CHAPTER - V

BIPOLAR VALUED MULTI FUZZY IDEALS OF A SEMIRING

5.1 Introduction: In this chapter, we introduce the concept of bipolar valued multi fuzzy ideals of a semiring and establish some results on these. We also made an attempt to study the properties of bipolar valued multi fuzzy ideals and bipolar valued multi fuzzy normal ideals of the semiring under homomorphism and anti-homomorphism.

5.1.1 Definition: Let \((R, +, .)\) be a semiring. A bipolar valued multi fuzzy subset \(\mathcal{A} = A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^-\) of \(R\) is said to be a bipolar valued multi fuzzy ideal of \(R\) if the following conditions are satisfied,

(i) \(A_i^+(x+y) \geq \min\{ A_i^+(x), A_i^+(y) \}\)

(ii) \(A_i^+(xy) \geq \max\{ A_i^+(x), A_i^+(y) \}\)

(iii) \(A_i^-(x+y) \leq \max\{ A_i^-(x), A_i^-(y) \}\)

(iv) \(A_i^-(xy) \leq \min\{ A_i^-(x), A_i^-(y) \}\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

5.1.2 Definition: Let \((R, +, .)\) be a semiring. A bipolar valued multi fuzzy ideal \(\mathcal{A} = A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^-\) of \(R\) is said to be a bipolar valued multi fuzzy normal ideal of \(R\) if the following conditions are satisfied

(i) \(A_i^+(x + y) = A_i^+(y + x)\)

(ii) \(A_i^+(xy) = A_i^+(yx)\)

(iii) \(A_i^-(x + y) = A_i^-(y + x)\)
(iv) \(A_i^- (xy) = A_i^- (yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

### 5.2 - PROPERTIES OF BIPOLAR VALUED MULTI FUZZY IDEALS:

#### 5.2.1 Theorem:
Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy ideal of a semiring \(R\). (i) If \(A_i^+(x+y) = 0\) then either \(A_i^+(x) = 0\) or \(A_i^+(y) = 0\) for all \(x\) and \(y\) in \(R\) and for all \(i\). (ii) If \(A_i^+(xy) = 0\) then either \(A_i^+(x) = 0\) and \(A_i^-(y) = 0\) for all \(x\) and \(y\) in \(R\) and for all \(i\). (iii) If \(A_i^-(x+y) = 0\) then either \(A_i^-(x) = 0\) or \(A_i^-(y) = 0\) for all \(x\) and \(y\) in \(R\) and for all \(i\). (iv) If \(A_i^-(xy) = 0\) then either \(A_i^-(x) = 0\) and \(A_i^-(y) = 0\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

**Proof:** Let \(x\) and \(y\) be in \(R\).

(i) By the definition \(A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \}\) which implies that \(0 \geq \min \{ A_i^+(x), A_i^+(y) \}\). Therefore either \(A_i^+(x) = 0\) or \(A_i^+(y) = 0\) for all \(i\). (ii) By the definition \(A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \}\) which implies that \(0 \geq \max \{ A_i^+(x), A_i^+(y) \}\). Therefore either \(A_i^+(x) = 0\) and \(A_i^+(y) = 0\) for all \(i\). (iii) By the definition \(A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \}\) which implies that \(0 \leq \max \{ A_i^-(x), A_i^-(y) \}\). Therefore either \(A_i^-(x) = 0\) or \(A_i^-(y) = 0\) for all \(i\). (iv) By the definition \(A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \}\) which implies that \(0 \leq \min \{ A_i^-(x), A_i^-(y) \}\). Therefore either \(A_i^- (x) = 0\) and \(A_i^- (y) = 0\) for all \(i\).

#### 5.2.2 Theorem:
If \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) is a bipolar valued multi fuzzy ideal of a semiring \(R\), then \(H = \{ x \in R \mid A_i^+(x) = 1, A_i^- (x) = -1 \ \text{for all} \ i \}\) is either empty or is a ideal of \(R\).

**Proof:** If no element satisfies this condition then \(H\) is empty.
If \( x \) and \( y \) in \( H \), then \( A_i^+(x) = 1, A_i^+(y) = 1, A_i^-(x) = -1, A_i^-(y) = -1, \)
\[
A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \} = \min \{ 1, 1 \} = 1.
\]
Therefore \( A_i^+(x+y) = 1 \) for all \( i \).

And \( A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \} = \max \{ 1, 1 \} = 1. \)

Therefore \( A_i^+(xy) = 1 \) for all \( i \).

Also \( A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \} = \max \{-1, -1 \} = -1. \)

Therefore \( A_i^-(x+y) = -1 \) for all \( i \).

And \( A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \} = \min \{-1, -1 \} = -1. \)

Therefore \( A_i^-(xy) = -1 \) for all \( i \).

We get \( x+y \in H \) and \( xy \in H \). Hence \( H \) is a ideal of \( R \).

**5.2.3 Theorem:** If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^- , \ldots, A_i^- \rangle \) and \( B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_i^-, B_2^- , \ldots, B_i^- \rangle \) are two bipolar valued multi fuzzy ideals of a semiring \( R \), then their intersection \( A \cap B \) is a bipolar valued multi fuzzy ideal of \( R. \)

**Proof:** Let \( C = A \cap B \) and \( C = \langle C_1^+, C_2^+, \ldots, C_i^+, C_i^-, C_2^- , \ldots, C_i^- \rangle. \)

Now \( C_i^+(x+y) = \min \{ A_i^+(x+y), B_i^+(x+y) \} \)
\[
\geq \min \{ \min \{ A_i^+(x), A_i^+(y) \}, \min \{ B_i^+(x), B_i^+(y) \} \}
\]
\[
\geq \min \{ \min \{ A_i^+(x), B_i^+(x) \}, \min \{ A_i^+(y), B_i^+(y) \} \}
\]
\[
= \min \{ C_i^+(x), C_i^+(y) \} \text{ for all } i.
\]

Therefore \( C_i^+(x+y) \geq \min \{ C_i^+(x), C_i^+(y) \} \) for \( x \) and \( y \) in \( R \) and for all \( i \).
And $C_i^+(xy) = \min \{ A_i^+(xy), B_i^+(xy) \}$

\[ \geq \min \{ \max \{ A_i^+(xy), A_i^+(xy) \}, \max \{ B_i^+(xy), B_i^+(xy) \} \} \]

\[ \geq \max \{ \min \{ A_i^+(xy), B_i^+(xy) \}, \min \{ A_i^+(xy), B_i^+(xy) \} \} \]

\[ = \max \{ C_i^+(xy), C_i^+(xy) \} \text{ for all } i. \]

Therefore $C_i^+(xy) \geq \max \{ C_i^+(xy), C_i^+(xy) \}$ for $x$ and $y$ in $R$ and for all $i$.

Also $C_i^-(x+y) = \max \{ A_i^-(x+y), B_i^-(x+y) \}$

\[ \leq \max \{ \max \{ A_i^-(x), A_i^-(y) \}, \max \{ B_i^-(x), B_i^-(y) \} \} \]

\[ \leq \max \{ \max \{ A_i^-(x), B_i^-(x) \}, \max \{ A_i^-(y), B_i^-(y) \} \} \]

\[ = \max \{ C_i^-(x), C_i^-(y) \} \text{ for all } i. \]

Therefore $C_i^-(x+y) \leq \max \{ C_i^-(x), C_i^-(y) \}$ for $x$ and $y$ in $R$ and for all $i$.

And $C_i^-(xy) = \max \{ A_i^-(xy), B_i^-(xy) \}$

\[ \leq \max \{ \min \{ A_i^-(xy), A_i^-(xy) \}, \min \{ B_i^-(xy), B_i^-(xy) \} \} \]

\[ \leq \min \{ \max \{ A_i^-(xy), B_i^-(xy) \}, \min \{ A_i^-(xy), B_i^-(xy) \} \} \]

\[ = \min \{ C_i^-(xy), C_i^-(xy) \} \text{ for all } i. \]

Therefore $C_i^-(xy) \leq \min \{ C_i^-(xy), C_i^-(xy) \}$ for $x$ and $y$ in $R$ and for all $i$.

Hence $A \bowtie B$ is a bipolar valued multi fuzzy ideal of $R$.

**5.2.4 Theorem:** The intersection of a family of bipolar valued multi fuzzy ideals of a semiring $R$ is a bipolar valued multi fuzzy ideal of $R.$
**Proof:** This theorem can easily prove by Theorem 5.2.3.

**5.2.5 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are any two bipolar valued multi fuzzy ideals of the semirings $R_1$ and $R_2$ respectively, then $A \times B$ is a bipolar valued multi fuzzy ideal of $R_1 \times R_2$.

**Proof:** Let $x_1$, $x_2$ be in $R_1$, $y_1$ and $y_2$ be in $R_2$. Then $(x_1, y_1)$ and $(x_2, y_2)$ are in $R_1 \times R_2$.

Now $(A_i \times B_i)^+(x_1, y_1) + (x_2, y_2) = (A_i \times B_i)^+(x_1 + x_2, y_1 + y_2)$

$$= \min \left\{ A_i^+(x_1 + x_2), B_i^+(y_1 + y_2) \right\}$$

$$\geq \min \left\{ \min \left\{ A_i^+(x_1), A_i^+(x_2) \right\}, \min \left\{ B_i^+(y_1), B_i^+(y_2) \right\} \right\}$$

$$= \min \left\{ \min \left\{ A_i^+(x_1), B_i^+(y_1) \right\}, \min \left\{ A_i^+(x_2), B_i^+(y_2) \right\} \right\}$$

$$= \min \left\{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \right\} \text{ for all } i.$$

Therefore $(A_i \times B_i)^+[x_1, y_1] + (x_2, y_2) \geq \min \left\{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \right\}$ for all $(x_1, y_1)$ and $(x_2, y_2)$ in $R_1 \times R_2$ and for all $i$.

And $(A_i \times B_i)^+[x_1, y_1, x_2, y_2] = (A_i \times B_i)^+(x_1 x_2, y_1 y_2)$

$$= \min \left\{ A_i^+(x_1 x_2), B_i^+(y_1 y_2) \right\}$$

$$\geq \min \left\{ \max \left\{ A_i^+(x_1), A_i^+(x_2) \right\}, \max \left\{ B_i^+(y_1), B_i^+(y_2) \right\} \right\}$$

$$\geq \max \left\{ \min \left\{ A_i^+(x_1), B_i^+(y_1) \right\}, \min \left\{ A_i^+(x_2), B_i^+(y_2) \right\} \right\}$$

$$= \max \left\{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \right\} \text{ for all } i.$$

Therefore $(A_i \times B_i)^+[x_1, y_1, x_2, y_2] \geq \max \left\{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^+(x_2, y_2) \right\}$ for all $(x_1, y_1)$ and $(x_2, y_2)$ in $R_1 \times R_2$ and for all $i$.

Also $(A_i \times B_i)^-[x_1, y_1] + (x_2, y_2) = (A_i \times B_i)^-(x_1 + x_2, y_1 + y_2)$

$$= \max \left\{ A_i^-(x_1 + x_2), B_i^-(y_1 + y_2) \right\}$$

$$\leq \max \left\{ \max \left\{ A_i^-(x_1), A_i^-(x_2) \right\}, \max \left\{ B_i^-(y_1), B_i^-(y_2) \right\} \right\}$$. 
\[ = \max \{ \max \{ A_i(x_1), B_i(y_1) \}, \max \{ A_i(x_2), B_i(y_2) \} \} \]

\[ = \max \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \} \text{ for all } i. \]

Therefore \( (A_i \times B_i)^+ \leq [ (x_1, y_1) + (x_2, y_2) ] \leq \max \{ (A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \} \) for all \( (x_1, y_1) \) and \( (x_2, y_2) \) in \( R_1 \times R_2 \) and for all \( i \).

And \((A_i \times B_i)^-[ (x_1, y_1)(x_2, y_2) ] = (A_i \times B_i)^-(x_1x_2, y_1y_2) \)

\[ = \max \{ A_i^-(x_1x_2), B_i^-(y_1y_2) \} \]

\[ \leq \max \{ \min \{ A_i^-(x_1), A_i^-(x_2) \}, \min \{ B_i^-(y_1), B_i^-(y_2) \} \} \]

\[ \leq \min \{ \max \{ A_i^-(x_1), B_i^-(y_1) \}, \max \{ A_i^-(x_2), B_i^-(y_2) \} \} \]

\[ = \min \{ (A_i \times B_i)^+(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \} \text{ for all } i. \]

Therefore \((A_i \times B_i)^- \leq [ (x_1, y_1)(x_2, y_2) ] \leq \min \{ (A_i \times B_i)^-(x_1, y_1), (A_i \times B_i)^-(x_2, y_2) \} \)
for all \( (x_1, y_1) \) and \( (x_2, y_2) \) in \( R_1 \times R_2 \) and for all \( i \).

Hence \( A \times B \) is a bipolar valued multi fuzzy ideal of \( R_1 \times R_2 \).

**5.2.6 Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^- \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy subset of a semiring \( R \) and \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_i^-, V_2^- \ldots, V_i^- \rangle \) be the strongest bipolar valued multi fuzzy relation of \( R \). If \( A \) is a bipolar valued multi fuzzy ideal of \( R \), then \( V \) is a bipolar valued multi fuzzy ideal of \( R \times R \).

**Proof:** Suppose that \( A \) is a bipolar valued multi fuzzy ideal of \( R \).

Then for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are in \( R \times R \).

We have \( V_i^+(x+y) = V_i^+[ (x_1, x_2)+(y_1, y_2) ] \)

\[ = V_i^+( x_1+y_1, x_2+y_2 ) \]

\[ = \min \{ A_i^+(x_1+y_1), A_i^+(x_2+y_2) \} \]

\[ \geq \min \{ \min \{ A_i^+(x_1), A_i^+(y_1) \}, \min \{ A_i^+(x_2), A_i^+(y_2) \} \} \]
\[ = \min \{ \min \{ A_i^+(x_1), A_i^+(x_2) \}, \min \{ A_i^+(y_1), A_i^+(y_2) \} \} \]

\[ = \min \{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \} \]

\[ = \min \{ V_i^+(x), V_i^+(y) \} \text{ for all } i. \]

Therefore \( V_i^+(x+y) \geq \min \{ V_i^+(x), V_i^+(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

And \( V_i^+(xy) = V_i^+[ (x_1, x_2)(y_1, y_2) ] \)

\[ = V_i^+(x_1y_1, x_2y_2) \]

\[ = \min \{ A_i^+(x_1y_1), A_i^+(x_2y_2) \} \]

\[ \geq \min \{ \max \{ A_i^+(x_1), A_i^+(y_1) \}, \max \{ A_i^+(x_2), A_i^+(y_2) \} \} \]

\[ \geq \max \{ \min \{ A_i^+(x_1), A_i^+(x_2) \}, \min \{ A_i^+(y_1), A_i^+(y_2) \} \} \]

\[ = \max \{ V_i^+(x_1, x_2), V_i^+(y_1, y_2) \} \]

\[ = \max \{ V_i^+(x), V_i^+(y) \} \text{ for all } i. \]

Therefore \( V_i^+(xy) \geq \max \{ V_i^+(x), V_i^+(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

Also we have \( V_i^-(x+y) = V_i^-[ (x_1, x_2)^+(y_1, y_2) ] \)

\[ = V_i^-(x_1+y_1, x_2+y_2) \]

\[ = \max \{ A_i^-(x_1+y_1), A_i^-(x_2+y_2) \} \]

\[ \leq \max \{ \max \{ A_i^-(x_1), A_i^-(y_1) \}, \max \{ A_i^-(x_2), A_i^-(y_2) \} \} \]

\[ = \max \{ \max \{ A_i^-(x_1), A_i^-(x_2) \}, \max \{ A_i^-(y_1), A_i^-(y_2) \} \} \]

\[ = \max \{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \} \]
\[ V_i^-(x) = \max \{ V_i^-(x), V_i^-(y) \} \text{ for all } i. \]

Therefore \( V_i^-(x+y) \leq \max \{ V_i^-(x), V_i^-(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

And \( V_i^-(xy) = V_i^-[ (x_1, x_2)(y_1, y_2) ] \]

\[ = V_i^-(x_1 y_1, x_2 y_2) \]

\[ = \max \{ A_i^-(x_1 y_1), A_i^-(x_2 y_2) \} \]

\[ \leq \max \{ \min \{ A_i^-(x_1), A_i^-(y_1) \}, \min \{ A_i^-(x_2), A_i^-(y_2) \} \} \]

\[ \leq \min \{ \max \{ A_i^-(x_1), A_i^-(x_2) \}, \max \{ A_i^-(y_1), A_i^-(y_2) \} \} \]

\[ = \min \{ V_i^-(x_1, x_2), V_i^-(y_1, y_2) \} \]

\[ = \min \{ V_i^-(x), V_i^-(y) \} \text{ for all } i. \]

Therefore \( V_i^-(xy) \leq \min \{ V_i^-(x), V_i^-(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

Hence \( V \) is a bipolar valued multi fuzzy ideal of \( \mathbb{R} \times \mathbb{R} \).

**5.2.7 Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( \mathbb{R} \). Then the pseudo bipolar valued multi fuzzy coset \( (aA)^p \) is a bipolar valued multi fuzzy ideal of the semiring \( \mathbb{R} \), for every \( a \) in \( \mathbb{R} \) and \( p \) in \( \mathbb{P} \).

**Proof:** Let \( A \) be a bipolar valued multi fuzzy ideal of the semiring \( \mathbb{R} \).

For every \( x \) and \( y \) in \( \mathbb{R} \),

we have \( (aA_i^+)^p(x+y) = p(a)A_i^+(x+y) \)

\[ \geq p(a) \min \{ A_i^+(x), A_i^+(y) \} \]

\[ = \min \{ p(a)A_i^+(x), p(a)A_i^+(y) \} \]
Therefore \((aA_i^+)^p(x) \geq \min \{ (aA_i^+)^p(x), (aA_i^+)^p(y) \}\) for all \(i\).

And \((aA_i^+)^p(xy) = p(a)A_i^+(xy)\)

\[ \geq p(a) \max \{ A_i^+(x), A_i^+(y) \} \]

\[ = \max \{ p(a)A_i^+(x), p(a)A_i^+(y) \} \]

\[ = \max \{ (aA_i^+)^p(x), (aA_i^+)^p(y) \}\) for all \(i\).

Therefore \((aA_i^+)^p(xy) \geq \max \{ (aA_i^+)^p(x), (aA_i^+)^p(y) \}\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also \((aA_i^-)^p(x+y) = p(a)A_i^-(x+y)\)

\[ \leq p(a) \max \{ A_i^-(x), A_i^-(y) \} \]

\[ = \max \{ p(a)A_i^-(x), p(a)A_i^-(y) \} \]

\[ = \max \{ (aA_i^-)^p(x), (aA_i^-)^p(y) \}\) for all \(i\).

Therefore \((aA_i^-)^p(x+y) \leq \max \{ (aA_i^-)^p(x), (aA_i^-)^p(y) \}\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \((aA_i^-)^p(xy) = p(a)A_i^-(xy)\)

\[ \leq p(a) \min \{ A_i^-(x), A_i^-(y) \} \]

\[ = \min \{ p(a)A_i^-(x), p(a)A_i^-(y) \} \]

\[ = \min \{ (aA_i^-)^p(x), (aA_i^-)^p(y) \}\) for all \(i\).

Therefore \((aA_i^-)^p(xy) \leq \min \{ (aA_i^-)^p(x), (aA_i^-)^p(y) \}\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Hence \((aA)^p\) is a bipolar valued multi fuzzy ideal of the semiring \(R\).

5.2.8 Theorem: If \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) is a bipolar valued multi fuzzy ideal of a semiring \(R\), then \(?(A) = \langle ?(A_1^+), \ldots, A_i^+, \ldots, A_i^-, \ldots, A_1^- \rangle\) is a bipolar valued multi fuzzy ideal of \(R\).

Proof: For every \(x\) and \(y\) in \(R\),
we have \( (A_i^+)(x+y) = \min \{ \frac{1}{2}, A_i^+(x+y) \} \)

\[
\geq \min \{ \frac{1}{2}, \min \{ A_i^+(x), A_i^+(y) \} \}
\]

\[
= \min \{ \min \{ \frac{1}{2}, A_i^+(x) \}, \min \{ \frac{1}{2}, A_i^+(y) \} \}
\]

\[
= \min \{ (A_i^+)(x), (A_i^+)(y) \} \text{ for all } i.
\]

Therefore \( (A_i^+)(x+y) \geq \min \{ (A_i^+)(x), (A_i^+)(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

Also \( (A_i^+)(xy) = \min \{ \frac{1}{2}, A_i^+(xy) \} \)

\[
\geq \min \{ \frac{1}{2}, \max \{ A_i^+(x), A_i^+(y) \} \}
\]

\[
= \max \{ \min \{ \frac{1}{2}, A_i^+(x) \}, \min \{ \frac{1}{2}, A_i^+(y) \} \}
\]

\[
= \max \{ (A_i^+)(x), (A_i^+)(y) \} \text{ for all } i.
\]

Therefore \( (A_i^+)(xy) \geq \max \{ (A_i^+)(x), (A_i^+)(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

And \( (A_i^-)(x+y) = \max \{ -\frac{1}{2}, A_i^-(x+y) \} \)

\[
\leq \max \{ -\frac{1}{2}, \max \{ A_i^-(x), A_i^-(y) \} \}
\]

\[
= \max \{ \max \{ -\frac{1}{2}, A_i^-(x) \}, \max \{ -\frac{1}{2}, A_i^-(y) \} \}
\]

\[
= \max \{ (A_i^-)(x), (A_i^-)(y) \} \text{ for all } i.
\]

Therefore \( (A_i^-)(x+y) \leq \max \{ (A_i^-)(x), (A_i^-)(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

Also \( (A_i^-)(xy) = \max \{ -\frac{1}{2}, A_i^-(xy) \} \)

\[
\leq \max \{ -\frac{1}{2}, \min \{ A_i^-(x), A_i^-(y) \} \}
\]

\[
= \min \{ \max \{ -\frac{1}{2}, A_i^-(x) \}, \max \{ -\frac{1}{2}, A_i^-(y) \} \}
\]
= \min \{ ?(A^{-i})(x), ?(A^{-i})(y) \} \text{ for all } i.

Therefore \(?(A^{-i})(xy) \leq \min \{ ?(A^{-i})(x), ?(A^{-i})(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Hence \(?(A)\) is a bipolar valued multi fuzzy ideal of \(R\).

5.2.9 Theorem: If \(A = \langle A^{-1}, A^{-2}, \ldots, A^{-i}, A^{-1}, A^{-2}, \ldots, A^{-i} \rangle\) is a bipolar valued multi fuzzy ideal of a semiring \(R\), then \(!\langle A \rangle = \langle !(A^{-1}), !(A^{-2}), \ldots, !(A^{-i}), !(A^{-1}), !(A^{-2}), \ldots, !(A^{-i}) \rangle\) is a bipolar valued multi fuzzy ideal of \(R\).

Proof: For every \(x\) and \(y\) in \(R\),

we have 

\[ !(A^{+i})(x+y) = \max \{ \frac{1}{2}, A^{+i}(x+y) \} \]

\[ \geq \max \{ \frac{1}{2}, \min \{ A^{+i}(x), A^{+i}(y) \} \} \]

\[ = \min \{ \max \{ \frac{1}{2}, A^{+i}(x) \}, \max \{ \frac{1}{2}, A^{+i}(y) \} \} \]

\[ = \min \{ !(A^{+i})(x), !(A^{+i})(y) \} \text{ for all } i. \]

Therefore \( !(A^{+i})(x+y) \geq \min \{ !(A^{+i})(x), !(A^{+i})(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And

\[ !(A^{+i})(xy) = \max \{ \frac{1}{2}, A^{+i}(xy) \} \]

\[ \geq \max \{ \frac{1}{2}, \max \{ A^{+i}(x), A^{+i}(y) \} \} \]

\[ = \max \{ \max \{ \frac{1}{2}, A^{+i}(x) \}, \max \{ \frac{1}{2}, A^{+i}(y) \} \} \]

\[ = \max \{ !(A^{+i})(x), !(A^{+i})(y) \} \text{ for all } i. \]

Therefore \( !(A^{+i})(xy) \geq \max \{ !(A^{+i})(x), !(A^{+i})(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also

\[ !(A^{-i})(x+y) = \min \{ -\frac{1}{2}, A^{-i}(x+y) \} \]
\[ \leq \min \{ -\frac{1}{2}, \max \{ A_i^-(x), A_i^-(y) \} \} \]

\[ = \max \{ \min \{ -\frac{1}{2}, A_i^-(x) \}, \min \{ -\frac{1}{2}, A_i^-(y) \} \} \]

\[ = \max \{ !(A_i^-)(x), !(A_i^-)(y) \} \text{ for all } i. \]

Therefore \( !(A_i^-)(x+y) \leq \max \{ !(A_i^-)(x), !(A_i^-)(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( !(A_i^-)(xy) = \min \{ -\frac{1}{2}, A_i^-(xy) \} \)

\[ \leq \min \{ -\frac{1}{2}, \min \{ A_i^-(x), A_i^-(y) \} \} \]

\[ = \min \{ \min \{ -\frac{1}{2}, A_i^-(x) \}, \min \{ -\frac{1}{2}, A_i^-(y) \} \} \]

\[ = \min \{ !(A_i^-)(x), !(A_i^-)(y) \} \text{ for all } i. \]

Therefore \( !(A_i^-)(xy) \leq \min \{ !(A_i^-)(x), !(A_i^-)(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( !(A) \) is a bipolar valued multi fuzzy ideal of \( R \).

**5.2.10 Theorem:** If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of a semiring \( R \), then

\[ Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A) = Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_1^+), Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_2^+), \ldots, Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+), Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_1^-), Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_2^-), \ldots, Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-) \] is a bipolar valued multi fuzzy ideal of \( R \).

**Proof:** For every \( x \) and \( y \) in \( R \), \( \alpha \) in \([0, 1]\) and \( \beta \) in \([-1, 0]\),

we have \( Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+)(x+y) = \min \{ \alpha_i, A_i^+(x+y) \} \)

\[ \geq \min \{ \alpha_i, \min \{ A_i^+(x), A_i^+(y) \} \} \]
\[
\begin{align*}
&= \min \{ \min \{ \alpha_i, A_i^+(x) \}, \min \{ \alpha_i, A_i^+(y) \} \} \\
&= \min \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(y)) \} \text{ for all } i.
\end{align*}
\]

Therefore
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(x+y)) \geq \min \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(y)) \} \text{ for all } x \text{ and } y \in \mathbb{R} \text{ and for all } i.
\]

And
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(xy)) = \min \{ \alpha_i, A_i^+(xy) \}
\]

\[
\geq \min \{ \alpha_i, \max \{ A_i^+(x), A_i^+(y) \} \}
\]

\[
= \max \{ \min \{ \alpha_i, A_i^+(x) \}, \min \{ \alpha_i, A_i^+(y) \} \}
\]

\[
= \max \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(y)) \} \text{ for all } i.
\]

Therefore
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(xy)) \leq \max \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^+(y)) \} \text{ for all } x \text{ and } y \in \mathbb{R} \text{ and for all } i.
\]

Also
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(x+y) = \max \{ \beta_i, A_i^- (x+y) \}
\]

\[
\leq \max \{ \beta_i, \max \{ A_i^-(x), A_i^-(y) \} \}
\]

\[
= \max \{ \max \{ \beta_i, A_i^-(x) \}, \max \{ \beta_i, A_i^-(y) \} \}
\]

\[
= \max \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(y) \} \text{ for all } i.
\]

Therefore
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(x+y) \leq \max \{ Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(x)), Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(y) \} \text{ for all } x \text{ and } y \in \mathbb{R} \text{ and for all } i.
\]

And
\[
Q_{\langle \alpha_1, \alpha_2, \ldots, \alpha_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle}(A_i^-)(xy) = \max \{ \beta_i, A_i^- (xy) \}
\]
\[ \leq \max \{ \beta_i, \min \{ A_i^-(x), A_i^-(y) \} \} \]

\[ = \min \{ \max \{ \beta_i, A_i^-(x) \}, \max \{ \beta_i, A_i^-(y) \} \} \]

\[ = \min \{ Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-)(x), Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-)(y) \} \quad \text{for all } i. \]

Therefore \[ Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-)(x) \]
\[ \leq \min \{ Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-)(x), \]
\[ Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-)(y) \} \quad \text{for all } x \text{ and } y \text{ in } R \text{ and for all } i.

Hence \[ Q((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A) \] is a bipolar valued multi fuzzy ideal of \( R. \)

5.2.11 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of a semiring \( R, \) then \[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A) = \langle P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_1^+), \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_2^+), \ldots, \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+), \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_1^-), \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_2^-), \ldots, \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^-) \rangle \] is a bipolar valued multi fuzzy ideal of \( R. \)

Proof: For every \( x \) and \( y \) in \( R, \) \( \alpha \) in \([0, 1] \) and \( \beta \) in \([−1, 0], \)

we have \[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(x+y) = \max \{ \alpha_i, A_i^+(x+y) \} \]

\[ \geq \max \{ \alpha_i, \min \{ A_i^+(x), A_i^+(y) \} \} \]

\[ = \min \{ \max \{ \alpha_i, A_i^+(x) \}, \max \{ \alpha_i, A_i^+(y) \} \} \]

\[ = \min \{ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(x), P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(y) \} \quad \text{for all } i. \]

Therefore \[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(x+y) \]
\[ \geq \min \{ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(x), \]
\[ P((\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i)) (A_i^+)(y) \} \quad \text{for all } x \text{ and } y \text{ in } R \text{ and for all } i.
And $P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (xy) = \max \{ \alpha_i, A_i^+ (xy) \}$

$\geq \max \{ \alpha_i, \max \{ A_i^+ (x), A_i^+ (y) \} \}$

$= \max \{ \max \{ \alpha_i, A_i^+ (x) \}, \max \{ \alpha_i, A_i^+ (y) \} \}$

$= \max \{ P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (x), P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (y) \}$ for all $i$.

Therefore $P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (xy) \geq \max \{ P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (x), P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^+) (y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Also $P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (x+y) = \min \{ \beta_i, A_i^- (x+y) \}$

$\leq \min \{ \beta_i, \max \{ A_i^- (x), A_i^- (y) \} \}$

$= \max \{ \min \{ \beta_i, A_i^- (x) \}, \min \{ \beta_i, A_i^- (y) \} \}$

$= \max \{ P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (x), P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (y) \}$ for all $i$.

Therefore $P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (x+y) \leq \max \{ P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (x), P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

And $P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (xy) = \min \{ \beta_i, A_i^- (xy) \}$

$\leq \min \{ \beta_i, \min \{ A_i^- (x), A_i^- (y) \} \}$

$= \min \{ \min \{ \beta_i, A_i^- (x) \}, \min \{ \beta_i, A_i^- (y) \} \}$

$= \min \{ P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (x), P_{\langle \alpha_1, \alpha_2, ..., \alpha_i, \beta_1, \beta_2, ..., \beta_i \rangle} (A_i^-) (y) \}$ for all $i$. 
Therefore \( P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i)(xy) \) \leq \min \{ P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i)(x), P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i)(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A) \) is a bipolar valued multi fuzzy ideal of \( R \).

5.2.12 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of a semiring \( R \), then

\[
G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A) = \langle G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_1^+), G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_2^+), \ldots, G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^+), G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_1^-), G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_2^-), \ldots, G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^-) \rangle
\]

is a bipolar valued multi fuzzy ideal of \( R \).

Proof: For every \( x \) and \( y \) in \( R \), \( \alpha \) in \([0, 1]\) and \( \beta \) in \([-1, 0]\),

we have

\[
G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i)(x+y) = \alpha_i A_i^+(x+y) \geq \alpha_i ( \min \{ A_i^+(x), A_i^+(y) \} )
\]

\[
= \min \{ \alpha_i A_i^+(x), \alpha_i A_i^+(y) \}
\]

\[
= \min \{ G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i)(x), G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i)(y) \} \text{ for all } i.
\]

Therefore

\[
G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^+)(x+y) \geq \min \{ G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^+)(x), G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^+)(y) \} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]

And

\[
G_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))}(A_i^+)(xy) = \alpha_i A_i^+(xy) \geq \alpha_i ( \max \{ A_i^+(x), A_i^+(y) \} )
\]

\[
= \max \{ \alpha_i A_i^+(x), \alpha_i A_i^+(y) \}
\]

Therefore

\[
P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i^+)(xy) \leq \min \{ P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i^+)(x), P_{((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))} (A_i^+)(y) \} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]
\[
\max \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+(x)), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+(y)) \right\} \text{ for all } i.
\]

Therefore
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+(xy)) \geq \max \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+(x)), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+(y)) \right\} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]

Also
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x+y) = -\beta_i A_i^- (x+y)
\]

\[
\leq -\beta_i \left( \max \left\{ A_i^-(x), A_i^-(y) \right\} \right)
\]

\[
= \max \left\{ -\beta_i A_i^-(x), -\beta_i A_i^-(y) \right\}
\]

\[
= \max \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(y) \right\} \text{ for all } i.
\]

Therefore
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x+y) \leq \max \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(y) \right\} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]

And
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(xy) = -\beta_i A_i^- (xy)
\]

\[
\leq -\beta_i \left( \min \left\{ A_i^-(x), A_i^-(y) \right\} \right)
\]

\[
= \min \left\{ -\beta_i A_i^-(x), -\beta_i A_i^-(y) \right\}
\]

\[
= \min \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(y) \right\} \text{ for all } i.
\]

Therefore
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(xy) \leq \min \left\{ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(x), \ G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^-)(y) \right\} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]

Hence
\[
G_{\langle a_1, a_2, \ldots, a_i, (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \text{ is a bipolar valued multi fuzzy ideal of } R.
\]
5.2.13 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy ideals of a semiring $R$, then $(A \cap B) = !(A) \cap !(B)$ is also a bipolar valued multi fuzzy ideal of $R$. 

Proof: By Theorem 5.2.3 and 5.2.9, it is true.

5.2.14 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy ideals of a semiring $R$, then $?(A \cap B) = ?(A) \cap ?(B)$ is also a bipolar valued multi fuzzy ideal of $R$. 

Proof: By Theorem 5.2.3 and 5.2.8, it is true.

5.2.15 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy ideal of a semiring $R$, then $!(?(A)) = ?!(A)$ is also a bipolar valued multi fuzzy ideal of $R$. 

Proof: By Theorem 5.2.8 and 5.2.9, it is true.

5.2.16 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy ideals of a semiring $R$, then $P_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (A \cap B) = P_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \cap P_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (B)$ is also a bipolar valued multi fuzzy ideal of $R$. 

Proof: By Theorem 5.2.3 and 5.2.11, it is true.

5.2.17 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy ideals of a semiring $R$, then $Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (A \cap B) = Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \cap Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_i) \rangle} (B)$ is also a bipolar valued multi fuzzy ideal of $R$. 


**Proof:** By Theorem 5.2.3 and 5.2.10, it is true.

**5.2.18 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy ideal of a semiring $R$, then $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A)$ is also a bipolar valued multi fuzzy ideal of $R$.

**Proof:** By Theorem 5.2.10 and 5.2.11, it is true.

**5.2.19 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy ideals of a semiring $R$, then $G_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A \cap B) = G_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A) \cap G_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))} (B)$ is also a bipolar valued multi fuzzy ideal of $R$.

**Proof:** By Theorem 5.2.3 and 5.2.12, it is true.

**5.2.20 Theorem:** Let $(R, +, )$ and $(R^1, +, )$ be any two semirings. The homomorphic image of a bipolar valued multi fuzzy ideal of $R$ is a bipolar valued multi fuzzy ideal of $R^1$.

**Proof:** Let $V = f(A)$, where $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy ideal of $R$. We have to prove that $V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle$ is a bipolar valued multi fuzzy ideal of $R^1$. Let $f(x)$ and $f(y)$ in $R^1$.

Now $V_i^+(f(x)+f(y)) = V_i^+(f(x)+y)$

$\geq A_i^+(x+y)$

$\geq \min \{ A_i^+(x), A_i^+(y) \}$

$= \min \{ V_i^+(f(x)), V_i^+(f(y)) \}$
which implies that $V_i^+ ( f(x)+f(y) ) \geq \min \{ V_i^+( f(x) ), V_i^+( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

And $V_i^+ ( f(x)f(y) ) = V_i^+ ( f(xy) )$
\[ \geq A_i^+(xy) \]
\[ \geq \max \{ A_i^+(x), A_i^+(y) \} \]
\[ = \max \{ V_i^+( f(x) ), V_i^+( f(y) ) \} \]

which implies that $V_i^+ ( f(x)f(y) ) \geq \max \{ V_i^+( f(x) ), V_i^+( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

Also $V_i^- ( f(x)+f(y) ) = V_i^- ( f(x+y) )$
\[ \leq A_i^-(x+y) \]
\[ \leq \max \{ A_i^-(x), A_i^-(y) \} = \max \{ V_i^-(f(x)), V_i^-(f(y)) \} \]

which implies that $V_i^- ( f(x)+f(y) ) \leq \max \{ V_i^- ( f(x) ), V_i^- ( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

And $V_i^- ( f(x)f(y) ) = V_i^- ( f(xy) )$
\[ \leq A_i^-(xy) \]
\[ \leq \min \{ A_i^-(x), A_i^-(y) \} \]
\[ = \min \{ V_i^- ( f(x) ), V_i^- ( f(y) ) \} \]

which implies that $V_i^- ( f(x)f(y) ) \leq \min \{ V_i^- ( f(x) ), V_i^- ( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

Hence $V$ is a bipolar valued multi fuzzy ideal of $R^l$.

5.2.21 Theorem: Let $(R, +, .)$ and $(R^l, +, .)$ be any two semirings. The homomorphic preimage of a bipolar valued multi fuzzy ideal of $R^l$ is a bipolar valued multi fuzzy ideal of $R$. 
Proof: Let $V = f(A)$, where $V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle$ is a bipolar valued multi fuzzy ideal of $R^i$. We have to prove that $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy ideal of $R$. Let $x$ and $y$ in $R$.

Now $A_i^+(x+y) = V_i^+( f(x+y) )$

$$= V_i^+( f(x)+f(y) )$$

$$\geq \min \{ V_i^+( f(x) ), V_i^+( f(y) ) \}$$

$$= \min \{ A_i^+(x), A_i^+(y) \}$$

which implies that $A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

And $A_i^+(xy) = V_i^+( f(xy) )$

$$= V_i^+( f(x)f(y) )$$

$$\geq \max \{ V_i^+( f(x) ), V_i^+( f(y) ) \}$$

$$= \max \{ A_i^+(x), A_i^+(y) \}$$

which implies that $A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Also $A_i^-(x+y) = V_i^-( f(x+y) )$

$$= V_i^-( f(x)+f(y) )$$

$$\leq \max \{ V_i^-( f(x) ), V_i^-( f(y) ) \}$$

$$= \max \{ A_i^-(x), A_i^-(y) \}$$

which implies that $A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

And $A_i^-(xy) = V_i^-( f(xy) )$

$$= V_i^-( f(x)f(y) )$$

$$\leq \min \{ V_i^-( f(x) ), V_i^-( f(y) ) \} = \min \{ A_i^-(x), A_i^-(y) \}$$

which implies that $A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $A$ is a bipolar valued multi fuzzy ideal of $R$. 
5.2.22 Theorem: Let \(( R, +, . )\) and \(( R^l, +, . )\) be any two semirings. The anti-homomorphic image of a bipolar valued multi fuzzy ideal of \(R\) is a bipolar valued multi fuzzy ideal of \(R^l\).

Proof: Let \(V = f(A)\), where \(A = \langle A^+_1, A^+_2, ..., A^+_i, A^-_1, A^-_2, ..., A^-_i \rangle\) is a bipolar valued multi fuzzy ideal of \(R\). We have to prove that \(V = \langle V^+_1, V^+_2, ..., V^+_i, V^-_1, V^-_2, ..., V^-_i \rangle\) is a bipolar valued multi fuzzy ideal of \(R^l\). Let \(f(x)\) and \(f(y)\) in \(R^l\).

Now \(V^+_i(f(x) + f(y)) = V^+_i(f(y + x))\)

\[
\geq A^+_i(y + x)
\]

\[
\geq \min \{ A^+_i(x), A^+_i(y) \}
\]

\[
= \min \{ V^+_i(f(x)), V^+_i(f(y)) \}
\]

which implies that \(V^+_i(f(x) + f(y)) \geq \min \{ V^+_i(f(x)), V^+_i(f(y)) \}\) for all \(f(x)\) and \(f(y)\) in \(R^l\) and for all \(i\).

And \(V^+_i(f(x)f(y)) = V^+_i(f(yx))\)

\[
\geq A^+_i(yx)
\]

\[
\geq \max \{ A^+_i(x), A^+_i(y) \} = \max \{ V^+_i(f(x)), V^+_i(f(y)) \}
\]

which implies that \(V^+_i(f(x)f(y)) \geq \max \{ V^+_i(f(x)), V^+_i(f(y)) \}\) for all \(f(x)\) and \(f(y)\) in \(R^l\) and for all \(i\).

Also \(V^-_i(f(x) + f(y)) = V^-_i(f(y + x))\)

\[
\leq A^-_i(y + x)
\]

\[
\leq \max \{ A^-_i(x), A^-_i(y) \}
\]

\[
= \max \{ V^-_i(f(x)), V^-_i(f(y)) \}
\]

which implies that \(V^-_i(f(x) + f(y)) \leq \max \{ V^-_i(f(x)), V^-_i(f(y)) \}\) for all \(f(x)\) and \(f(y)\) in \(R^l\) and for all \(i\).

And \(V^-_i(f(x)f(y)) = V^-_i(f(yx))\)
\[ A_i^-(yx) \leq \min \{ A_i^-(x), A_i^-(y) \} = \min \{ V_i^-(f(x)), V_i^-(f(y)) \} \]

which implies that \( V_i^-(f(x)f(y)) \leq \min \{ V_i^-(f(x)), V_i^-(f(y)) \} \) for all \( f(x) \) and \( f(y) \) in \( R^l \) and for all \( i \).

Hence \( V \) is a bipolar valued multi fuzzy ideal of \( R^l \).

5.2.23 **Theorem:** Let \( (R, +, \cdot) \) and \( (R^l, +, \cdot) \) be any two semirings. The anti-homomorphically preimage of a bipolar valued multi fuzzy ideal of \( R^l \) is a bipolar valued multi fuzzy ideal of \( R \).

**Proof:** Let \( V = f(A) \), where \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_i^-, V_2^-, \ldots, V_i^- \rangle \) is a bipolar valued multi fuzzy ideal of \( R^l \). We have to prove that \( A = \langle A_1^+, A_1^-, \ldots, A_i^+, A_i^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of \( R \). Let \( x \) and \( y \) in \( R \).

Now \( A_i^+(x+y) = V_i^+(f(x+y)) \)

\[ = V_i^+(f(y)+f(x)) \geq \min \{ V_i^+(f(x)), V_i^+(f(y)) \} = \min \{ A_i^+(x), A_i^+(y) \} \]

which implies that \( A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^+(xy) = V_i^+(f(xy)) \)

\[ = V_i^+(f(y)f(x)) \geq \max \{ V_i^+(f(x)), V_i^+(f(y)) \} = \max \{ A_i^+(x), A_i^+(y) \} \]

which implies that \( A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( A_i^-(x+y) = V_i^-(f(x+y)) \)

\[ = V_i^-(f(y)+f(x)) \leq \max \{ V_i^-(f(x)), V_i^-(f(y)) \} = \max \{ A_i^-(x), A_i^-(y) \} \]

which implies that \( A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).
And $A_i^-(xy) = V_i^-(f(xy)) = V_i^-(f(y)f(x))$

$$\leq \min \{ V_i^-(f(x)), V_i^-(f(y)) \} = \min \{ A_i^-(x), A_i^-(y) \}$$

which implies that $A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $A$ is a bipolar valued multi fuzzy ideal of $R$.

**5.2.24 Theorem:** Let $A = (A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^-)$ be a bipolar valued multi fuzzy ideal of a semiring $H$ and $f$ is an isomorphism from a semiring $R$ onto $H$.

Then $A \circ f = (A_1^+ f, A_2^+ f, \ldots, A_i^+ f, A_1^- f, A_2^- f, \ldots, A_i^- f)$ is a bipolar valued multi fuzzy ideal of $R$.

**Proof:** Let $x$ and $y$ in $R$ and $A \circ f$ be a bipolar valued multi fuzzy ideal of the semiring $H$.

Then $(A_i^+ f)(x+y) = A_i^+(f(x+y))$

$$= A_i^+(f(x)+f(y))$$

$$\geq \min \{ A_i^+(f(x)), A_i^+(f(y)) \} \geq \min \{ (A_i^+ f)(x), (A_i^+ f)(y) \},$$

which implies that $(A_i^+ f)(x+y) \geq \min\{(A_i^+ f)(x), (A_i^+ f)(y)\}$ for all $x$ and $y$ in $R$ and for all $i$.

And $(A_i^+ f)(xy) = A_i^+(f(xy))$

$$= A_i^+(f(x)f(y))$$

$$\geq \max \{ A_i^+(f(x)), A_i^+(f(y)) \} \geq \max \{ (A_i^+ f)(x), (A_i^+ f)(y) \},$$

which implies that $(A_i^+ f)(xy) \geq \max \{ (A_i^+ f)(x), (A_i^+ f)(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Also $(A_i^- f)(x+y) = A_i^-(f(x+y))$

$$= A_i^-(f(x)+f(y))$$

$$\leq \max \{ A_i^-(f(x)), A_i^-(f(y)) \} \leq \max \{ (A_i^- f)(x), (A_i^- f)(y) \},$$
which implies that \((A_i^{-}\circ f)(x+y) \leq \max \{ (A_i^{-}\circ f)(x), (A_i^{-}\circ f)(y) \}\) for all x and y in R and for all i.

And \((A_i^{-}\circ f)(xy) = A_i^{-}( f(xy) ) = A_i^{-}( f(x)f(y) ) \leq \min \{ (A_i^{-}\circ f)(x), (A_i^{-}\circ f)(y) \}, \)

which implies that \((A_i^{-}\circ f)(xy) \leq \min \{ (A_i^{-}\circ f)(x), (A_i^{-}\circ f)(y) \}\) for all x and y in R and for all i.

Therefore \(A\circ f\) is a bipolar valued multi fuzzy ideal of R.

5.2.25 Theorem: Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy ideal of a semiring H and f is an anti-isomorphism from a semiring R onto H. Then \(A\circ f = \langle A_1^+\circ f, A_2^+\circ f, \ldots, A_i^+\circ f, A_i^-\circ f, A_2^-\circ f, \ldots, A_i^-\circ f \rangle\) is a bipolar valued multi fuzzy ideal of R.

Proof: Let x and y in R and \(A\circ f\) be a bipolar valued multi fuzzy ideal of the semiring H.

Then \((A_i^+\circ f)(x+y) = A_i^+( f(x+y) ) = A_i^+( f(y)+f(x) ) \geq \min \{ A_i^+( f(x) ), A_i^+( f(y) ) \} \geq \min \{ (A_i^+\circ f)(x), (A_i^+\circ f)(y) \}, \)

which implies that \((A_i^+\circ f)(x+y) \geq \min \{ (A_i^+\circ f)(x), (A_i^+\circ f)(y) \}\) for all x and y in R and for all i.

And \((A_i^+\circ f)(xy) = A_i^+( f(xy) ) = A_i^+( f(y)f(x) ) \geq \max \{ A_i^+( f(x) ), A_i^+( f(y) ) \} \)
\[ \geq \max \{ (A_i^+ f)(x), (A_i^+ f)(y) \}, \]

which implies that \((A_i^+ f)(xy) \geq \max \{ (A_i^+ f)(x), (A_i^+ f)(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also \((A_i^- f)(x+y) = A_i^-(f(x+y))\)
\[
= A_i^-(f(y)+f(x)) \\
\leq \max \{ A_i^-(f(x)), A_i^-(f(y)) \} \leq \max \{ (A_i^- f)(x), (A_i^- f)(y) \},
\]

which implies that \((A_i^- f)(x+y) \leq \max \{ (A_i^- f)(x), (A_i^- f)(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \((A_i^- f)(xy) = A_i^-(f(xy))\)
\[
= A_i^-(f(y)f(x)) \\
\leq \min \{ A_i^-(f(x)), A_i^-(f(y)) \} \leq \min \{ (A_i^- f)(x), (A_i^- f)(y) \},
\]

which implies that \((A_i^- f)(xy) \leq \min \{ (A_i^- f)(x), (A_i^- f)(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Therefore \(A \circ f\) is a bipolar valued multi fuzzy ideal of \(R\).

**5.2.26 Theorem:** If \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^- A_2^- \ldots, A_i^- \rangle\) is a bipolar valued multi fuzzy ideal of a semiring \(R\), then \(\oplus A = \langle \oplus A_1^+, \oplus A_2^+, \ldots, \oplus A_i^+, \oplus A_1^-, \oplus A_2^-, \ldots, \oplus A_i^- \rangle\) is a bipolar valued multi fuzzy ideal of the semiring \(R\).

**Proof:** Let \(x\) and \(y\) in \(R\).

We have \(\oplus A_i^+(x+y) = A_i^+(x+y) + 1 - H(A_i^+)\)
\[
\geq \min \{ A_i^+(x), A_i^+(y) \} + 1 - H(A_i^+) \\
= \min \{ A_i^+(x) + 1 - H(A_i^+), A_i^+(y) + 1 - H(A_i^+) \} \\
= \min \{ \oplus A_i^+(x), \oplus A_i^+(y) \}
\]

which implies \(\oplus A_i^+(x+y) \geq \min \{ \oplus A_i^+(x), \oplus A_i^+(y) \} \) for all \(x\) and \(y\) in \(R\) and for all \(i\).
And \( A_i^+(xy) = A_i^+(xy) + 1 - H(A_i^+) \)

\[
\geq \max \{ A_i^+(x), A_i^+(y) \} + 1 - H(A_i^+)
\]

\[
= \max \{ A_i^+(x) + 1 - H(A_i^+), A_i^+(y) + 1 - H(A_i^+) \}
\]

\[
= \max \{ A_i^+(x), A_i^+(y) \}
\]

which implies \( A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( A_i^-(x+y) = A_i^-(x+y) - 1 - H(A_i^-) \)

\[
\leq \max \{ A_i^-(x), A_i^-(y) \} - 1 - H(A_i^-)
\]

\[
= \max \{ A_i^-(x) - 1 - H(A_i^-), A_i^-(y) - 1 - H(A_i^-) \}
\]

\[
= \max \{ A_i^-(x), A_i^-(y) \}
\]

which implies \( A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^-(xy) = A_i^-(xy) - 1 - H(A_i^-) \)

\[
\leq \min \{ A_i^-(x), A_i^-(y) \} - 1 - H(A_i^-)
\]

\[
= \min \{ A_i^-(x) - 1 - H(A_i^-), A_i^-(y) - 1 - H(A_i^-) \}
\]

\[
= \min \{ A_i^-(x), A_i^-(y) \}
\]

which implies \( A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( A \) is a bipolar valued multi fuzzy ideal of \( R \).

**5.2.27 Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). Then

(i) \( H(A_i^+) = 1 \) if and only if \( A_i^+(x) = A_i^+(x) \) for all \( x \) in \( R \) and for all \( i \).

(ii) \( H(A_i^-) = -1 \) if and only if \( A_i^-(x) = A_i^-(x) \) for all \( x \) in \( R \) and for all \( i \).

(iii) \( A_i^+(x) = 1 \) if and only if \( H(A_i^+) = A_i^+(x) \) for all \( x \) in \( R \) and for all \( i \).

(iv) \( A_i^-(x) = -1 \) if and only if \( H(A_i^-) = A_i^-(x) \) for all \( x \) in \( R \) and for all \( i \).

(v) \( (A) = A \).
Proof: It is trivial.

5.2.28 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of a semiring \( R \), then \( 0A = \langle 0A_1^+, 0A_2^+, \ldots, 0A_i^+, 0A_1^-, 0A_2^-, \ldots, 0A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of the semiring \( R \).

Proof: For any \( x \) in \( R \), we have

\[
0A_i^+(x+y) = A_i^+(x+y)H(A_i^+)
\]

\[
\geq \min \{ A_i^+(x), A_i^+(y) \} H(A_i^+)
\]

\[
= \min \{ A_i^+(x)H(A_i^+), A_i^+(y)H(A_i^+) \}
\]

\[
= \min \{ 0A_i^+(x), 0A_i^+(y) \}
\]

which implies that \( 0A_i^+(x+y) \geq \min \{ 0A_i^+(x), 0A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( 0A_i^+(xy) = A_i^+(xy)H(A_i^+) \)

\[
\geq \max \{ A_i^+(x), A_i^+(y) \} H(A_i^+)
\]

\[
= \max \{ A_i^+(x)H(A_i^+), A_i^+(y)H(A_i^+) \}
\]

\[
= \max \{ 0A_i^+(x), 0A_i^+(y) \}
\]

which implies that \( 0A_i^+(xy) \geq \max \{ 0A_i^+(x), 0A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( 0A_i^-(x+y) = -A_i^-(x+y)H(A_i^-) \)

\[
\leq (-)\max \{ A_i^-(x), A_i^-(y) \} H(A_i^-)
\]

\[
= \max \{ -A_i^-(x)H(A_i^-), -A_i^-(y)H(A_i^-) \}
\]

\[
= \max \{ 0A_i^-(x), 0A_i^-(y) \}
\]

which implies that \( 0A_i^-(x+y) \leq \max \{ 0A_i^-(x), 0A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( 0A_i^-(xy) = -A_i^-(xy)H(A_i^-) \)

\[
\leq (-)\min \{ A_i^-(x), A_i^-(y) \} H(A_i^-)
\]
\[ \text{which implies that } 0^i A_i \subseteq \min \{ 0^i A_i(x), 0^i A_i(y) \} \text{ for all x and y in } R \text{ and for all i.} \]

Hence \(0^i A\) is a bipolar valued multi fuzzy ideal of the semiring \(R\).

5.2.29 Theorem: If \(A = (A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^-)\) is a bipolar valued multi fuzzy ideal of a semiring \(R\), then \(A = (A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^-)\) is a bipolar valued multi fuzzy ideal of \(R\).

\textbf{Proof}: For any \(x\) in \(R\),

we have \(A_i^+(x+y) = A_i^+(x+y) / H(A_i^+)\)

\[ \geq \min \{ A_i^+(x), A_i^+(y) \} / H(A_i^+) \]

\[ = \min \{ A_i^+(x) / H(A_i^+), A_i^+(y) / H(A_i^+) \} \]

\[ = \min \{ \lambda A_i^+(x), \lambda A_i^+(y) \} \]

which implies that \(\lambda A_i^+(x+y) \geq \min \{ \lambda A_i^+(x), \lambda A_i^+(y) \} \text{ for all x and y in } R \text{ and for all i.} \)

And \(\lambda A_i^+(xy) = A_i^+(xy) / H(A_i^+)\)

\[ \geq \max \{ A_i^+(x), A_i^+(y) \} / H(A_i^+) \]

\[ = \max \{ A_i^+(x) / H(A_i^+), A_i^+(y) / H(A_i^+) \} \]

\[ = \max \{ \lambda A_i^+(x), \lambda A_i^+(y) \} \]

which implies that \(\lambda A_i^+(xy) \geq \max \{ \lambda A_i^+(x), \lambda A_i^+(y) \} \text{ for all x and y in } R \text{ and for all i.} \)

Also \(\lambda A_i^-(x+y) = -A_i^-(x+y) / H(A_i^-)\)

\[ \leq (-) \max \{ A_i^-(x), A_i^-(y) \} / H(A_i^-) \]

\[ = \max \{ -A_i^-(x) / H(A_i^-), -A_i^-(y) / H(A_i^-) \} \]

\[ = \max \{ \lambda A_i^-(x), \lambda A_i^-(y) \} \]
which implies that $\tilde{A}_i(x+y) \leq \max \{ \tilde{A}_i(x), \tilde{A}_i(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

And $\tilde{A}_i(xy) = \frac{\tilde{A}_i(x) / H(A_i)}{\tilde{A}_i(y) / H(A_i)}$

which implies that $\tilde{A}_i(xy) \leq \min \{ \tilde{A}_i(x), \tilde{A}_i(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $\tilde{A}$ is a bipolar valued multi fuzzy ideal of the semiring $R$.

**5.2.30 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued normal multi fuzzy ideal of a semiring $R$, then $^0A = A$.

**Proof:** It is trivial.

**5.2.31 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued normal multi fuzzy ideal of a semiring $R$, then $\tilde{A} = A$.

**Proof:** It is trivial.

**5.2.32 Theorem:** Let $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ be a bipolar valued multi fuzzy ideal of a semiring $R$. (i) If $H(A_i^+) < 1$, then $^0A_i^+ < A_i^+$ for all $i$. (ii) $H(A_i^-) > -1$, then $^0A_i^- > A_i^-$ for all $i$. (iii) $H(A_i^+) < 1$ and $H(A_i^-) > -1$, then $^0A < A$ for all $i$.

**Proof:** It is trivial.

**5.2.33 Theorem:** Let $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ be a bipolar valued multi fuzzy ideal of a semiring $R$. (i) If $H(A_i^+) < 1$, then $\tilde{A}_i^+ > A_i^+$ for all $i$. (ii) $H(A_i^-) > -1$, then $\tilde{A}_i^- < A_i^-$ for all $i$. (iii) $H(A_i^+) < 1$ and $H(A_i^-) > -1$, then $\tilde{A} > A$ for all $i$. (iii) $H(A_i^+) < 1$ and $H(A_i^-) > -1$ for all $i$, then $\tilde{A}$ is a bipolar valued normal multi fuzzy ideal of $R$. 
**Proof:** It is trivial.

### 5.3 - PROPERTIES OF BIPOLAR VALUED MULTI FUZZY NORMAL IDEALS:

#### 5.3.1 Theorem:

Let \((R, +, \cdot)\) be a semiring. If \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) and \(B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle\) are two bipolar valued multi fuzzy normal ideals of \(R\), then \(A \cap B\) is a bipolar valued multi fuzzy normal ideal of \(R\).

**Proof:**

Let \(x\) and \(y\) in \(R\). Let \(C = A \cap B\) and \(C = \langle C_1^+, C_2^+, \ldots, C_i^+, C_1^-, C_2^-, \ldots, C_i^- \rangle\). By Theorem 5.2.3, \(C\) is a bipolar valued multi fuzzy ideal of \(R\), since \(A\) and \(B\) are two bipolar valued multi fuzzy ideals of \(R\).

Now \(C_i^+(x+y) = \min \{ A_i^+(x+y), B_i^+(x+y) \} \)

\[= \min \{ A_i^+(y+x), B_i^+(y+x) \} \]

\[= C_i^+(y+x) \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.\]

Therefore \(C_i^+(x+y) = C_i^+(y+x)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \(C_i^+(xy) = \min \{ A_i^+(xy), B_i^+(xy) \} \)

\[= \min \{ A_i^+(yx), B_i^+(yx) \} \]

\[= C_i^+(yx) \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.\]

Therefore \(C_i^+(xy) = C_i^+(yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).
Also \( C_i(y+x) = \max \{ A_i(y+x), B_i(y+x) \} \)

\[ = \max \{ A_i(y+x), B_i(y+x) \} \]

\[ = C_i(y+x) \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i. \]

Therefore \( C_i(y+x) = C_i(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( C_i(xy) = \max \{ A_i(xy), B_i(xy) \} \)

\[ = \max \{ A_i(xy), B_i(xy) \} \]

\[ = C_i(xy) \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i. \]

Therefore \( C_i(xy) = C_i(xy) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( A \bowtie B \) is a bipolar valued multi fuzzy normal ideal of \( R \).

5.3.2 Theorem: Let \((R, +, \cdot)\) be a semiring. The intersection of a family of bipolar valued multi fuzzy normal ideals of \( R \) is a bipolar valued multi fuzzy normal ideal of \( R \).

Proof: It is trivial.

5.3.3 Theorem: Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^-, \ldots, A_i^- \rangle \) and \( B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_i^-, B_2^-, \ldots, B_i^- \rangle \) be bipolar valued multi fuzzy ideals of the semirings \( R \) and \( H \), respectively. If \( A \) and \( B \) are bipolar valued multi fuzzy normal ideals, then \( A \times B \) is a bipolar valued multi fuzzy normal ideal of \( R \times H \).

Proof: By Theorem 5.2.5, \( A \times B \) is a bipolar valued multi fuzzy ideal of \( R \times H \). Let \( x_1 \) and \( x_2 \) be in \( R \), \( y_1 \) and \( y_2 \) be in \( H \).

Then \((x_1, y_1)\) and \((x_2, y_2)\) are in \( R \times H \).
Now \((A_i \times B_i)^+[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^+ (x_1 + x_2, y_1 + y_2)\)
\[= \min \{ A_i^+ (x_1 + x_2), B_i^+ (y_1 + y_2) \} \]
\[= \min \{ A_i^+ (x_2 + x_1), B_i^+ (y_2 + y_1) \} \]
\[= (A_i \times B_i)^+(x_2 + x_1, y_2 + y_1) \]
\[= (A_i \times B_i)^+[(x_2, y_2) + (x_1, y_1)] \text{ for all } i. \]

Therefore \((A_i \times B_i)^+[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^+ (x_2, y_2) + (x_1, y_1) \) for all \((x_1, y_1)\) and \((x_2, y_2)\) in \(R \times H\) and for all \(i\).

And \((A_i \times B_i)^+[(x_1, y_1)(x_2, y_2)] = (A_i \times B_i)^+(x_1 x_2, y_1 y_2)\)
\[= \min \{ A_i^+ (x_1 x_2), B_i^+ (y_1 y_2) \} \]
\[= \min \{ A_i^+ (x_2 x_1), B_i^+ (y_2 y_1) \} \]
\[= (A_i \times B_i)^+(x_2 x_1, y_2 y_1) \]
\[= (A_i \times B_i)^+[(x_2, y_2)(x_1, y_1)] \text{ for all } i. \]

Therefore \((A_i \times B_i)^+[(x_1, y_1)(x_2, y_2)] = (A_i \times B_i)^+ (x_2, y_2)(x_1, y_1) \) for all \((x_1, y_1)\) and \((x_2, y_2)\) in \(R \times H\) and for all \(i\).

Also \((A_i \times B_i)^-[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^-(x_1 + x_2, y_1 + y_2)\)
\[= \max \{ A_i^-(x_1 + x_2), B_i^-(y_1 + y_2) \} \]
\[= \max \{ A_i^-(x_2 + x_1), B_i^-(y_2 + y_1) \} \]
\[= (A_i \times B_i)^-(x_2 + x_1, y_2 + y_1) \]
\[= (A_i \times B_i)^+[(x_2, y_2) + (x_1, y_1)] \text{ for all } i. \]

Therefore \((A_i \times B_i)^-[(x_1, y_1) + (x_2, y_2)] = (A_i \times B_i)^- (x_2, y_2) + (x_1, y_1) \) for all \((x_1, y_1)\) and \((x_2, y_2)\) in \(R \times H\) and for all \(i\).

And \((A_i \times B_i)^-[(x_1, y_1)(x_2, y_2)] = (A_i \times B_i)^-(x_1 x_2, y_1 y_2)\)
\[= \max \{ A_i^-(x_1 x_2), B_i^-(y_1 y_2) \} \]
\[= \max \{ A_i^-(x_2 x_1), B_i^-(y_2 y_1) \} \]
\[ = (A_i \times B_i)^\tau (x_{2i}, y_{1i}) \]
\[ = (A_i \times B_i)^\tau [ (x_2, y_2)(x_1, y_1) ] \text{ for all } i. \]

Therefore \((A_i \times B_i)^\tau [ (x_1, y_1)(x_2, y_2) ] = (A_i \times B_i)^\tau [ (x_2, y_2)(x_1, y_1) ] \) for all \((x_1, y_1)\) and \((x_2, y_2)\) in \(R \times H\) and for all \(i\).

Hence \(A \times B\) is a bipolar valued multi fuzzy normal ideal of \(R \times H\).

**5.3.4 Theorem:** Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy subset in a semiring \(R\) and \(V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle\) be the strongest bipolar valued multi fuzzy relation on \(R\). If \(A\) is a bipolar valued multi fuzzy normal ideal of \(R\), then \(V\) is a bipolar valued multi fuzzy normal ideal of \(R \times R\).

**Proof:** Suppose that \(A\) is a bipolar valued multi fuzzy normal ideal of \(R\). Then for any \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\) are in \(R \times R\). By Theorem 5.2.6, \(V\) is a bipolar valued multi fuzzy ideal of \(R\).

We have \(V_i^+(x+y) = V_i^+[(x_1, x_2) + (y_1, y_2)]\)
\[ = V_i^+(x_1+y_1, x_2+y_2) \]
\[ = \min \{ A_i^+(x_1+y_1), A_i^+(x_2+y_2) \} \]
\[ = \min \{ A_i^+(y_1+x_1), A_i^+(y_2+x_2) \} \]
\[ = V_i^+(y_1+x_1, y_2+x_2) \]
\[ = V_i^+[(y_1, y_2) + (x_1, x_2)] = V_i^+(y+x) \text{ for all } i. \]

Therefore \(V_i^+(x+y) = V_i^+(y+x)\) for all \(x\) and \(y\) in \(R \times R\) and for all \(i\).

And \(V_i^+(xy) = V_i^+[(x_1, x_2)(y_1, y_2)]\)
\[ V_i^+(x_1 y_1, x_2 y_2) = V_i^+(y_1 x_1, y_2 x_2) \]

\[ V_i^+[(x_1, y_2)(x_1, x_2)] = V_i^+(x_1 y_1, x_2 y_2) \]

Therefore \( V_i^+(xy) = V_i^+(yx) \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

Also \( V_i^-[(x_1, y_2) + (y_1, x_2)] \)

\[ V_i^-(x_1 + y_1, x_2 + y_2) = V_i^-[(x_1, x_2) + (y_1, y_2)] \]

= \[ V_i^-[(y_1 + x_1, y_2 + x_2)] = V_i^-[(y_1, y_2) + (x_1, x_2)] = V_i^-[(y_1, x_2) + (x_1, y_2)] \]

Therefore \( V_i^-[(x_1, x_2) + (y_1, y_2)] = V_i^-[(y_1, x_2) + (x_1, y_2)] \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).

And \( V_i^-(xy) = V_i^-[(x_1, x_2)(y_1, y_2)] \)

\[ V_i^-[(x_1 y_1, x_2 y_2)] = V_i^-[(y_1 x_1, y_2 x_2)] \]

Therefore \( V_i^-(xy) = V_i^-[(x_1, x_2)(y_1, y_2)] = V_i^-[(y_1 x_1, y_2 x_2)] \) for all \( x \) and \( y \) in \( \mathbb{R} \times \mathbb{R} \) and for all \( i \).
Hence $V$ is a bipolar valued multi fuzzy normal ideal of $R \times R$.

**5.3.5 Theorem:** Let $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ be a bipolar valued multi fuzzy normal ideal of a semiring $R$. Then the pseudo bipolar valued multi fuzzy coset $(aA)^p$ is a bipolar valued multi fuzzy normal ideal of $R$, for every $a$ in $R$ and $p$ in $P$.

**Proof:** By Theorem 5.2.7, $(aA)^p$ is a bipolar valued multi fuzzy ideal of $R$. For every $x$ and $y$ in $R$,

we have $( (aA_i^+)^p )(x+y) = p(a)A_i^+(x+y) = p(a)A_i^+(y+x) = ( (aA_i^+)^p )(y+x)$ for all $i$.

Therefore $( (aA_i^+)^p )(x+y) = ( (aA_i^+)^p )(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.

And $( (aA_i^+)^p )(xy) = p(a) A_i^+(xy) = p(a) A_i^+(yx) = ((aA_i^+)^p)(yx)$ for all $i$.

Therefore $( (aA_i^+)^p )(xy) = ( (aA_i^+)^p )(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Also $( (aA_i^-)^p )(x+y) = p(a)A_i^-(x+y) = p(a)A_i^-(y+x) = ((aA_i^-)^p)(y+x)$ for all $i$. Therefore

$( (aA_i^-)^p )(x+y) = ( (aA_i^-)^p )(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.

And $( (aA_i^-)^p )(xy) = p(a) A_i^-(xy) = p(a) A_i^-(yx) = ( (aA_i^-)^p )(yx)$ for all $i$.

Therefore $( (aA_i^-)^p )(xy) = ( (aA_i^-)^p )(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $(aA)^p$ is a bipolar valued multi fuzzy normal ideal of $R$.

**5.3.6 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy normal ideal of a semiring $R$, then $? (A)$ is a bipolar valued multi fuzzy normal ideal of $R$.
**Proof:** By Theorem 5.2.8, \(?(A)\) is a bipolar valued multi fuzzy ideal of \(R\). For every \(x\) and \(y\) in \(R\),

we have \(?(A_i^+)(x+y) = \min\{ \frac{1}{2}, A_i^+(x+y) \} \)

\[= \min\{ \frac{1}{2}, A_i^+(y+x) \} = ?(A_i^+)(y+x) \text{ for all } i.\]

Therefore \(?(A_i^+)(x+y) = ?(A_i^+)(y+x) \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \(?(A_i^+)(xy) = \min\{ \frac{1}{2}, A_i^+(xy) \} \)

\[= \min\{ \frac{1}{2}, A_i^+(yx) \} = ?(A_i^+)(yx) \text{ for all } i.\]

Therefore \(?A_i^+(xy) = ?A_i^+(yx) \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also \(?(A_i^-)(x+y) = \max\{ -\frac{1}{2}, A_i^- (x+y) \} \)

\[= \max\{ -\frac{1}{2}, A_i^- (y+x) \} = ?(A_i^-)(y+x) \text{ for all } i.\]

Therefore \(?(A_i^-)(x+y) = ?(A_i^-)(y+x) \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \(?(A_i^-)(xy) = \max\{ -\frac{1}{2}, A_i^- (xy) \} \)

\[= \max\{ -\frac{1}{2}, A_i^- (yx) \} = ?(A_i^-)(yx) \text{ for all } i.\]

Therefore \(?(A_i^-)(xy) = ?(A_i^-)(yx) \) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Hence \(?(A)\) is a bipolar valued multi fuzzy normal ideal of \(R\).

**5.3.7 Theorem:** If \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_i^-, A_2^-, \ldots, A_i^- \rangle\) is a bipolar valued multi fuzzy normal ideal of a semiring \(R\), then \(!(A)\) is a bipolar valued multi fuzzy normal ideal of \(R\).
**Proof:** Let \( x \) and \( y \) in \( R \). By Theorem 5.2.9, \(! (A)\) is a bipolar valued multi fuzzy ideal of \( R \).

We have \( ! (A_i^+) (x+y) = \max \{ \frac{1}{2}, A_i^+ (x+y) \} \)

\[ = \max \{ \frac{1}{2}, A_i^+ (y+x) \} = ! (A_i^+) (y+x) \text{ for all } i. \]

Therefore \( ! (A_i^+) (x+y) = ! (A_i^+) (y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( ! (A_i^+) (xy) = \max \{ \frac{1}{2}, A_i^+ (xy) \} \)

\[ = \max \{ \frac{1}{2}, A_i^+ (yx) \} = ! (A_i^+) (yx) \text{ for all } i. \]

Therefore \( ! (A_i^+) (xy) = ! (A_i^+) (yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( ! (A_i^-) (x+y) = \min \{ -\frac{1}{2}, A_i^- (x+y) \} \)

\[ = \min \{ -\frac{1}{2}, A_i^- (y+x) \} = ! (A_i^-) (y+x) \text{ for all } i. \]

Therefore \( ! (A_i^-) (x+y) = ! (A_i^-) (y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( ! (A_i^-) (xy) = \min \{ -\frac{1}{2}, A_i^- (xy) \} \)

\[ = \min \{ -\frac{1}{2}, A_i^- (yx) \} = ! (A_i^-) (yx) \text{ for all } i. \]

Therefore \( ! (A_i^-) (xy) = ! (A_i^-) (yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( ! (A) \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**5.3.8 Theorem:** If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) and \( B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle \) are bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( ! (A \cap B) = ! (A) \cap ! (B) \) is also a bipolar valued multi fuzzy normal ideal of \( R \).
Proof: It is trivial.

5.3.9 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle$ are bipolar valued multi fuzzy normal ideal of a semiring $R$, then $?(A \cap B) = ?(A) \cap ?(B)$ is also a bipolar valued multi fuzzy normal ideal of $R$.

Proof: It is trivial.

5.3.10 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy normal ideal of a semiring $R$, then $\!(?((A)) = ?(\!(A))$ is also a bipolar valued multi fuzzy normal ideal of $R$.

Proof: It is trivial.

5.3.11 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy normal ideal of a semiring $R$, then $Q_{\langle a_1, a_2, \ldots, a_i \lambda(b_1, b_2, \ldots, b_i) \rangle}(A)$ is a bipolar valued multi fuzzy normal ideal of $R$.

Proof: By Theorem 5.2.10, $Q_{\langle a_1, a_2, \ldots, a_i \lambda(b_1, b_2, \ldots, b_i) \rangle}(A)$ is a bipolar valued multi fuzzy ideal of $R$.

For every $x$ and $y$ in $R$, for $\alpha$ in $[0, 1]$ and $\beta$ in $[-1, 0]$,

we have $Q_{\langle a_1, a_2, \ldots, a_i \lambda(b_1, b_2, \ldots, b_i) \rangle}(A_i^+) (x+y) = \min \{ \alpha_i, A_i^+ (x+y) \}$

$= \min \{ \alpha_i, A_i^+ (y+x) \}$

$= Q_{\langle a_1, a_2, \ldots, a_i \lambda(b_1, b_2, \ldots, b_i) \rangle}(A_i^+) (y+x)$ for all $i$. 
Therefore \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(x+y) = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(y+x) \) for all \( x, y \) in \( R \) and for all \( i \).

And \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(xy) = \min \{ \alpha_i, A_i^+(xy) \} \)

\[ = \min \{ \alpha_i, A_i^+(yx) \} \]

\[ = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(yx) \) for all \( i \).

Therefore \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(xy) = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(yx) \) for all \( x, y \) in \( R \) and for all \( i \).

Also \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^+)(x+y) = \max \{ \beta_i, A_i^+(x+y) \} \)

\[ = \max \{ \beta_i, A_i^+(y+x) \} \]

\[ = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(y+x) \) for all \( i \).

Therefore \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(x+y) = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(xy) = \max \{ \beta_i, A_i^-(xy) \} \)

\[ = \max \{ \beta_i, A_i^-(yx) \} \]

\[ = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(yx) \) for all \( i \).

Therefore \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(xy) = Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A_i^-)(yx) \) for all \( x, y \) in \( R \) and for all \( i \).

Hence \( Q_{((a_1, a_2, \ldots, a_i), (\beta_1, \beta_2, \ldots, \beta_i))} (A) \) is a bipolar valued multi fuzzy normal ideal of \( R \).
5.3.12 Theorem: If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy normal ideal of a semiring $R$, then $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A)$ is a bipolar valued multi fuzzy normal ideal of $R$.

Proof: Let $x$ and $y$ in $R$ and $\alpha$ in $[0, 1]$, $\beta$ in $[-1, 0]$.

By Theorem 5.2.11, $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A)$ is a bipolar valued multi fuzzy ideal of $R$.

We have $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(x+y) = \max \{ \alpha_i, A_i^+(x+y) \}$

$$= \max \{ \alpha_i, A_i^+(y+x) \}$$

$$= P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(y+x) \text{ for all } i.$$ 

Therefore $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(x+y) = P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.

And $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(xy) = \max \{ \alpha_i, A_i^+(xy) \}$

$$= \max \{ \alpha_i, A_i^+(yx) \}$$

$$= P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(yx) \text{ for all } i.$$ 

Therefore $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(xy) = P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^+)(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Also $P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(x+y) = \min \{ \beta_i, A_i^-(x+y) \}$

$$= \min \{ \beta_i, A_i^-(y+x) \}$$

$$= P_{((\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(y+x) \text{ for all } i.$$
Therefore \( P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(x+y) = P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \). 

And \( P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(xy) = \min \{ \beta_i, A_i^-(xy) \} \)

\[
= \min \{ \beta_i, A_i^-(yx) \}
\]

\[
= P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(yx) \text{ for all } i.
\]

Therefore \( P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(xy) = P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A_i^-)(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A) \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**5.3.13 Theorem:** If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) and \( B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle \) are bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A \cap B) = P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A) \cap P_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(B) \)

\( (B) \) is also a bipolar valued multi fuzzy normal ideal of \( R \).

**Proof:** It is trivial.

**5.3.14 Theorem:** If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) and \( B = \langle B_1^+, B_2^+, \ldots, B_i^+, B_1^-, B_2^-, \ldots, B_i^- \rangle \) are bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( Q_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A \cap B) = Q_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(A) \cap Q_{((a_1, a_2, \ldots, a_i, \beta_1, \beta_2, \ldots, \beta_i))}(B) \)

\( (B) \) is also a bipolar valued multi fuzzy normal ideal of \( R \).

**Proof:** It is trivial.
5.3.15 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of a semiring \( R \), then

\[
P_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A)) = Q_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} \left( P_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \right)
\]

is also a bipolar valued multi fuzzy normal ideal of \( R \).

**Proof:** It is trivial.

5.3.16 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**Proof:** Let \( x \) and \( y \) in \( R \) and \( \alpha \) in \([0, 1]\), \( \beta \) in \([-1, 0]\).

By Theorem 5.2.12, \( G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A) \) is a bipolar valued multi fuzzy ideal of \( R \).

We have

\[
G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (x+y) = \alpha_i A_i^+ (x+y)
\]

\[
= \alpha_i A_i^+ (y+x)
\]

\[
= G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (y+x) \text{ for all } i.
\]

Therefore

\[
G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (x+y) = G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (y+x) \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i.
\]

And

\[
G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (xy) = \alpha_i A_i^+ (xy)
\]

\[
= \alpha_i A_i^+ (yx)
\]

\[
= G_{\langle (\alpha_1, \alpha_2, \ldots, \alpha_i), (\beta_1, \beta_2, \ldots, \beta_i) \rangle} (A_i^+) (yx) \text{ for all } i.
\]
Therefore $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^+)(xy) = G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^+)(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Also $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(x+y) = -\beta_i A_i^-(x+y)$

$= -\beta_i A_i^-(y+x)$

$= G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(y+x)$ for all $i$.

Therefore $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(x+y) = G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.

And $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(xy) = -\beta_i A_i^-(xy)$

$= -\beta_i A_i^-(yx)$

$= G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(yx)$ for all $i$.

Therefore $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(xy) = G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A_i^-)(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A)$ is a bipolar valued multi fuzzy normal ideal of $R$.

5.3.17 Theorem: If $A = \langle A_1^+, A_2^+, ..., A_i^+, A_1^-, A_2^-, ..., A_i^- \rangle$ and $B = \langle B_1^+, B_2^+, ..., B_i^+, B_1^-, B_2^-, ..., B_i^- \rangle$ are bipolar valued multi fuzzy normal ideal of a semiring $R$, then $G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A \cap B) = G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(A) \cap G_{((a_1, a_2, ..., a_i)(\beta_1, \beta_2, ..., \beta_i))}(B)$ is also a bipolar valued multi fuzzy normal ideal of $R$.

Proof: It is trivial.
**5.3.18 Theorem:** Let \((R, +, \cdot)\) and \((R^1, +, \cdot)\) be any two semirings. The homomorphic image of a bipolar valued multi fuzzy normal ideal of \(R\) is a bipolar valued multi fuzzy normal ideal of \(R^1\).

**Proof:** Let \(V = f(A)\), where \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) is a bipolar valued multi fuzzy normal ideal of \(R\). We have to prove that \(V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle\) is a bipolar valued multi fuzzy normal ideal of \(R^1\). By Theorem 5.2.20, \(V\) is a bipolar valued multi fuzzy ideal of \(R^1\), since \(A\) is a bipolar valued multi fuzzy ideal of \(R\). For \(f(x), f(y)\) in \(R^1\), we have \(V_i^+( f(x)+f(y) ) = V_i^+( f(x+y) ) \)
\[
\geq A_i^+(x+y) 
\]
\[
= A_i^+(y+x) \leq V_i^+( f(y+x) ) = V_i^+( f(y)+f(x) ) 
\]
which implies that \(V_i^+( f(x)+f(y) ) = V_i^+(f(y)+f(x))\) for all \(f(x)\) and \(f(y)\) in \(R^1\) and for all \(i\).

And \(V_i^+( f(x)f(y) ) = V_i^+( f(xy) ) \)
\[
\geq A_i^+(xy) 
\]
\[
= A_i^+(yx) \leq V_i^+( f(yx) ) = V_i^+( f(y)f(x) ) 
\]
which implies that \(V_i^+( f(x)f(y) ) = V_i^+(f(y)f(x))\) for all \(f(x)\) and \(f(y)\) in \(R^1\) and for all \(i\).

Also \(V_i^-( f(x)+f(y) ) = V_i^- f(x+y) \)
\[
\leq A_i^- (x+y) 
\]
\[
= A_i^- (y+x) \geq V_i^- ( f(y+x) ) = V_i^- ( f(y)+f(x) ) 
\]
which implies that \(V_i^- ( f(x)+f(y) ) = V_i^- (f(y)+f(x))\) for all \(f(x)\) and \(f(y)\) in \(R^1\) and for all \(i\).

And \(V_i^- ( f(x)f(y) ) = V_i^- f(xy) \)
\[
\leq A_i^- (xy) 
\]
\[ = A_i^+(yx) \geq V_i^-( f(yx) ) = V_i^-( f(y)f(x) ) \]
which implies that \( V_i^- ( f(x)f(y) ) = V_i^- ( f(y)f(x) ) \) for all \( f(x) \) and \( f(y) \) in \( R^1 \) and for all \( i \).
Hence \( V \) is a bipolar valued multi fuzzy normal ideal of \( R^1 \).

**5.3.19 Theorem:** Let \( (R, +, .) \) and \( (R^1, +, .) \) be any two semirings. The homomorphic pre-image of a bipolar valued multi fuzzy normal ideal of \( R^1 \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**Proof:** Let \( V = f(A) \), where \( V = \langle V_1^+, V_2^+, \ldots, V_1^-, V_2^-, \ldots, V_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R^1 \). We have to prove that \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R \). By Theorem 5.2.21, \( A \) is a bipolar valued multi fuzzy ideal of \( R \), since \( V \) is a bipolar valued multi fuzzy ideal of \( R^1 \). Let \( x \) and \( y \) in \( R \).

Now \( A_i^+(x+y) = V_i^+( f(x+y) ) \)
\[ = V_i^+( f(x)+f(y) ) \]
\[ = V_i^+( f(y)+f(x) ) = V_i^+( f(y+x) ) = A_i^+(y+x) \]
which implies that \( A_i^+(x+y) = A_i^+(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^+(xy) = V_i^+( f(xy) ) \)
\[ = V_i^+( f(x)f(y) ) \]
\[ = V_i^+( f(y)f(x) ) = V_i^+( f(yx) ) = A_i^+(yx) \]
which implies that \( A_i^+(xy) = A_i^+(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( A_i^-(x+y) = V_i^- ( f(x+y) ) \)
\[ = V_i^- ( f(x)+f(y) ) \]
\[ = V_i^- ( f(y)+f(x) ) = V_i^- ( f(y+x) ) = A_i^-(y+x) \]
which implies that \( A_i^-(x+y) = A_i^-(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^-(xy) = V_i^- ( f(xy) ) \)
\[ = V_i^-(f(x)f(y)) \]
\[ = V_i^-(f(y)f(x)) = V_i^-(f(yx)) = A_i^-(yx) \]

which implies that \( A_i^-(xy) = A_i^-(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( A \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**5.3.20 Theorem:** Let \( (R, +, \cdot) \) and \( (R^1, +, \cdot) \) be any two semirings. The anti-homomorphic image of a bipolar valued multi fuzzy normal ideal of \( R \) is a bipolar valued multi fuzzy normal ideal of \( R^1 \).

**Proof:** Let \( V = f(A) \), where \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R \). We have to prove that \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_i^-, V_2^-, \ldots, V_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R^1 \). By Theorem 5.2.22, \( V \) is a bipolar valued multi fuzzy ideal of \( R^1 \), since \( A \) is a bipolar valued multi fuzzy ideal of \( R \). Let \( f(x) \) and \( f(y) \) in \( R^1 \).

Now \( V_i^+(f(x)+f(y)) = V_i^+(f(y)+f(x)) \)
\[ \geq A_i^+(y+x) \]
\[ = A_i^+(x+y) \leq V_i^+(f(x)+f(y)) = V_i^+(f(y)+f(x)) \]
which implies that \( V_i^+(f(x)+f(y)) = V_i^+(f(y)+f(x)) \) for all \( f(x) \) and \( f(y) \) in \( R^1 \) and for all \( i \).

And \( V_i^+(f(x)f(y)) = V_i^+(f(y)f(x)) \)
\[ \geq A_i^+(xy) \]
\[ = A_i^+(xy) \leq V_i^+(f(xy)) = V_i^+(f(y)f(x)) \]
which implies that \( V_i^+(f(x)f(y)) = V_i^+(f(y)f(x)) \) for all \( f(x) \) and \( f(y) \) in \( R^1 \) and for all \( i \).

Also \( V_i^-(f(x)+f(y)) = V_i^-(f(y)+f(x)) \)
\[ \leq A_i^-(y+x) \]
\[ = A_i^-(x+y) \geq V_i^-(f(x)+f(y)) = V_i^-(f(y)+f(x)) \]
which implies that \( V_i^- (f(x)+f(y)) = V_i^- (f(y)+f(x)) \) for all \( f(x) \) and \( f(y) \) in \( R^i \) and for all \( i \).

And \( V_i^- (f(x)f(y)) = V_i^- (f(y)f(x)) \leq A_i^-(yx) = A_i^-(xy) \geq V_i^- (f(xy)) = V_i^- (f(y)f(x)) \)

which implies that \( V_i^- (f(x)f(y)) = V_i^- (f(y)f(x)) \) for all \( f(x) \) and \( f(y) \) in \( R^i \) and for all \( i \).

Hence \( V \) is a bipolar valued multi fuzzy normal ideal of \( R^i \).

5.3.21 Theorem: Let \( (R, +, \cdot) \) and \( (R^i, +, \cdot) \) be any two semirings. The anti-homomorphically pre-image of a bipolar valued multi fuzzy normal ideal of \( R^i \) is a bipolar valued multi fuzzy normal ideal of \( R \).

Proof: Let \( V = f(A) \), where \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R^i \). We have to prove that \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of \( R \). By Theorem 5.2.23, \( A \) is a bipolar valued multi fuzzy ideal of \( R \), since \( V \) is a bipolar valued multi fuzzy ideal of \( R^i \). Let \( x \) and \( y \) in \( R \).

Now \( A_i^-(x+y) = V_i^- (f(x+y)) \)
\[ = V_i^+ (f(y)+f(x)) \]
\[ = V_i^+ (f(x)+f(y)) = V_i^- (f(y+x)) = A_i^+(y+x) \]
which implies that \( A_i^-(x+y) = A_i^+(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^+(xy) = V_i^+ (f(xy)) \)
\[ = V_i^+ (f(y)f(x)) \]
\[ = V_i^+ (f(x)f(y)) = V_i^+ (f(y)x) = A_i^+(yx) \]
which implies that \( A_i^+(xy) = A_i^+(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( A_i^-(x+y) = V_i^- (f(x+y)) \)
\[ = V_i(-f(y)+f(x)) \]
\[ = V_i(-f(x)+f(y)) = V_i(-f(y+x)) = A_i^{-}(y+x) \]

which implies that \( A_i^{-}(x+y) = A_i^{-}(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^{-}(xy) = V_i(-f(xy)) \)
\[ = V_i(-f(y)f(x)) \]
\[ = V_i(-f(x)f(y)) = V_i(-f(yx)) = A_i^{-}(yx) \]

which implies that \( A_i^{-}(xy) = A_i^{-}(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Hence \( A \) is a bipolar valued multi fuzzy normal ideal of \( R \).

**5.3.22 Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( H \) and \( f \) is an isomorphism from a semiring \( R \) onto \( H \). If \( A \) is a bipolar valued multi fuzzy normal ideal of the semiring \( H \), then \( A \circ f = \langle A_1^+ \circ f, A_2^+ \circ f, \ldots, A_i^+ \circ f, A_1^- \circ f, A_2^- \circ f, \ldots, A_i^- \circ f \rangle \) is a bipolar valued multi fuzzy normal ideal of the semiring \( R \).

**Proof:** Let \( A \) be a bipolar valued multi fuzzy normal ideal of a semiring \( H \). By Theorem 5.2.24, \( A \circ f \) is a bipolar valued multi fuzzy ideal of the semiring \( R \). Let \( x \) and \( y \) in \( R \).

Now \( (A_i^+ \circ f)(x+y) = A_i^+(f(x+y)) \)
\[ = A_i^+(f(x)+f(y)) \]
\[ = A_i^+(f(y)+f(x)) = A_i^+(f(y+x)) = (A_i^+ \circ f)(y+x) \]

which implies that \( (A_i^+ \circ f)(x+y) = (A_i^+ \circ f)(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( (A_i^+ \circ f)(xy) = A_i^+(f(xy)) \)
\[ = A_i^+(f(x)f(y)) \]
= A_i^+(f(y)f(x)) = A_i^+(f(yx)) = (A_i^+\circ f)(yx)

which implies that \((A_i^+\circ f)(xy) = (A_i^+\circ f)(yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also \((A_i^-\circ f)(x+y) = A_i^-(f(x+y))\)

= A_i^-(f(x)+f(y))

= A_i^-(f(y)+f(x)) = A_i^-(f(y+x)) = (A_i^-\circ f)(y+x)

which implies that \((A_i^-\circ f)(x+y) = (A_i^-\circ f)(y+x)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \((A_i^-\circ f)(xy) = A_i^-(f(xy))\)

= A_i^-(f(x)f(y))

= A_i^-(f(y)f(x)) = A_i^-(f(yx)) = (A_i^-\circ f)(yx)

which implies that \((A_i^-\circ f)(xy) = (A_i^-\circ f)(yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Hence \(A\circ f\) is a bipolar valued multi fuzzy normal ideal of a semiring \(R\).

**5.3.23 Theorem:** Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy ideal of a semiring \(H\) and \(f\) is an anti-isomorphism from a semiring \(H\) onto \(R\). If \(A\) is a bipolar valued multi fuzzy normal ideal of the semiring \(H\), then \(A\circ f = \langle A_1^+\circ f, A_2^+\circ f, \ldots, A_i^+\circ f, A_1^-\circ f, A_2^-\circ f, \ldots, A_i^-\circ f \rangle\) is a bipolar valued multi fuzzy normal ideal of the semiring \(R\).

**Proof:** Let \(A\) be a bipolar valued multi fuzzy normal ideal of a semiring \(H\). By Theorem 5.2.25, \(A\circ f\) is a bipolar valued multi fuzzy ideal of a semiring \(R\). Let \(x\) and \(y\) in \(R\).
Now \((A_i^+ \circ f)(x+y) = A_i^+ (f(x+y))\)

\[= A_i^+ (f(y)+f(x))\]

\[= A_i^+ (f(x)+f(y)) = A_i^+ (f(y+x)) = (A_i^+ \circ f)(y+x)\]

which implies that \((A_i^+ \circ f)(x+y) = (A_i^+ \circ f)(y+x)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \((A_i^+ \circ f)(xy) = A_i^+ (f(xy))\)

\[= A_i^+ (f(y)f(x))\]

\[= A_i^+ (f(x)f(y)) = (A_i^+ \circ f)(yx)\]

which implies that \((A_i^+ \circ f)(xy) = (A_i^+ \circ f)(yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Also \((A_i^- \circ f)(x+y) = A_i^- (f(x+y))\)

\[= A_i^- (f(y)+f(x))\]

\[= A_i^- (f(x)+f(y)) = A_i^- (f(y+x)) = (A_i^- \circ f)(y+x)\]

which implies that \((A_i^- \circ f)(x+y) = (A_i^- \circ f)(y+x)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

And \((A_i^- \circ f)(xy) = A_i^- (f(xy))\)

\[= A_i^- (f(y)f(x))\]

\[= A_i^- (f(x)f(y)) = (A_i^- \circ f)(yx)\]

which implies that \((A_i^- \circ f)(xy) = (A_i^- \circ f)(yx)\) for all \(x\) and \(y\) in \(R\) and for all \(i\).

Hence \(A \circ f\) is a bipolar-valued multi fuzzy normal ideal of a semiring \(R\).
5.3.24 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( ^\oplus A \) is a bipolar valued multi fuzzy normal ideal of the semiring \( R \).

Proof: By Theorem 5.2.26, \( ^\oplus A \) is a bipolar valued multi fuzzy ideal of the semiring \( R \).

Let \( x \) and \( y \) in \( R \).

We have \( ^\oplus A_i^+(x+y) = A_i^+(x+y) + 1 - H(A_i^+) \)

\[ = A_i^+(y+x) + 1 - H(A_i^+) = ^\oplus A_i^+(y+x) \]

which implies \( ^\oplus A_i^+(x+y) = ^\oplus A_i^+(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( ^\oplus A_i^+(xy) = A_i^+(xy) + 1 - H(A_i^+) \)

\[ = A_i^+(yx) + 1 - H(A_i^+) = ^\oplus A_i^+(yx) \]

which implies \( ^\oplus A_i^+(xy) = ^\oplus A_i^+(yx) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( ^\oplus A_i^-(x+y) = A_i^-(x+y) - 1 - H(A_i^-) \)

\[ = A_i^-(y+x) - 1 - H(A_i^-) = ^\oplus A_i^-(y+x) \]

which implies \( ^\oplus A_i^-(x+y) = ^\oplus A_i^-(y+x) \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( ^\oplus A_i^-(xy) = A_i^-(xy) - 1 - H(A_i^-) \)

\[ = A_i^-(yx) - 1 - H(A_i^-) = ^\oplus A_i^-(yx) \]

which implies \( ^\oplus A_i^-(xy) = ^\oplus A_i^-(yx) \) for all \( x \) and \( y \) in \( R \).

Hence \( ^\oplus A \) is a bipolar valued multi fuzzy normal ideal of \( R \).

5.3.25 Theorem: If \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy normal ideal of a semiring \( R \), then \( ^0 A \) is a bipolar valued multi fuzzy normal ideal of the semiring \( R \).

Proof: By Theorem 5.2.28, \( ^0 A \) is a bipolar valued multi fuzzy ideal of the semiring \( R \).

For \( x \) and \( y \) in \( R \),
we have $0A_i^+(x+y) = A_i^+(x+y)H(A_i^+) = A_i^+(y+x)H(A_i^+) = 0A_i^+(y+x)$
which implies that $0A_i^+(x+y) = 0A_i^+(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.
And $0A_i^+(xy) = A_i^+(xy)H(A_i^+) = A_i^+(yx)H(A_i^+) = 0A_i^+(yx)$
which implies that $0A_i^+(xy) = 0A_i^+(yx)$ for all $x$ and $y$ in $R$ and for all $i$.
Also $0A_i^-(x+y) = -A_i^-(x+y)H(A_i^-) = -A_i^-(y+x)H(A_i^-) = 0A_i^-(y+x)$
which implies that $0A_i^-(x+y) = 0A_i^-(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.
And $0A_i^-(xy) = -A_i^-(xy)H(A_i^-) = -A_i^-(yx)H(A_i^-) = 0A_i^-(yx)$
which implies that $0A_i^-(xy) = 0A_i^-(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $0A$ is a bipolar valued multi fuzzy normal ideal of the semiring $R$.

**5.3.26 Theorem:** If $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy normal ideal of a semiring $R$, then $^\Delta A$ is a bipolar valued multi fuzzy normal ideal of $R$.

**Proof:** By Theorem 5.2.29, $^\Delta A$ is a bipolar valued multi fuzzy ideal of $R$. For all $x$ and $y$ in $R$,
we have $^\Delta A_i^+(x+y) = A_i^+(x+y) / H(A_i^+) = A_i^+(y+x) / H(A_i^+) = ^\Delta A_i^+(y+x)$
which implies that $^\Delta A_i^+(x+y) = ^\Delta A_i^+(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.
And $^\Delta A_i^+(xy) = A_i^+(xy) / H(A_i^+) = A_i^+(yx) / H(A_i^+) = ^\Delta A_i^+(yx)$
which implies that $^\Delta A_i^+(xy) = ^\Delta A_i^+(yx)$ for all $x$ and $y$ in $R$ and for all $i$.
Also $^\Delta A_i^-(x+y) = -A_i^-(x+y) / H(A_i^-) = -A_i^-(y+x) / H(A_i^-) = ^\Delta A_i^-(y+x)$
which implies that $^\Delta A_i^-(x+y) = ^\Delta A_i^-(y+x)$ for all $x$ and $y$ in $R$ and for all $i$.
And $^\Delta A_i^-(xy) = -A_i^-(xy) / H(A_i^-) = -A_i^-(yx) / H(A_i^-) = ^\Delta A_i^-(yx)$
which implies that $^\Delta A_i^-(xy) = ^\Delta A_i^-(yx)$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $^\Delta A$ is a bipolar valued multi fuzzy normal ideal of the semiring $R$. 
5.4–PROPERTIES OF ( (α₁, α₂, ...αᵢ), (β₁, β₂, ...βᵢ) )-LEVEL SUBSETS OF BIPOLAR VALUED MULTI FUZZY IDEALS:

5.4.1 Theorem: Let \( A = (A_1^+, A_2^+, ..., A_i^+, A_i^-, A_2^-, ..., A_i^-) \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). Then for \( \alpha_i \) in \([0, 1]\) and \( \beta_i \) in \([-1, 0]\) such that \( \alpha_i \leq A_i^+(e) \) and \( \beta_i \geq A_i^-(e) \) for all \( i \), \( A_{((\alpha_1, \alpha_2, ..., \alpha_i), (\beta_1, \beta_2, ..., \beta_i))} \) is a \(( (\alpha_1, \alpha_2, ...\alpha_i), (\beta_1, \beta_2, ...\beta_i) )\)-level subideal of \( R \).

Proof: It is trivial.

5.4.2 Theorem: Let \( A = (A_1^+, A_2^+, ..., A_i^+, A_i^-, A_2^-, ..., A_i^-) \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). Then for \( \alpha_i, \delta_i \) in \([0, 1]\), \( \beta_i, \phi_i \) in \([-1, 0]\), \( \alpha_i \leq A_i^+(e), \delta_i \leq A_i^+(e), \beta_i \geq A_i^-(e), \phi_i \geq A_i^-(e), \delta_i \leq \alpha_i \) and \( \beta_i \leq \phi_i \) for all \( i \), the two \((\alpha, \beta)\)-level subideals \( A_{((\alpha_1, \alpha_2, ..., \alpha_i), (\beta_1, \beta_2, ..., \beta_i))} \) and \( A_{((\delta_1, \delta_2, ..., \delta_i), (\phi_1, \phi_2, ..., \phi_i))} \) of \( A \) are equal if and only if there is no \( x \) in \( R \) such that \( \alpha_i > A_i^+(x) > \delta_i \) and \( \beta_i < A_i^-(x) < \phi_i \) for all \( i \).

Proof: It is trivial.

5.4.3 Theorem: Let \( A = (A_1^+, A_2^+, ..., A_i^+, A_i^-, A_2^-, ..., A_i^-) \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). If any two \((\alpha, \beta)\)-level subideals of \( A \) belongs to \( R \), then their intersection is also \((\alpha, \beta)\)-level subideal of \( A \) in \( R \).

Proof: It is trivial.
5.4.4 **Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). The intersection of a collection of \((\alpha, \beta)\)-level subideals of \( A \) is also a \((\alpha, \beta)\)-level subideal of \( A \).

**Proof:** It is trivial.

5.4.5 **Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). If any two \((\alpha, \beta)\)-level subideals of \( A \) belongs to \( R \), then their union is also \((\alpha, \beta)\)-level subideal of \( A \) in \( R \).

**Proof:** It is trivial.

5.4.6 **Theorem:** Let \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) be a bipolar valued multi fuzzy ideal of a semiring \( R \). The union of a collection of \((\alpha, \beta)\)-level subideals of \( A \) is also a \((\alpha, \beta)\)-level subideal of \( A \).

**Proof:** It is trivial.

5.4.7 **Theorem:** The homomorphic image of a \((\alpha, \beta)\)-level subideal of a bipolar valued multi fuzzy ideal of a semiring \( R \) is a \((\alpha, \beta)\)-level subideal of a bipolar valued multi fuzzy ideal of a semiring \( R^l \).

**Proof:** Let \( V = f(A) \). Here \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi fuzzy ideal of \( R \). Clearly \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle \) is a bipolar valued multi fuzzy ideal of \( R^l \). Let \( x \) and \( y \) in \( R \). Then \( f(x) \) and \( f(y) \) in \( R^l \). Let \( A_{(\alpha, \beta)} \) be a \((\alpha, \beta) = ( (\alpha_1, \alpha_2, \ldots \alpha_i), (\beta_1, \beta_2, \ldots \beta_i) \)-level subideal of \( A \).

That is, \( A_{1}^{i}(x) \geq \alpha_i \) and \( A_{1}^{i}(x) \leq \beta_i ; A_{1}^{i}(y) \geq \alpha_i \) and \( A_{1}^{i}(y) \leq \beta_i ; \)

\( A_{1}^{i}(x+y) \geq \alpha_i, A_{1}^{i}(x+y) \leq \beta_i, A_{1}^{i}(xy) \geq \alpha_i, A_{1}^{i}(xy) \leq \beta_i \) for all \( i \).
We have to prove that $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subideal of $V$.

Now $V_i^+( f(x) ) \geq A_i^+(x) \geq \alpha_i$ which implies that $V_i^+( f(x) ) \geq \alpha_i$;

and $V_i^+( f(y) ) \geq A_i^+(y) \geq \alpha_i$ which implies that $V_i^+( f(y) ) \geq \alpha_i$ for all $i$.

Then $V_i^+( f(x)+f(y) ) = V_i^+( f(x+y) ) \geq A_i^+(x+y) \geq \alpha_i$,

which implies that $V_i^+( f(x)+f(y) ) \geq \alpha_i$ for all $i$.

And $V_i^+( f(x)f(y) ) = V_i^+( f(xy) ) \geq A_i^+(xy) \geq \alpha_i$,

which implies that $V_i^+( f(x)f(y) ) \geq \alpha_i$ for all $i$.

And $V_i^-( f(x)) \leq A_i^-(x) \leq \beta_i$ which implies that $V_i^-( f(x)) \leq \beta_i$;

and $V_i^-( f(y) ) \leq A_i^-(y) \leq \beta_i$ which implies that $V_i^-( f(y) ) \leq \beta_i$ for all $i$.

Then $V_i^-( f(x)+f(y) ) = V_i^-( f(x+y) ) \leq A_i^-(x+y) \leq \beta_i$,

which implies that $V_i^-( f(x)+f(y) ) \leq \beta_i$ for all $i$.

And $V_i^-( f(x)f(y) ) = V_i^-( f(xy) ) \leq A_i^-(xy) \leq \beta_i$,

which implies that $V_i^-( f(x)f(y) ) \leq \beta_i$ for all $i$.

Hence $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subideal of a bipolar valued multi fuzzy ideal $V$ of $R^1$.

5.4.8 **Theorem:** The homomorphic pre-image of a $(\alpha, \beta)$-level subideal of a bipolar valued multi fuzzy ideal of a semiring $R^1$ is a $(\alpha, \beta)$-level subideal of a bipolar valued multi fuzzy ideal of a semiring $R$.

**Proof:** It is trivial.

5.4.9 **Theorem:** The anti-homomorphic image of a $(\alpha, \beta)$-level subideal of a bipolar valued multi fuzzy ideal of a semiring $R$ is a $(\alpha, \beta)$-level subideal of a bipolar valued multi fuzzy ideal of a semiring $R^1$.

**Proof:** It is trivial.
5.4.10 Theorem: The anti-homomorphic pre-image of a \((\alpha, \beta)\)-level subideal of a bipolar valued multi fuzzy ideal of a semiring \(R^1\) is a \((\alpha, \beta)\)-level subideal of a bipolar valued multi fuzzy ideal of the semiring \(R\).

Proof: It is trivial.

5.4.11 Theorem: Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy ideal of a semiring \(R\). Then for \(\alpha_i\) in \([0, 1]\), \(A_i^+\)-level \(\alpha\)-cut \(P(A_i^+, \alpha)\) is a \(A_i^+\)-level \(\alpha\)-cut subideal of \(R\).

Proof: It is trivial.

5.4.12 Theorem: Let \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) be a bipolar valued multi fuzzy ideal of a semiring \(R\). Then for \(\beta_i\) in \([-1, 0]\), \(A_i^-\)-level \(\beta\)-cut \(N(A_i^-, \beta)\) is a \(A_i^-\)-level \(\beta\)-cut subideal of \(R\).

Proof: It is trivial.

5.4.13 Theorem: If \(M\) and \(N\) are two bipolar valued multi fuzzy translations of bipolar valued multi fuzzy ideal \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) of a semiring \(R\), then their intersection \(M \cap N\) is also a bipolar valued multi fuzzy translation of \(A\).

Proof: It is trivial.

5.4.14 Theorem: The intersection of a family of bipolar valued multi fuzzy translations of bipolar valued multi fuzzy ideal \(A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle\) of a semiring \(R\) is a bipolar valued multi fuzzy translation of \(A\).
**Proof:** It is trivial.

5.4.15 **Theorem:** Union of any two bipolar valued multi fuzzy translations of bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of a semiring \( R \) is a bipolar valued multi fuzzy translation of \( A \).

**Proof:** It is trivial.

5.4.16 **Theorem:** The union of a family of bipolar valued multi fuzzy translations of bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of a semiring \( R \) is a bipolar valued multi fuzzy translation of \( A \).

**Proof:** It is trivial.

5.4.17 **Theorem:** A bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of a semiring \( R \) is a bipolar valued multi fuzzy ideal of \( R \).

**Proof:** Assume that \( T = \langle T_1^+, T_2^+, \ldots, T_i^+, T_1^-, T_2^-, \ldots, T_i^- \rangle \) is a bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of a semiring \( R \).

Let \( x \) and \( y \) in \( R \).

We have \( T_i^+(x+y) = A_i^+(x+y) + \alpha_i \)

\[ \geq \min \{ A_i^+(x), A_i^+(y) \} + \alpha_i \]

\[ = \min \{ A_i^+(x) + \alpha_i, A_i^+(y) + \alpha_i \} \]

\[ = \min \{ T_i^+(x), T_i^+(y) \} \text{ for all } i. \]

Therefore \( T_i^+(x+y) \geq \min \{ T_i^+(x), T_i^+(y) \} \text{ for all } x \text{ and } y \text{ in } R \text{ and for all } i. \)
And $T_i^+(xy) = A_i^+(xy) + \alpha_i$

$$\geq \max \{ A_i^+(x), A_i^+(y) \} + \alpha_i$$

$$= \max \{ A_i^+(x) + \alpha_i, A_i^+(y) + \alpha_i \}$$

$$= \max \{ T_i^+(x), T_i^+(y) \} \text{ for all } i.$$ 

Therefore $T_i^+(xy) \geq \max \{ T_i^+(x), T_i^+(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Also $T_i^-(x+y) = A_i^-(x+y) + \beta_i$

$$\leq \max \{ A_i^-(x), A_i^-(y) \} + \beta_i$$

$$= \max \{ A_i^-(x) + \beta_i, A_i^-(y) + \beta_i \}$$

$$= \max \{ T_i^-(x), T_i^-(y) \} \text{ for all } i.$$ 

Therefore $T_i^-(x+y) \leq \max \{ T_i^-(x), T_i^-(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

And $T_i^-(xy) = A_i^-(xy) + \beta_i$

$$\leq \min \{ A_i^-(x), A_i^-(y) \} + \beta_i$$

$$= \min \{ A_i^-(x) + \beta_i, A_i^-(y) + \beta_i \}$$

$$= \min \{ T_i^-(x), T_i^-(y) \} \text{ for all } i.$$ 

Therefore $T_i^-(xy) \leq \min \{ T_i^-(x), T_i^-(y) \}$ for all $x$ and $y$ in $R$ and for all $i$.

Hence $T$ is a bipolar valued multi fuzzy ideal of $R$.

**5.4.18 Theorem:** Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two semirings and $f$ be a homomorphism. Then the homomorphic image of a bipolar valued multi fuzzy
translation of a bipolar valued multi fuzzy ideal $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ of $R$ is also a bipolar valued multi fuzzy ideal of $R^1$.

**Proof:** Let $V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle = f(T^d_{(\alpha,\beta)})$, where $T^d_{(\alpha,\beta)}$ is a bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ of $R$. We have to prove that $V$ is a bipolar valued multi fuzzy ideal of $R^1$.

For all $f(x)$ and $f(y)$ in $R^1$,
we have $V_i^+ [ f(x)+f(y) ] = V_i^+ [ f(x+y) ]$

\[ \geq T^{+A_{\alpha_i}} (x+y) \]

\[ = A_i^+ (x+y) + \alpha_i \]

\[ \geq \min \{ A_i^+ (x), A_i^+ (y) \} + \alpha_i \]

\[ = \min \{ A_i^+ (x) + \alpha_i, A_i^+ (y) + \alpha_i \} \]

\[ = \min \{ T^{+A_{\alpha_i}} (x), T^{+A_{\alpha_i}} (y) \} \]

which implies that $V_i^+ [ f(x)+f(y) ] \geq \min \{ V_i^+ ( f(x) ), V_i^+ ( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^1$ and for all $i$.

And $V_i^+ [ f(x)f(y) ] = V_i^+ [ f(xy) ]$

\[ \geq T^{+A_{\alpha_i}} (xy) \]

\[ = A_i^+ (xy) + \alpha_i \]

\[ \geq \max \{ A_i^+ (x), A_i^+ (y) \} + \alpha_i \]

\[ = \max \{ A_i^+ (x) + \alpha_i, A_i^+ (y) + \alpha_i \} \]

\[ = \max \{ T^{+A_{\alpha_i}} (x), T^{+A_{\alpha_i}} (y) \} \]

which implies that $V_i^+ [ f(x)f(y) ] \geq \max \{ V_i^+ ( f(x) ), V_i^+ ( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^1$ and for all $i$. 
Also $V_i^-[ f(x)+f(y) ] = V_i^-[ f(x+y) ]$

$\leq T^{-f}(x+y)$

$= A_i^-(x+y) + \beta_i$

$\leq \max \{ A_i^-(x), A_i^-(y) \} + \beta_i$

$= \max \{ A_i^-(x) + \beta_i, A_i^-(y) + \beta_i \}$

$= \max \{ T^{-f}(x), T^{-f}(y) \}$

which implies that $V_i^-[ f(x)+f(y) ] \leq \max \{ V_i^-( f(x) ), V_i^-( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

And $V_i^-[ f(x)f(y) ] = V_i^-[ f(xy) ]$

$\leq T^{-f}(xy)$

$= A_i^-(xy) + \beta_i$

$\leq \min \{ A_i^-(x), A_i^-(y) \} + \beta_i$

$= \min \{ A_i^-(x) + \beta_i, A_i^-(y) + \beta_i \}$

$= \min \{ T^{-f}(x), T^{-f}(y) \}$

which implies that $V_i^-[ f(x)f(y) ] \leq \min \{ V_i^-( f(x) ), V_i^-( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

Therefore $V$ is a bipolar valued multi fuzzy ideal of $R^l$.

5.4.19 Theorem: Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two semirings and $f$ be a homomorphism. Then the homomorphic pre-image of bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal $V = \langle V_1^+, V_2^+, ..., V_i^+, V_1^-, V_2^-, ..., V_i^- \rangle$ of $R^l$ is a bipolar valued multi fuzzy ideal of $R$.

Proof: Let $T = T_{(a,\beta)} = f(A)$, where $T_{(a,\beta)}$ is a bipolar valued multi fuzzy translation of bipolar valued multi fuzzy ideal $V = \langle V_1^+, V_2^+, ..., V_i^+, V_1^-, V_2^-, ..., V_i^- \rangle$ of $R^l$. We
have to prove that \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) is a bipolar valued multi-fuzzy ideal of \( R \).

Let \( x \) and \( y \) in \( R \).

Then \( A_i^+(x+y) = T_{\alpha_i}^+( f(x+y) ) \)

\[
= T_{\alpha_i}^+( f(x)+f(y) )
\]

\[
= V_i^+[ f(x)+f(y) ] + \alpha_i
\]

\[ \geq \min \{ V_i^+( f(x) ), V_i^+( f(y) ) \} + \alpha_i \]

\[ = \min \{ V_i^+( f(x) ) + \alpha_i, V_i^+( f(y) ) + \alpha_i \} \]

\[ = \min \{ T_{\alpha_i}^+( f(x) ), T_{\alpha_i}^+( f(y) ) \} \]

\[ = \min \{ A_i^+(x), A_i^+(y) \} \]

which implies that \( A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^+(xy) = T_{\alpha_i}^+( f(xy) ) \)

\[
= T_{\alpha_i}^+( f(x)f(y) )
\]

\[
= V_i^+[ f(x)f(y) ] + \alpha_i
\]

\[ \geq \max \{ V_i^+( f(x) ), V_i^+( f(y) ) \} + \alpha_i \]

\[ = \max \{ V_i^+( f(x) ) + \alpha_i, V_i^+( f(y) ) + \alpha_i \} \]

\[ = \max \{ T_{\alpha_i}^+( f(x) ), T_{\alpha_i}^+( f(y) ) \} \]

\[ = \max \{ A_i^+(x), A_i^+(y) \} \]

which implies that \( A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Also \( A_i^-(x+y) = T_{\beta_i}^-( f(x+y) ) \)

\[
= T_{\beta_i}^-( f(x)+f(y) )
\]

\[
= V_i^-[ f(x)+f(y) ] + \beta_i
\]

\[ \leq \max \{ V_i^- f(x) ), V_i^- f(y) ) \} + \beta_i \]
\[ \max \{ V_i^- (f(x)) + \beta_i, V_i^- (f(y)) + \beta_i \} \]
\[ = \max \{ T_{\mu^-}^\alpha (f(x)), T_{\mu^-}^\alpha (f(y)) \} \]
\[ = \max \{ A_i^-(x), A_i^-(y) \} \]

which implies \( A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

And \( A_i^-(xy) = T_{\mu^-}^\alpha (f(xy)) \)
\[ = T_{\mu^-}^\alpha (f(x)f(y)) \]
\[ = V_i^- [f(x)f(y)] + \beta_i \]
\[ \leq \min \{ V_i^- (f(x)), V_i^- (f(y)) \} + \beta_i \]
\[ = \min \{ V_i^- (f(x)) + \beta_i, V_i^- (f(y)) + \beta_i \} \]
\[ = \min \{ T_{\mu^-}^\alpha (f(x)), T_{\mu^-}^\alpha (f(y)) \} \]
\[ = \min \{ A_i^-(x), A_i^-(y) \} \]

which implies \( A_i^-(xy) \leq \min \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( R \) and for all \( i \).

Therefore \( A \) is a bipolar valued multi fuzzy ideal of \( R \).

**5.4.20 Theorem:** Let \(( R, +, )\) and \(( R', +, )\) be any two semirings and \( f \) be an anti-homomorphism. Then the anti-homomorphic image of a bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of \( R \) is also a bipolar valued multi fuzzy ideal of \( R' \).

**Proof:** Let \( V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle = f(T_{(\alpha, \beta)}^d) \), where \( T_{(\alpha, \beta)}^d \) is a bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal \( A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle \) of \( R \). We have to prove that \( V \) is a bipolar valued multi fuzzy ideal of \( R' \).

For all \( f(x) \) and \( f(y) \) in \( R' \),
we have $V_i^+[ f(x)+f(y) ] = V_i^+[ f(y+x) ]$

\[
\geq T^{+\alpha}_\alpha(y+x)
\]
\[
= A_i^+(y+x) + \alpha_i
\]
\[
\geq \min \{ A_i^+(y), A_i^+(x) \} + \alpha_i
\]
\[
= \min \{ A_i^+(x) + \alpha_i, A_i^+(y) + \alpha_i \}
\]
\[
= \min \{ T^{+\alpha}_\alpha(x), T^{+\alpha}_\alpha(y) \}
\]

which implies that

\[
V_i^+[ f(x)+f(y) ] \geq \min \{ V_i^+( f(x) ), V_i^+( f(y) ) \}
\]

for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

And $V_i^+[ f(x)f(y) ] = V_i^+[ f(yx) ]$

\[
\geq T^{+\alpha}_\alpha(yx)
\]
\[
= A_i^+(yx) + \alpha_i
\]
\[
\geq \max \{ A_i^+(y), A_i^+(x) \} + \alpha_i
\]
\[
= \max \{ A_i^+(x) + \alpha_i, A_i^+(y) + \alpha_i \}
\]
\[
= \max \{ T^{+\alpha}_\alpha(x), T^{+\alpha}_\alpha(y) \}
\]

which implies that

\[
V_i^+[ f(x)f(y) ] \geq \max \{ V_i^+( f(x) ), V_i^+( f(y) ) \}
\]

for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

Also $V_i^-[ f(x)+f(y) ] = V_i^-[ f(y+x) ]$

\[
\leq T^{-\beta}_\beta(y+x)
\]
\[
= A_i^-(y+x) + \beta_i
\]
\[
\leq \max \{ A_i^-(y), A_i^-(x) \} + \beta_i
\]
\[
= \max \{ A_i^-(x) + \beta_i, A_i^-(y) + \beta_i \}
\]
\[
= \max \{ T^{-\beta}_\beta(x), T^{-\beta}_\beta(y) \}
\]
which implies that $V_i^{-}[ f(x)+f(y) ] \leq \max \{ V_i^{-}( f(x) ), V_i^{-}( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

And $V_i^{-}[ f(x)f(y) ] = V_i^{-}[ f(y)x ]$

$$\leq T^{-,\lambda}_{R^l}(yx)$$

$$= A_i^{-}(yx) + \beta_i$$

$$\leq \min \{ A_i^{-}(x), A_i^{-}(y) \} + \beta_i$$

$$= \min \{ A_i^{-}(x) + \beta_i, A_i^{-}(y) + \beta_i \}$$

$$= \min \{ T^{-,\lambda}_{\lambda_i}(x), T^{-,\lambda}_{\lambda_i}(y) \}$$

which implies that $V_i^{-}[ f(x)f(y) ] \leq \min \{ V_i^{-}( f(x) ), V_i^{-}( f(y) ) \}$ for all $f(x)$ and $f(y)$ in $R^l$ and for all $i$.

Therefore $V$ is a bipolar valued multi fuzzy ideal of $R^l$.

**5.4.21 Theorem:** Let $( R, +, \cdot )$ and $( R^l, +, \cdot )$ be any two semirings and $f$ be an anti-homomorphism. Then the anti-homomorphic pre-image of bipolar valued multi fuzzy translation of a bipolar valued multi fuzzy ideal $V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle$ of $R^l$ is a bipolar valued multi fuzzy ideal of $R$.

**Proof:** Let $T = T^{\nu}_{(\alpha, \beta)} = f(A)$, where $T^{\nu}_{(\alpha, \beta)}$ is a bipolar valued multi fuzzy translation of bipolar valued multi fuzzy ideal $V = \langle V_1^+, V_2^+, \ldots, V_i^+, V_1^-, V_2^-, \ldots, V_i^- \rangle$ of $R^l$. We have to prove that $A = \langle A_1^+, A_2^+, \ldots, A_i^+, A_1^-, A_2^-, \ldots, A_i^- \rangle$ is a bipolar valued multi fuzzy ideal of $R$.

Let $x$ and $y$ in $R$.

Then $A_i^{-}(x+y) = T^{-,\lambda}_{\alpha_i}( f(x+y) )$

$$= T^{+,\lambda}_{\alpha_i}( f(y)+f(x) )$$

$$= V_i^+[ f(y)+f(x) ] + \alpha_i$$
\[
\begin{align*}
\geq & \min \{ V_i^+ (f(y)), V_i^+(f(x)) \} + \alpha_i \\
= & \min \{ V_i^+ (f(x)) + \alpha_i, V_i^+ (f(y)) + \alpha_i \} \\
= & \min \{ T^{+\nu}_\alpha (f(x)), T^{+\nu}_\alpha (f(y)) \} = \min \{ A_i^+(x), A_i^+(y) \}
\end{align*}
\]
which implies that \( A_i^+(x+y) \geq \min \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

And \( A_i^+(xy) = T^{+\nu}_\alpha (f(xy)) \)
\[
\begin{align*}
= & T^{+\nu}_\alpha (f(y)f(x)) \\
= & V_i^+ [f(y)f(x)] + \alpha_i \\
\geq & \max \{ V_i^+ (f(y)), V_i^+(f(x)) \} + \alpha_i \\
= & \max \{ V_i^+ (f(x)) + \alpha_i, V_i^+ (f(y)) + \alpha_i \} \\
= & \max \{ T^{+\nu}_\alpha (f(x)), T^{+\nu}_\alpha (f(y)) \} = \max \{ A_i^+(x), A_i^+(y) \}
\end{align*}
\]
which implies that \( A_i^+(xy) \geq \max \{ A_i^+(x), A_i^+(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

Also \( A_i^-(x+y) = T^{-\nu}_\beta (f(x+y)) \)
\[
\begin{align*}
= & T^{-\nu}_\beta (f(y)+f(x)) \\
= & V_i^- [f(y)+f(x)] + \beta_i \\
\leq & \max \{ V_i^- (f(y)), V_i^- (f(x)) \} + \beta_i \\
= & \max \{ V_i^- (f(x)) + \beta_i, V_i^- (f(y)) + \beta_i \} \\
= & \max \{ T^{-\nu}_\beta (f(x)), T^{-\nu}_\beta (f(y)) \} \\
= & \max \{ A_i^-(x), A_i^-(y) \}
\end{align*}
\]
which implies \( A_i^-(x+y) \leq \max \{ A_i^-(x), A_i^-(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

And \( A_i^-(xy) = T^{-\nu}_\beta (f(xy)) \)
\[
\begin{align*}
= & T^{-\nu}_\beta (f(y)f(x)) \\
= & V_i^- [f(y)f(x)] + \beta_i
\end{align*}
\]
\[ \leq \min \{ V_i^{-}( f(y) ), V_i^{-}( f(x) ) \} + \beta_i \]

\[ = \min \{ V_i^{-}( f(x) ) + \beta_i, V_i^{-}( f(y) ) + \beta_i \} \]

\[ = \min \{ T_{\rho}^{-}( f(x) ), T_{\rho}^{-}( f(y) ) \} \]

\[ = \min \{ A_i^{-}(x), A_i^{-}(y) \} \]

which implies \( A_i^{-}(xy) \leq \min\{ A_i^{-}(x), A_i^{-}(y) \} \) for all \( x \) and \( y \) in \( \mathbb{R} \) and for all \( i \).

Therefore \( A \) is a bipolar valued multi fuzzy ideal of \( \mathbb{R} \).