $\tau^*(\mathcal{F})$ (or, simply $\tau^*$). We will write interior with respect to $\tau^*$ by $(\text{int})^*$. Clearly $\tau \subseteq \tau^*(\mathcal{F})$ for any fuzzy ideal $\mathcal{F}$ on $X$. Also, for any two fuzzy ideal $\mathcal{F}, \mathcal{G}$ on $X$, $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \tau^*(\mathcal{G}) \subseteq \tau^*(\mathcal{F})$. It is also known that $A'(\mathcal{F}, \tau) = A'(\mathcal{G}, \tau^*(\mathcal{F}))$ for any fuzzy set $A$ of $X$ and $[\tau^*(\mathcal{F})](\mathcal{G}) = \tau^*(\mathcal{F})$ [6].

3. Some characterizations

Theorem 3.1. Let $(X, \tau)$ be a fts with $\mathcal{F}$ be a fuzzy ideal on $X$. Let for any fuzzy set $A$ of $X$ and for any $\mu \in \tau$, $B$ be a fuzzy set such that for every $x \in X$,

$$B(x) = \begin{cases}
A(x), & \text{if } \mu(x) + A(x) > 1 \\
0, & \text{otherwise.}
\end{cases}$$

Then, $A'(x) + \mu(x) > 1$ implies $x \in B$, where $\lambda = A'(x)$ and $x \in X$. 

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Definition 2.1, [6|. A non-empty collection of fuzzy sets $\mathcal{F}$ of a set $X$ is called a fuzzy ideal on $X$ if and only if (i) $\mu \in \mathcal{F}$ and $\nu \subseteq \mu \Rightarrow \nu \in \mathcal{F}$ [heredity], (ii) $\mu \in \mathcal{F}$ and $\nu \in \mathcal{F} \Rightarrow \mu \cup \nu \in \mathcal{F}$ [finite additivity].

Definition 2.2, [6|. Let $(X, \tau)$ be a fts and $\mathcal{F}$ be a fuzzy ideal on $X$. Let $A$ be any fuzzy set of $X$. Then the fuzzy local function $A'(\mathcal{F}, \tau)$ of $A$ is the union of all fuzzy points $x_i$ such that if $\nu \in N(x_i)$ and $I \in \mathcal{F}$ then there is at least one $y \in X$ for which $\nu(y) + A(y) - I(y) > 0$.

Therefore, any $x_i \in A'(\mathcal{F}, \tau)$ implies that there is at least one $\mu \in N(x_i)$ such that for every $y \in X$, $\mu(y) + A(y) - I(y) > 0$ for some $I \in \mathcal{F}$.

We will write $A'(\mathcal{F})$ or simply $A'$ in place of $A'(\mathcal{F}, \tau)$ and it will cause no ambiguity.

Theorem 2.1, [6|. Let $(X, \tau)$ be a fts and $\mathcal{F}, \mathcal{G}$ be two fuzzy ideals on $X$. Then for any fuzzy sets $A, B$ of $X$,

\begin{enumerate}
  \item $A \subseteq B \Rightarrow A'(\mathcal{F}, \tau) \subseteq B'(\mathcal{F}, \tau)$,
  \item $\mathcal{F} \subseteq \mathcal{G} \Rightarrow A'(\mathcal{G}, \tau) \subseteq A'(\mathcal{F}, \tau)$,
  \item $A' = \text{cl}(A') \subseteq \text{cl}(A)$,
  \item $(A')' \subseteq A'$,
  \item $(A \cup B)' = A' \cup B'$,
  \item $I \in \mathcal{F} \Rightarrow (A \cup I)' = A'$.
\end{enumerate}

Now the operator $\text{cl}^*$ defined by $\text{cl}^*(A) = A \cup A'$ for any fuzzy set $A$ of $(X, \tau)$ is obviously a fuzzy closure operator [6, 9] and the fuzzy topology generated by it is denoted by $\tau^*(\mathcal{F})$ (or, simply $\tau^*$). We will write interior with respect to $\tau^*$ by $(\text{int})^*$. Clearly $\tau \subseteq \tau^*(\mathcal{F})$ for any fuzzy ideal $\mathcal{F}$ on $X$. Also, for any two fuzzy ideal $\mathcal{F}, \mathcal{G}$ on $X$, $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \tau^*(\mathcal{G}) \subseteq \tau^*(\mathcal{F})$. It is also known that $A'(\mathcal{F}, \tau) = A'(\mathcal{G}, \tau^*(\mathcal{F}))$ for any fuzzy set $A$ of $X$ and $[\tau^*(\mathcal{F})](\mathcal{G}) = \tau^*(\mathcal{F})$ [6].
Proof. Since $\lambda = A'(z)$ implies $x \in A'^*$. Therefore, if $v \in N(z)$ and $I \in \mathcal{F}$, then there is at least one $y \in X$ such that $v(y) + A(y) - 1 > I(y)$. Now, $\mu \in \tau$ and $y \in A'$ implies $\mu \in N(x)$, so that $\mu \cap v \in N(z)$. Therefore, for $\mu \cap v$, if $I \in \mathcal{F}$, then there is at least one $p \in X$ such that $\mu \cap v(p) + A(p) - 1 > I(p)$. i.e., $\mu(p) + A(p) - 1 > I(p) \geq 0$ (as $\mu \cap v \subseteq \mu$). Let $\lambda' = A(p)$, i.e., $p \not\in A$ Then, by definition of $B$, $B \subseteq \mathcal{B}$ and $\lambda' = B(p)$.

So $\mu \cap v(p) + B(p) - 1 > I(p)$ and hence $v(p) + B(p) - 1 > I(p)$. Since our choice of $v$ and $I$ are arbitrary, therefore $x \in A'$ and this completes the proof.

Theorem 3.2. Let $(X, \tau)$ be a fts with $\mathcal{F}$ be a fuzzy ideal on $X$. Then the following are equivalent:

(i) $1_x = (1_x)^*$

(ii) $\tau \cap \mathcal{F} = \{0_x\}$

(iii) $I \in \mathcal{F}$ if $\tau \cap \mathcal{F} = \{0_x\}$

Proof. (i) $\Rightarrow$ (ii). Let $x \in X$ Then $v \in N(x)$ and $I \in \mathcal{F}$ imply there is at least one $y \in X$ such that $v(y) + \lambda > I(y)$, i.e., $v(y) > I(y)$. Now, for every $\mu \in \tau$ with $\mu \not\subseteq \lambda$, $\mu$ will be a q-nbd of at least one $x_0 \in (1_x)^*$, because $(1_x)^* = 1_x$ (by hypothesis) and $\lambda = 1$ is permissible. Hence, if $I \in \mathcal{F}$ and $\mu \in \tau$, then there is at least one $p \in X$ such that $\mu(p) > I(p)$, i.e., $\mu \in \mathcal{F}$. Therefore $\tau \cap \mathcal{F} = \{0_x\}$.

(ii) $\Rightarrow$ (iii). Let $\tau \cap \mathcal{F} = \{0_x\}$ and $I \in \mathcal{F}$. Since $\int I$ is a fuzzy open set, therefore, $\int I \subseteq I \in \mathcal{F}$ (by heredity). So, by hypothesis, $\int I = 0_x$ and as $I$ is any member of $\mathcal{F}$, therefore the result follows.

(iii) $\Rightarrow$ (i). Let $x \in X$ be any fuzzy point and $\mu$ be any q-nbd of $x$. Then there is a fuzzy open set $\nu \subseteq \mu$ such that $\nu \not\subseteq x$. Since $I \in \mathcal{F}$ implies $\int I = 0_x$ (by hypothesis) and $\int v = v \not\subseteq x$, therefore $v \not\subseteq \mathcal{F}$. So if $I \in \mathcal{F}$ then there is at least one $y \in X$ such that $\nu(y) > I(y)$, i.e., $\mu(y) > v(y) > I(y)$, or $1 + \mu(y) - 1 > I(y)$. Since $\mu$ is any q-nbd of $x$, therefore $x \in (1_x)^*$, and hence $1_x \subseteq (1_x)^*$ as $(1_x)^* \subseteq 1_x$ is always true.

Theorem 3.3. Let $(X, \tau)$ be a fts with $\mathcal{F}$ be a fuzzy ideal on $X$ and let $\mu \in \tau$ implies there is at least one $x \in X$ such that $\mu(x) > 0$. Then $1_x = (1_x)^*$.

Proof. Clearly $(1_x)^* \subseteq 1_x$. Let $x \in 1_x$ and $\mu \in N(z)$. Then there is $\nu \in \tau$ with $\nu \subseteq \mu$ such that $\nu \not\subseteq \mu$. Now, by hypothesis, $\nu \in \tau$ implies there is at least one $y \in X$ such that $\nu(y) > 0$. Let $t = \nu(y)$ then $\nu(t) > 0$. So, if $I \in \mathcal{F}$ and $v \in N(x)$, then there is at least one $z \in X$ such that $v(z) + 1 - x > I(z)$, i.e., $v(z) + 1 - x > I(z)$ or $1 + v(z) - 1 > I(z)$. Since $\nu \subseteq \mu$ we have that if $I \in \mathcal{F}$ then there is at least one $z \in X$
such that \(1 + \mu(z) - I(z)\), i.e., \(x_2 \in (l_x)^{'}\) as \(\mu\) is any q-nbd of \(x_2\). Therefore \(l_x \subseteq (l_x)^{'}\). This completes the proof.

Remark 3.1. Since for any fuzzy set \(A\) of \(X\), \(A'(\mathcal{F}, \tau) = A'(\mathcal{F}, \tau'(\mathcal{F}))\) [6]. Therefore, in Theorem 3.2, we may replace \(\tau\) by \(\tau'(\mathcal{F})\) and \(\text{int} I = 0_x\) by \((\text{int}) I = 0_x\).

Also the results of Theorem 3.2 and Theorem 3.3 clearly show the departure from the results of general topology [8].

Now we construct a fuzzy set for any two fuzzy sets using the quasi-coincident idea of fuzzy set theory. This was defined by Giles [7] as bold intersection of any two fuzzy sets. Let \(A, B\) be any two fuzzy sets of \(X\). Then we can have a fuzzy set \(C_{AB}\) defined by \(C_{AB}(x) = \max\{A(x) + B(x) - 1, 0\}\). Clearly \(C_{AB} = C_{BA}\). The following results are immediate consequence of the above construction. We omit the proofs as they are easy to check.

**Theorem 3.4.** Let \(A, B\) be any two fuzzy sets of \(X\). Then the following results hold true:

1. if \(A \# B\), then \(C_{AB} \neq 0_x\) and vice versa,
2. \(C_{AB} \subseteq A \cap B\),
3. \(C_{BA} = C_{AB}\).

**Theorem 3.5.** Let \(\mu, \nu\) be any two fuzzy open sets of a fts \((X, \tau)\). Then \(C_{\nu \mu} \neq 0_x \Leftrightarrow \exists x \in X\) such that \(\mu \in N(x)\) and \(\nu \in N(x)\).

**Theorem 3.6.** Let \((X, \tau)\) be a fts with \(\mathcal{F}\) be a fuzzy ideal on \(X\). Let \(A\) be any fuzzy set of \(X\). Then \(x_2 \in A'(\mathcal{F}, \tau) \Leftrightarrow C_{\mu} \in \mathcal{F}\) for every \(\nu \in N(x)\).

Proof. Let \(x_2 \in A'(\mathcal{F}, \tau)\) then \(\nu \in N(x)\) and \(I \in \mathcal{F}\) implies there is at least one \(y \in X\) such that \(v(y) + A(y) - 1 > I(y)\), i.e., \(C_{\mu}(y) > I(y)\). This is true for every pair \((\nu, I)\) where \(\nu \in N(x)\) and \(I \in \mathcal{F}\). i.e., for the fuzzy set \(C_{\mu}\) corresponding to every \(I \in \mathcal{F}\), there is at least one \(y \in X\) such that \(C_{\mu}(y) > I(y)\). Hence, \(C_{\mu} \in \mathcal{F}\). Since, \(\nu \in N(x)\) is arbitrary, therefore, \(x_2 \in A'(\mathcal{F}, \tau) \Rightarrow C_{\mu} \in \mathcal{F}\) for every \(\nu \in N(x)\).

Conversely, let \(C_{\mu} \in \mathcal{F}\) for every \(\nu \in N(x)\). i.e., for each \(I \in \mathcal{F}\), there is at least one \(y \in X\) such that \(C_{\mu}(y) > I(y)\). i.e., \(v(y) + A(y) - 1 > I(y)\). Therefore, for each \(\nu \in N(x)\) and \(I \in \mathcal{F}\) there is at least one \(y \in X\) such that \(v(y) + A(y) - 1 > I(y)\). i.e., \(x_2 \in A'(\mathcal{F}, \tau)\). This completes the proof.

Clearly, \(x_2 \in A' \Leftrightarrow C_{\mu} \in \mathcal{F}\) for at least one \(\nu \in N(x)\). [Here, \(C_{\mu}\) may be equal to \(0_x\).

Now, we define a criterion between fuzzy topologies and fuzzy ideals which will be helpful for our study of fuzzy semiregularization properties.
Definition 3.1. Let \((X, \tau)\) be a fts with \(\mathcal{I}\) be a fuzzy ideal on \(X\) such that \(\text{int} I = 0_X\) for every \(I \in \mathcal{I}\). Then \(\tau\) is said to have \(\mathcal{I}\)-interior if and only if for any \(\mu \in \tau\) and for any fuzzy set \(A\) of \(X\), \(\mu \cap \text{int} A \Rightarrow C_{\mu A} \in \mathcal{I}\).

Theorem 3.7. Let \((X, \tau)\) be a fts with \(\mathcal{I}\) be a fuzzy ideal on \(X\) such that \(\text{int} I = 0_X\) for every \(I \in \mathcal{I}\). Then \(\tau\) has \(\mathcal{I}\)-interior implies \(\mu \subseteq \mu'\) for every \(\mu \in \tau\).

Proof. Let \(\mu \in \tau\) and \(x_0 \notin \mu\) be any fuzzy point. Let \(v \in N(x_0)\) be any q-nbd of \(x_0\). Then obviously \(\mu \cap \text{int} v\), so that \(C_{\mu v} \in \mathcal{I}\) (by hypothesis). Therefore, by Theorem 3.6, \(x_0 \notin \mu'\), i.e., \(\mu \subseteq \mu'\).

The following example shows an interesting result. It also supports our previous constructions.

Example 3.1. Let \(\mathcal{I}_n = \{I : \text{int}(\text{cl} I) = 0_X\}\). Then \(\mathcal{I}_n\) is obviously a fuzzy ideal on fts \((X, \tau)\). Let \(A\) be any fuzzy set of \(X\). Let \(\tau\) has \(\mathcal{I}_n\)-interior. Then it is easy to verify that \(A(\mathcal{I}_n) = \text{cl}(\text{int} A) \subseteq [\text{cl} A](\mathcal{I}_n) = \text{cl}(\text{int}(A))\). Therefore \(\tau(\mathcal{I}_n)\) is generated by the \(\text{cl}'\) operator where \(\text{cl}' A = A \cup \text{cl}(\text{int} A)\) for every fuzzy set \(A\) of \(X\).

4. Fuzzy semiregularization properties and fuzzy local functions

In this section we investigate how fuzzy semiregular properties are shared by a fts and generated fts by a fuzzy ideal and also discuss some related topics. We first state a lemma which will be helpful for our discussion.

Lemma 4.1. \([10]\). Let \(\tau\) and \(\sigma\) be two fuzzy topologies on \(X\) and \(\tau \subseteq \sigma\). If \(\text{cl}_\sigma(v) = \text{cl}_\tau(v)\) for every \(v \in \sigma\), then \(\tau = \sigma\).

Theorem 4.1. Let \((X, \tau)\) be a fts with \(\mathcal{I}\) be a fuzzy ideal on \(X\) such that \(\text{int} I = 0_X\) for every \(I \in \mathcal{I}\). Then \(\tau\) has \(\mathcal{I}\)-interior implies \(\tau = \{\tau'\}_{\mathcal{I}}\).

Proof. Since \(\tau'\) has \(\mathcal{I}\)-interior, therefore, for any \(v \in \tau'\), \(v \subseteq v'\) [by Theorem 3.7 and as \(v'(\mathcal{I}, \tau) = v'(\mathcal{I}, \tau')\)]. So \(\text{cl}_\tau(v) = \text{cl}_\tau'(v) = v \cup v' = v' = \text{cl}_\tau(v' \subseteq \text{cl}_\tau'(v)\) for every \(v \in \tau'\) [by Theorem 2.1, (iii)]. Also \(v \subseteq v'\) implies \(\text{cl}_\tau(v) \subseteq \text{cl}_\tau'(v') = \text{cl}_\tau'(v)\). Hence \(\text{cl}_\tau(v) \subseteq \text{cl}_\tau'(v)\) for every \(v \in \tau'\). Therefore, by Lemma 4.1, the result follows.

Corollary 4.1. Let \((X, \tau)\) be a fts with \(\mathcal{I}\) be a fuzzy ideal on \(X\) such that \(\text{int} I = 0_X\) for every \(I \in \mathcal{I}\). Then \(\tau\) has \(\mathcal{I}\)-interior and \((X, \tau')\) is fuzzy semiregular implies \(\tau = \tau'\).

Proof. Since \(\tau' = (\tau')_\mathcal{I} = \tau, \tau \subseteq \tau \subseteq \tau'\), the result follows.
Definition 4.1, [12]. Let \((X, \tau)\) be a fts and \(Y \subseteq X\). Then the family \(\nu = \{f|_{\nu} : \mu \in \tau\}\) constitutes a fuzzy topology for \(Y\) and is called the relative fuzzy topology or the relativization of \(\tau\) to \(Y\) and \((Y, \nu)\) is called a subspace of \((X, \tau)\). More formally, a fts \((Y, \nu)\) is called a subspace of another fts \((X, \tau)\) if and only if \(Y \subseteq X\) and \(\nu\) is the relativization of \(\tau\) to \(Y\). Here we adopt the following conventions and it will cause no ambiguity. A fuzzy set \(\mu\) on \(Y\) is automatically considered as a fuzzy set on \(X\), in that case \(\mu\) takes the value 0 on \(X \setminus Y\). Conversely, any fuzzy set of \(X\) taking value 0 on \(X \setminus Y\), can also be considered as a fuzzy set on \(Y\) [12]. For a subspace we will write the relative fuzzy topology \(\nu\) as \(\tau|_{Y}\) on \(Y\).

Let \((X, \tau)\) be a fts with \(\mathfrak{I}\) be a fuzzy ideal on \(X\) and \(Y \subseteq X\). Then \(\mathfrak{I}_{Y} = \{I|_{Y} : I \in \mathfrak{I}\}\) is obviously a fuzzy ideal on \(Y\) and called relative fuzzy ideal or the relativization of \(\mathfrak{I}\) to \(Y\). The following lemmas and results are only stated as they are easy to check.

Lemma 4.2. Let \((X, \tau)\) be a fts with \(\mathfrak{I}\) be a fuzzy ideal on \(X\) and \(Y \subseteq X\). Then \((\tau|_{Y})(\mathfrak{I}_{Y}) = \tau^{*}(\mathfrak{I}|_{Y})\).

Lemma 4.3. Let \((X, \tau)\) be a fts with \(\mathfrak{I}\) be a fuzzy ideal on \(X\) such that \(\text{int}_{\tau}(I|_{Y}) = 0_{Y}\) for every \(I \in \mathfrak{I}\). Then \((\tau|_{Y})(\mathfrak{I}_{Y})\) has \(\mathfrak{I}_{Y}\)-interior \(\Leftrightarrow\) \(\tau^{*}(\mathfrak{I}|_{Y})\) has \(\mathfrak{I}_{Y}\)-interior.

Theorem 4.2. Let \((X, \tau)\) be a fts with \(\mathfrak{I}\) be a fuzzy ideal on \(X\) such that \(\text{int}_{\tau}(I|_{Y}) = 0_{Y}\) for every \(I \in \mathfrak{I}\) where \(Y \subseteq X\). Then \(\tau|_{Y}\) has \(\mathfrak{I}_{Y}\)-interior implies \((\tau|_{Y}) = \{\tau^{*}|_{Y}\}\).

Here we mention some fuzzy semiregular properties which are shared by \((X, \tau)\) and \((X, \tau^{*})\), namely, fuzzy \(T_{2}^{*}\)-property, fuzzy almost regularity, fuzzy Urysohn, fuzzy nearcompactness, fuzzy almost compactness, fuzzy \(S\)-closedness, the property of being fuzzy extremely disconnected, etc. [10, 11]. In this connection we refer [3] for general topological ideas of some semiregularization properties. The following corollary is an immediate consequence of previous discussion.

Corollary 4.2. Fuzzy semiregularization properties are shared by \((X, \tau)\) and \((X, \tau^{*})\) if \(\tau^{*}\) has \(\mathfrak{I}\)-interior where \(\mathfrak{I}\) be a fuzzy ideal on \(X\) such that \(\text{int}I = 0_{X}\) for every \(I \in \mathfrak{I}\).

Now we discuss some mappings between two fts. It has been proved that if the domain or codomain are replaced by their respective fuzzy semiregularization spaces, they show many interesting results [10, 11].

Definition 4.2, [10, 11]. A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is called:

(a) fuzzy almost continuous (in short, FAC),
(b) fuzzy δ-continuous (FDC),
(c) fuzzy super continuous (FSC),
(d) fuzzy weakly continuous (FWC),

if for any fuzzy point \(x_2\) in \(X\) and any fuzzy open set \(v \in \sigma\) containing \(f(x_2)\), there is a fuzzy open set \(\mu \in \tau\) containing \(x_2\) such that

(a) \(f(\mu) \subseteq \text{int}(\text{cl}(v))\),
(b) \(f(\text{int}(\text{cl}(\mu))) \subseteq \text{int}(\text{cl}(v))\),
(c) \(f(\text{int}(\text{cl}(\mu))) \subseteq v\),
(d) \(f(\mu) \subseteq \text{cl}(v)\),

respectively. Equivalent forms of definitions can be found in [2, 13].

Theorem 4.3, [11]. Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a function. Then

(a) \(f\) is FAC if and only if \(f : (X, \tau) \rightarrow (Y, \sigma)\) is continuous,
(b) \(f\) is FDC if and only if \(f : (X, \tau) \rightarrow (Y, \sigma)\) is continuous,
(c) \(f\) is FSC if and only if \(f : (X, \tau) \rightarrow (Y, \sigma)\) is continuous.

Corollary 4.3, [11]. Let \(X\) and \(Y\) be fuzzy semiregular spaces, then for any function \(f : X \rightarrow Y\) the following are equivalent: FSC \(\iff\) FC \(\iff\) FDC \(\iff\) FAC. [FC means fuzzy continuous].

Corollary 4.4. Let \((X, \tau)\) be a fts with \(\mathcal{J}\) be a fuzzy ideal on \(X\) such that \(\text{int}_\tau I = 0_x\) for every \(I \in \mathcal{J}\). Let \((Y, \sigma)\) be another fts. Let \(\tau'\) has \(\mathcal{J}\)-interior. Then

\[ f : (X, \tau) \rightarrow (Y, \sigma)\] is FSC if and only if \(f : (X, \tau') \rightarrow (Y, \sigma)\) is FSC.

Proof. Follows from Theorem 4.1 and Theorem 4.3 (c).

Corollary 4.5. Let \((X, \tau)\) be a fts with \(\mathcal{J}\) be a fuzzy ideal on \(X\) such that \(\text{int}_\tau I = 0_x\) for every \(I \in \mathcal{J}\). Let \((Y, \sigma)\) be another fts with \(\mathcal{J}\) be a fuzzy ideal on \(Y\) such that \(\text{int}_\sigma J = 0_y\) for every \(J \in \mathcal{J}\). Let \(\tau'(\mathcal{J})\) has \(\mathcal{J}\)-interior and \(\sigma'(\mathcal{J})\) has \(\mathcal{J}\)-interior. Then \(f : (X, \tau) \rightarrow (Y, \sigma)\) is FDC if and only if \(f : (X, \tau') \rightarrow (Y, \sigma')\) is FDC.

Proof. Follows from Theorem 4.1 and Theorem 4.3, (b).

Corollary 4.6. Let \((X, \tau)\) be a fts and \((Y, \sigma)\) be another fts with \(\mathcal{J}\) be a fuzzy ideal on \(Y\) such that \(\text{int}_\sigma J = 0_y\) for every \(J \in \mathcal{J}\). Let \(\sigma'\) has \(\mathcal{J}\)-interior. Then

\[ f : (X, \tau) \rightarrow (Y, \sigma)\] is FAC if and only if \(f : (X, \tau) \rightarrow (Y, \sigma')\) is FAC.
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Proof. Follows from Theorem 4.1 and Theorem 4.3 (a).

Corollary 4.7. Let \((X, \tau)\) and \((Y, \sigma)\) be two fts with \(\mathcal{I}\) and \(\mathcal{J}\) be two fuzzy ideals on \(X\) and \(Y\) respectively such that \(\text{int}_I I = 0_x\) for every \(I \in \mathcal{I}\) and \(\text{int}_J J = 0_y\) for every \(J \in \mathcal{J}\). Let \(\tau'(\mathcal{I})\) has \(\mathcal{I}\)-interior and \(\sigma'(\mathcal{J})\) has \(\mathcal{J}\)-interior and also \((X, \tau')\) and \((Y, \sigma')\) are fuzzy semiregular spaces. Then for any function \(f:(X, \tau) \to (Y, \sigma)\), the following are equivalent: \(\text{FSC} \iff \text{FC} \iff \text{FD} \iff \text{FA}\). Proof. Follows from Corollary 4.1 and Corollary 4.3.

Theorem 4.4, [11]. Let \((X, \tau)\) be a fts and \((Y, \sigma)\) be fuzzy almost regular. Then \(f:(X, \tau) \to (Y, \sigma)\) is FWC if and only if it is FAC.

Corollary 4.8. Let \((X, \tau)\) be a fts and \((Y, \sigma)\) be another fts with a fuzzy ideal \(\mathcal{J}\) on \(Y\) such that \(\text{int}_J J = 0_y\) for every \(J \in \mathcal{J}\). Let \(\sigma'\) have \(\mathcal{J}\)-interior and \((Y, \sigma)\) be fuzzy almost regular. Then \(f:(X, \tau) \to (Y, \sigma)\) is FWC if and only if \(f:(X, \tau) \to (Y, \sigma')\) is FAC.

Proof. Follows from Theorem 4.1, Theorem 4.3 (a) and Theorem 4.4.

Corollary 4.9, [11]. Let \((X, \tau)\) be fuzzy semiregular and \((Y, \sigma)\) be fuzzy regular. Then for a function \(f:(X, \tau) \to (Y, \sigma)\), the following are equivalent: \(\text{FSC} \iff \text{FC} \iff \text{FD} \iff \text{FWC} \iff \text{FAC}\).

Corollary 4.10. Let \((X, \tau)\) be a fts with \(\mathcal{I}\) be a fuzzy ideal on \(X\) such that \(\text{int}_I I = 0_x\) for every \(I \in \mathcal{I}\). Let \((Y, \sigma)\) be a fuzzy regular space. Let \(\tau'\) is \(\mathcal{I}\)-interior and \((X, \tau')\) be fuzzy semiregular. Then, for a function \(f:(X, \tau) \to (Y, \sigma)\), the following are equivalent: \(\text{FSC} \iff \text{FC} \iff \text{FD} \iff \text{FA} \iff \text{FWC}\).

Proof. Follows from Corollary 4.1 and Corollary 4.9.

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