Chapter 2

Periodic Point Property

The results proved in this chapter are required in later chapters. Theorem 1 will be used in Chapter 3 whereas Proposition 2 and Theorem 2 will be used in Chapter 4.

§1

Introduction and Preliminaries:

The concept of periodic points has been well-studied in the Theory of Functions (See [31], [36]), Ergodic Theory (see [29]), Theory of Fractals (see [24]), Theory of Dynamical systems (see [3],[4] and [11]), Algebraic Topology (see [32]) etc. We say that a topological space $X$ has periodic point property (abbreviated as p.p.p) if every continuous self-map of $X$ has a periodic point. This is analogous to the fixed point property (see [34]) that has been studied in General Topology, Algebraic Topology and Functional Analysis, Theory of Differential Equations, etc.

In this chapter, we find that the study of p.p.p. takes us to the class of countable compact spaces. This class of topological spaces has attracted the attention of many mathematicians like Sierpinski, Mazurkiewicz, J.de Groot, M. Katetov, Bessaga, Pelczynski, and Rajagopalan. A summary of their contributions is available in [19], It is already known that every countable compact metric space $X$ has p.p.p.[33].It
follows that every closed subspace of such $X$ also has p.p.p. It is natural to ask whether the converse is true, i.e. if a metric space $X$ has the property that every closed subspace of $X$ has p.p.p. then should $X$ be countable and compact? We answer this in the affirmative. This may be compared with the main result of [16] where a similar converse to Banach's contraction mapping theorem has been proved. The only result on countable compact spaces that we need for our discussion, has been proved in [25] and in a different way in [18]. It states that every countable compact space is a well-ordered space.

Now we fix some notations. $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \to X$ be any self-map of $X$. A point $x$ is called a periodic point of $f$ if $f^n(x) = x$ for some $n$ in $\mathbb{N}$. If this happens with $n = 1$, then it is called a fixed point of $f$. We repeat that a topological space $X$ is said to have p.p.p. if every continuous self-map of $X$ has a periodic point. We use the word 'clopen' as an abbreviated form of 'both open and closed'.

**Proposition 1.1:** The following spaces satisfy p.p.p.:-

(a) All finite spaces.

(b) All compact convex subsets of $\mathbb{R}^n$; in particular the closed interval $[0,1]$ and the closed unit disc $D$ in $\mathbb{R}^2$.

(c) All strongly rigid spaces.

(d) All well-ordered compact spaces, and in particular $[1,\omega]$, where $\omega$ is the first uncountable ordinal number.
Proof: (a) Let $X$ be a finite discrete space having exactly $n$ elements. Let $f : X \to X$ be any function. Let $x_0 \in X$. Among the $n + 1$ terms of the sequence $x_0, f(x_0), f^2(x_0), \cdots, f^n(x_0)$, some two should be equal by pigeon-hole principle, say $f^r(x_0) = f^s(x_0)$ with $r < s$. Then $f^r(x_0)$ is a periodic point of $f$.

(b) It is a well-known theorem that every compact convex subset of $\mathbb{R}^n$ has the fixed point property (f.p.p), namely that every continuous self-map has a fixed point (See [34]). It is easily seen that f.p.p implies p.p.p.

(c) A Hausdorff space $X$ is said to be strongly rigid if every continuous self-map of $X$ is either a constant map or the identity map. Obviously, these spaces have f.p.p. and therefore p.p.p.

(d) Every well-ordered compact space is of the form $[1, \alpha]$ for some ordinal number $\alpha$. When $\alpha$ is a finite ordinal, the result follows from part (a). If at all there is an ordinal $\alpha$ for which the result fails, let $\alpha_0$ be the least such. Let $X = [1, \alpha_0]$ and let $f : X \to X$ be a continuous function without periodic points. For each $x \in X$ let $A_x$ denote \{y \in X | y = f^n(x) \text{ for some } n \in \mathbb{N} \}. Let $A = \{x \in X | x \text{ is in the closure of } A_x \}$. We first claim that $A_0 \subseteq A$. If not, $A_{\alpha_0} \subseteq [1, \beta]$ for some $\beta < \alpha_0$. Because $f$ is continuous, $f$ takes $A_{\alpha_0}$ to itself. Noting that $A_{\alpha_0}$ is homeomorphic to the well-ordered compact space $[1, 7]$ for some $7 < \beta < \alpha_0$ it follows from the choice of $\alpha_0$ that the restriction $f|_{A_{\alpha_0}}$ has a periodic point in $A_{\alpha_0}$. The same point is a periodic point of $f$ in $X$, contradicting the choice of $f$. This proves that $\alpha_0 \in A$ and thus $A$ is non-empty. Now let $\alpha_1$ be the least element of $A$. Then $A_{\alpha_1} \subseteq A$ by the continuity of $f$. And so $\alpha_1 < \alpha_0$. Whereas $\alpha_1 \in A_{\alpha_1}$, because $\alpha_1$ is in $A$, and this is possible only when $\alpha_1 \in A_{\alpha_1}$.
(since otherwise the left ray $[1, \alpha_1]$ will be a neighbourhood of $\alpha_1$ disjoint from $A_{\alpha_1}$).

This implies that $\alpha_1 = f^n(\alpha_1)$ for some natural number $n$, and thus $\alpha_1$ is a periodic point of $f$. This is a contradiction to the choice of $f$, hence proves the result.

**Proposition 1.2:** The following spaces do not possess p.p.p.

(a) An infinite discrete space.

(b) The circle $S^1$.

(c) The infinite product $\prod_{n=1}^{\infty} X_n$ of finite spaces $X_n = \{1, 2, \ldots, n\}$.

(d) The Cantor set $A'$.

(e) The real line $\mathbb{R}$ (and more generally any topological group with an element of infinite order)

(f) The Stone-Cech compactification $\beta \mathbb{N}$ of the discrete space $\mathbb{N}$ of natural numbers.

Proof: (a) If $X$ is an infinite discrete space, let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a countably infinite subset of $X$. Define $f : X \to X$ by

$$f(x) = \begin{cases} x_1 & \text{if } x \notin A \\ x_{n+1} & \text{if } x = x_n, \ n \in \mathbb{N} \end{cases}$$

Then $f$ is a continuous self-map of $X$ without any periodic point.

(b) Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $\theta$ be an angle non-commensurate with $\pi$ (that is, $\theta$ is not a rational multiple of $\pi$). Then the rotation by $\theta$ radians,
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(this is same as multiplication by the complex number $e^{i\theta}$ if $S^1$ is viewed in the complex plane) is a continuous self-map of $S^1$ without periodic points.

(c) Let for every $n \in \mathbb{N}$, the set $X_n$ be equal to $\{m \in \mathbb{N} : m < \pi\}$, provide discrete topology to each $X_n$. Let $P = \prod_{n \in \mathbb{N}} X_n$ with the product topology. Consider $/ : P \to P$ defined by the rule: If $x = (x_n) \in P$, then $f(x) = y$ whose $n$th coordinate $y_n$ is given by

$$\begin{cases} x_{n+1} & \text{if } x_n \neq n \\ 1 & \text{if } x_n = n \end{cases}$$

In other words we are taking a cyclic permutation $p_n : X_n \to X_n$ of order $n$ for each $n \in \mathbb{N}$ and $/ = \prod_{n \in \mathbb{N}} p_n$. The component functions $f_n : P \times X_n$ have the property that $f_n^{-1}(k) = \prod_{m=1}^{\infty} Y_m$

where $Y_m = \begin{cases} X_m & \text{if } m \neq n \\ \text{a singleton, namely } p_n^{-1}(k) & \text{if } m = n. \end{cases}$

So, $f_n^{-1}(k)$ is a basic open set in the product space $P$, for each $k$ in $X_n$. Therefore each $f_n$ is continuous. Therefore $/ \text{ is continuous. But } / \text{ does not have a periodic point. This is because if } x \in P \text{ and } n \in \mathbb{N}, \text{ then } x \text{ and } f^n(x) \text{necessarily differ in their } n + 1^t \text{ coordinate. In fact}

$$\begin{cases} x_{n+1} - 1 & \text{if } x_{n+1} \neq 1 \\ n + 1 & \text{if } x_{n+1} = 1. \end{cases}$$

(d) Let $K$ be the Cantor set consisting of those numbers in $[0,1]$ that admit a ternary representation $0.a_1a_2 \cdots a_n \cdots$ where each $a_n$ is either 0 or 2. Let $X_n$ have the meaning as in (c). Let $Q = \prod_{n=1}^{\infty} X_{2^n}$ with product topology. Define $h : K \to Q$ by the rule: if $x$ has ternary representation $0.x_1x_2 \cdots x_n \cdots$ where each $x_n$ is either 0 or 2, then the $n$th coordinate of $h(x)$ is 1 or 2 or $\cdots$ or $2^n$ according as the
block \( x_{2n-1}, x_{2n-1+1}, \ldots, x_{2n-1} \) is 00\cdots0, or 00\cdots01, or 00\cdots10, or 
\[ \cdots \text{ or } 110 \cdots \],
respectively. It is easily seen that \( h \) is a bijection. Also, if \( W \) is a basic open subset
of \( Q \), of the form \( \cap_{n=1}^\infty Y_n \) where

\[
\begin{align*}
X_{2n} & \quad \text{if } n \neq m \\
a \text{ singleton in } X_{2m} & \quad \text{if } n = m
\end{align*}
\]

then \( h^{-1}(Y_n) \) is a subset of \( K \) of the form \( \{x \in K : \) in the ternary expansion of \( x, \)
the finite number of terms \( x_r, x_{r+1}, \ldots, x_{r+s} \) coincide with the pre-assigned values\} for some \( r, s \) in \( \mathbb{N} \) and for some pre-assigned block of values from \( \{0, 2\} \). It follows
that \( h^{-1}(Y_n) \) is clopen in \( K \) for each basic open set in \( Q \). Therefore \( h \) is continuous.
Since \( K \) and \( Q \) are compact, \( h \) is a homeomorphism. Lastly by a proof similar to
that of (c), one can prove that \( Q \) does not have p.p.p. It follows that \( K \) does not
have p.p.p.

(e) The map \( f(x) = x + 1 \) for all \( x \) in \( \mathbb{R} \) is a continuous self-map of \( \mathbb{R} \) without
any periodic point.

(f) Let \( \beta \mathbb{N} \to \beta \mathbb{N} \) be the unique continuous extension of the map \( f : \mathbb{N} \to \mathbb{N} \)
defined by \( f(x) = x + 1 \) for all \( x \) in \( \mathbb{N} \). We claim that \( f \) has no periodic points. If \( n \)
is in \( \mathbb{N} \), the \( n_o \) subsets \( A_r = \{kn_o + r ; k \in \mathbb{N} \} \) for \( r = 0, 1, \ldots, n - 1 \) are pairwise
disjoint subsets of \( \mathbb{N} \). Their closures in \( \beta \mathbb{N} \) should be pairwise disjoint. If \( x \) is in
\( \beta \mathbb{N} \), then \( x \) belongs to \( A_r \) for a unique \( r \). But \( f^m(A_r) \neq A_s \) is disjoint from \( A_r \) if
\( m < n \). There \( f^m(x) \in A_s \), and hence \( f^m(x) \neq x \). Since \( n_o \) in \( \mathbb{N} \) is arbitrary, this
means that no \( x \) in \( \beta \mathbb{N} \) can be a periodic point of \( f \).
Theorem 1: The following are equivalent for a metric space \( X \):

(a) \( X \) is countable and compact.

(b) \( X \) is zero-dimensional and has p.p.p.

(c) Every closed subspace of \( X \) has p.p.p.

(d) Every continuous image of \( X \) has p.p.p.

Proof: Step 1: We first prove that every countable compact \textbf{Hausdorff} space \( X \) has p.p.p. Let \( f : X \rightarrow X \) be continuous. Call a subset \( A \) of \( X \) as \( f \)-invariant if \( f(A) \subseteq A \). By Zorn’s lemma, there is a minimal non-empty closed \( f \)-invariant subset \( A \) of \( X \) (such sets are called minimal sets in Topological Dynamics). Because of minimality, \( A \) has the property that for each \( x \) in \( A \), \( \{f^n(x) : n \in \mathbb{N} \} = A \). Applying Baire Category theorem to the countable compact \textbf{Hausdorff} space \( A \), we obtain an isolated point \( a \in A \). This \( a \) is in \( A \setminus \{f^n(a) : n \in \mathcal{N} \} \), and therefore \( a \in \{f^n(a) : n \in \mathcal{N} \} \), because \( a \) is isolated in \( A \). This implies that \( a \) is a periodic point of \( f \) (in fact this proves that \( A \) is a finite set).

Step 2: Whenever \( X \) is a countable compact space, so is every closed subspace of \( X \), and so is every continuous image of \( X \). Therefore it is immediate from step 1 that (1) implies both (3) and (4).

Step 3: It is well-known that every countable \( T_{3\frac{1}{2}} \) (i.e., completely regular, Hausdorff) space \( X \) is \textbf{zero-dimensional}. Indeed, if \( x \in X \) and if \( F \) is a closed
subset of $X$ not containing $x$, then there is a continuous function $/ : X \rightarrow [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y$ in $F$. The range of this $/$ has to be countable, since its domain is so. Therefore there is $t$ with $0 < t < 1$, that is not in the range of $/$. Now the set $f^{-1}([0,t)) = f^{-1}([0,t])$ is both open and closed in $X$, containing $x$, and disjoint from $F$.

Similar proof can be given to show that every countable $T_4$ space has the property that any two disjoint closed subsets can be separated by disjoint clopen subsets.

In particular every countable compact Hausdorff space is zero-dimensional. This observation together with step 1 proves that (1) implies (2). Thus so far we have proved that (1) implies each of the other three.

**Step 4:** We now prove that every zero-dimensional metric space $X$ with p.p.p. has to be compact. To prove this, let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$ be an any countably infinite subset of $X$; we shall prove that $A$ has a limit point. Suppose not. Then $A$ is a discrete closed subset of $X$. For each $n \in \mathbb{N}$ we choose a clopen set $V_n$ such that (i) these are pairwise disjoint (ii) $V_n \cap A = \{x_n\}$ and (iii) $V_n \subset B(x_n, \frac{1}{n})$. Let $V_0$ be the complement of $\bigcup_{n=1}^{\infty} V_n$. If $x \in V_0$, let $r > 0$ be such that $B(x, r) \cap A$ is empty. Then $B(x, \frac{r}{2}) \cap V_n$ is empty for all $n$ such that $\frac{1}{n} < \frac{r}{2}$. Therefore $x$ is not in the closure of $\bigcup_{k=n_0}^{\infty} V_k$ for a suitable $n_0$. Nor is $x$ in the closure of the clopen set $\bigcup_{k=1}^{n_0} V_k$. Therefore the set $\bigcup_{n=1}^{\infty} V_n$ is closed. It is obviously open also. Thus $\{V_0, V_1, V_2, \ldots, V_n, \ldots\}$ is a partition of $X$ into clopen subsets. Define $/ : X \rightarrow X$ by the rule $f(x) = x_{n+1}$ if $x \in V_n$. Then it is easily seen that $/$ is a continuous self-map of $X$ without any periodic point. But $X$ has p.p.p. by assumption. Therefore $A$ should have a limit point. Thus $X$ is compact.
Step 5: Next we prove that every uncountable zero-dimensional compact metric space fails to satisfy p.p.p. Let \( X \) be one such space. For the subset \( S = \{ x \in X \mid x \text{ admits a countable neighbourhood} \} \) the family \( \{ V_x : V_x \text{ is a countable neighbourhood of } x \text{ in } S \} \) is an open cover. Because \( S \) is second countable, there should be a countable subcover for this open cover. This implies that \( S \) is countable. Let \( Y \) be the complement of \( S \) in \( X \). Then \( Y \) is an uncountable, zero-dimensional compact metric space in which every point is a limit point. By a classical theorem (see[14]) \( Y \) is homeomorphic to the Cantor set \( K \). Therefore \( Y \) admits a base \( \{ W_1, W_2, \ldots, W_n, \ldots \} \) of nonempty clopen (in \( Y \)) subsets such that \( Y = W_1 \cup W_2 \) and \( W_n = W_{2n+1} \cup W_{2n+2} \) for all \( n \in \mathbb{N} \). (we have only to note that the standard base for the topology of the Cantor set \( K \) has this property.) Moreover every infinite subfamily of this base has at most one element in the intersection. Now the sets \( W_1 \) and \( W_2 \) are disjoint compact subsets of the zero-dimensional Hausdorff space \( Y \), and therefore can be separated by some clopen subsets \( W_1 \) and \( W_2 \) of \( Y \) whose union is \( Y \). Similarly for each \( n \in \mathbb{N} \), the disjoint compact subsets \( W_{2n+1} \) and \( W_{2n+2} \) of the previously defined zero-dimensional Hausdorff space \( W_n \) can be separated by disjoint clopen (in \( W_n \) and therefore in \( X \)) subsets \( W_{2n+1} \) and \( W_{2n+2} \) whose union is \( W_n \). Thus we recursively arrive finally at a family \( \{ W_n : n \in \mathbb{N} \} \) of clopen subsets of \( X \) such that \( W_n \cap Y = W_n \) for each \( n \). We use this family to define a function \( r : X \to Y \) by the rule \( r(x) \) is the unique element in \( Y \) of \( \cap \{ W_n : n \in \mathbb{N}, x \in W_n \} \). To show that this intersection set is a singleton, we first note that \( \{ W_{2n-1}, W_{2n}, \ldots, W_{2n+1-2} \} \) is a partition of \( X \) into clopen sets, for each \( n \in \mathbb{N} \). Therefore \( x \) must belong to one and only one of these sets.

In particular, for every \( m \in \mathbb{N} \), \( 3n > m \) such that \( x \in W_n \).
There are therefore infinitely many $n \in \mathbb{IN}$ such that $x \in W_n$. Therefore the intersection of the corresponding $W_n$'s cannot have more than one element. On the other hand any two such $W_n$'s being comparable, this family has finite intersection property, and hence by compactness of $X$, has nonempty intersection. Thus $r(x)$ gets defined uniquely for each $x \in X$. It is also easily seen that

(a) if $x \in Y$, then $x \in W_n \iff x \in W_n$. Therefore $r(x) = x$ for all $x$ in $Y$.

(b) $r^{-1}(W_n) = W_n$ for all $n$. Since $\{W_n : n \in \mathbb{IN}\}$ is a base for $Y$ this implies that $r$ is continuous.

We have thus proved that $r : X \to Y$ is a retraction map. Now from proposition 1.2(d), there is a continuous self-map $f : Y \to Y$ without periodic points. Then $f \circ r : X \to Y$, regarded as a map from $X$ to $X$, has no periodic points at all. Thus $X$ does not have p.p.p.

**Step 6:** Combining step 4 and step 5, we see that (2) implies (1). Now let $X$ satisfy (3): every closed subspace of $X$ has p.p.p. Then first, $X$ has to be compact, because otherwise, the infinite discrete space $\mathbb{IN}$ is homeomorphic to some closed subset, that does not have p.p.p. by proposition 1.2(a). Next, $X$ has to be countable also, since otherwise, by a known classical result (see [14]), $X$ will contain a closed set homeomorphic to the Cantor set $K$, that does not have p.p.p. by proposition 1.2(d). These together prove that (3) implies (1).

**Step 7:** Lastly, we prove that (4) implies (2). Let $X$ be a metric space every continuous image of which has p.p.p. We consider two cases.
Case 1: Suppose there is some \( x_0 \) in \( X \) and some \( \alpha > 0 \) in \( \mathbb{R} \) such that for every \( ft \) in \([0,\alpha]\), there is \( y \) in \( X \) such that \( d(x_0, y) = ft \). Then the function \( f : X \to S^1 \) defined by \( f(x) = \exp(i \cdot \frac{\pi}{\alpha} \cdot d(x_0, x)) \) is continuous and surjective. Thus \( S^1 \) becomes a continuous image of \( X \). But by proposition 1.2(6), \( S^1 \) does not have p.p.p. Therefore this case cannot arise. The only possibility is the next case.

Case 2: For every \( x \) in \( X \) and for every \( \alpha > 0 \) in \( \mathbb{R} \), there exists \( ft \) in the open interval \((0,\alpha)\) such that \( ft \) is not of the form \( d(x, y) \) for any \( y \) in \( X \). Then the open ball \( B(x, ft) \) is also the closed ball \( B(x, ft) \), and is therefore a clopen set inside \( B(x, \alpha) \). Thus \( X \) is zero-dimensional.

**Remark 1:** We proved in proposition 1.2 that neither the discrete space \( \mathbb{N} \) nor the Cantor set \( K \) has p.p.p. In a sense, we can say that these two are the main culprits preventing a strong version of p.p.p. in a metric space. This is because each of the four statements in the main theorem is equivalent to each of the following:

5. Neither \( \mathbb{N} \) nor \( K \) is homeomorphic to a closed subspace of \( X \).

6. \( X \) is zero-dimensional and neither \( \mathbb{N} \) nor \( K \) is a continuous image of \( X \).

**Remark 2:** We proved in step 1 that every countable compact space has p.p.p. Essentially the same ideas can be used to prove the stronger result that every compact scattered space has p.p.p. A space is said to be scattered if every subset of it contains a point that is isolated in its relative topology. There is no need to use Baire category theorem while imitating that proof. In fact, the following result holds: A compact space \( X \) is scattered if and only if every continuous image of \( X \) has p.p.p. For proving the second half, we use the following result of Rudin: If \( X \) is a compact non-scattered space, then the closed interval \([0, 1]\) is a continuous image
of $X$ (see [30]). For proving the first half, we need the result that every continuous image of a compact scattered space is again so (see [21]).

**Remark 3:** In §1 we stated a theorem of Mazurkiewicz and Sierpinski. This theorem, combined with proposition 1.1(d), gives another proof of the result of step 1. The advantage in this proof is that we do not use Baire category theorem. The disadvantage is that we use the theorem of [25] that is not as popular as Baire category theorem.

**Remark 4:** After reading §2, one may naturally seek a characterization of all metric spaces that satisfy p.p.p. But no neat characterization of this class is expected, because one can prove that this class is closed neither under continuous images, nor under closed subspaces, nor under arbitrary products. There are haphazard examples provided by Proposition 1.2(c). J.de Groot [10] has proved a surprising result that the Euclidean plane $\mathbb{R}^2$ contains many strongly rigid subspaces. These are neither compact nor countable. These are neither zero-dimensional nor path-connected. For more examples of strongly rigid spaces, see [20]. Every space is a closed subspace of one such space.

**Remark 5:** We leave this question open: What are all the Hausdorff spaces, every closed subspace of which has p.p.p.? Our main theorem answers this question among metric spaces. One can prove that every compact scattered space has the property that every closed subspace has p.p.p. But the converse is not true.

**Remark 6:** (a) The method of step 5 of §2 can be used to prove the following stronger results: (1) In every zero-dimensional Hausdorff space $X$, every compact metrizable subspace is a retract. (2) Every retract of every space with p.p.p, again
(b) The idea of proof of proposition 1.2(c) can be used to prove that for a family
\( \{Y_n : n \in \mathbb{N}\} \) of topological spaces, if \( \prod Y_n \) has p.p.p., then each \( Y_n \) has p.p.p.
and further there is \( n_0 \) in \( \mathbb{N} \) such that for every \( n > n_0 \), the period of every periodic
point of every continuous self-map of \( Y_n \) is \( < n_0 \). (i.e. there is a uniform bound \( n_0 \)
for these periods, eventually).

(c) In Step 7 of §2 we have actually proved that if a metric space \( X \) does not
admit \([0,1]\) as a continuous image, then \( X \) is zero-dimensional. This result can
be supplemented with a companion result. Every compact Hausdorff space \( X \) is
either zero-dimensional or admits \([0,1]\) as a continuous image. For, if the compact
space \( X \) is not zero-dimensional, a known result says that there exists a connected
subset of \( X \) containing two distinct elements \( x, y \). Then there exists a continuous
\( f : X \to [0,1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \). Since \( f(A) \) has to be connected,
\( f \) is onto.

Remark 7: The main theorem implies that the empty set cannot be the set
of all periods of any continuous self-map of any countable compact space. It is
natural to ask which subsets \( A \) of \( \mathbb{N} \) arise as the set of all periods for some such
function? This question has been completely answered by us, by describing the
family of all possible period sets, for each countable compact space. The proof in-
volves known results on the structure of such spaces and will appear in next chapter.

Proposition 2: Let \( X \subset \mathbb{R} \) be a compact set
Such that \( X \setminus (\text{int } X) \) is countable. Then \( X \) has p.p.p.

Proof: Let \( Y = X \setminus (\text{int } X) \). Let \( f : X \to X \) be continuous function. For each \( y \)
in $Y$, let
\[ g(y) = \begin{cases} 
\{f(y)\} & \text{if } f(y) \in Y \\
\{a_y, b_y\} & \text{if } f(y) \text{ belongs to the component interval } [a_y, b_y] \text{ of } X.
\end{cases} \]
Then $g$ is a multifunction from $Y$ to $Y$.

We first prove that $g$ is lower semicontinuous.

(Note: By a multifunction $\varphi$ from a topological space $X$ to $Y$ we mean that $\varphi(x)$ is a nonempty subset of $Y$ for each $x$ in $X$. A multifunction $\varphi : X \to Y$ is called lower semicontinuous if for every open subset $V$ in $Y$ the set \( \{x \in X | \varphi(x) \cap V \text{ is nonempty} \} \) is open in $X$.)

Let $(a, b)$ be any open interval in $\mathcal{R}$.

\[
\{y \in Y | g(y) \cap (a, b) \text{ is nonempty} \} = \begin{cases} 
f^{-1}((a, b) \cap Y) \cup S & \text{empty if (a,b) is disjoint from Y} \\
\text{Union of } f^{-1}(a, b) \text{ and sets of the form } f^{-1}(a_y, b_y) & \text{where either } a_y \text{ or } b_y \text{ belongs to } (a, b)
\end{cases}
\]

where $S = \{y \in Y | \text{either } a < a_y < b \text{ or } a < b_y < b \text{ where } f(y) \in (a_y, b_y)\}$

In all the cases it is an open set. Now we apply the following selection theorem of Michael in [26]: Let $Y$ be a zero-dimensional complete metric space.

Let $g : Y \to Y$ be a l.s.c. multifunction. Then $g$ admits a continuous selection.

By this, we obtain a continuous function $f : Y \to Y$. We have thus proved:

Let $X \subset \mathbb{R}$ be a closed set. Let $f : X \to X$ be continuous. Let $Y$ be the boundary of $X$. Then 3 continuous $f : Y \to Y$ such that

(a) $f(y) = f(y)$ whenever $f(y) \in Y$

(b) $f(y)$ and $f(y)$ always (i.e., $\forall y \in Y$) lie in the same component of $X.$
We next claim that if \( y \) is a periodic point of \( f \), then some element in the \( X \)-component of \( y \) must be a periodic point of \( f \). Let \( (f)^n(y) = y \). Then \( f^n(y) \) \( \subseteq \) the component of \( y \). [because, \( f \) takes components inside components]. Then

\[
f^n(\text{component of } y) \subseteq \text{component of } y.
\]

Therefore \( f^n \) should have some fixed point. This point is a periodic point of \( f \).

Thus we have proved that any compact subset of \( \mathbb{R} \) with countable boundary has p.p.p. In the next section we deal with compact subsets of \( \mathbb{R} \) with uncountable boundary.

\[\text{§3}\]

**Lemma 3:** Let \( X, Y \) be two closed subsets of \( \mathbb{R} \) such that \( Y \subseteq X \). Then the following are equivalent:
1. \( Y \) is a retract of \( X \).
2. \( a, b \in Y \) and \( [a, b] \subseteq X \) implies that \( [a, b] \subseteq Y \).

**Proof:** (1) \( \Rightarrow \) (2): Let \( r : X \to Y \) be a retraction. Assume that \( a \) and \( b \) are elements of \( Y \) and let \( [a, b] \subseteq X \). Then \( r(a) = a \) and \( r(b) = b \). Also \( r([a, b]) \) should be a connected set containing \( r(a) \) and \( r(b) \). Therefore it contains \( [a, b] \). Thus \( [a, b] \subseteq Y \).

(2) \( \Rightarrow \) (1): Let (2) hold. Then no bounded component of \( Y^c \) is contained in \( X \). Choose some \( x_n \) in each bounded component \( (a_n, b_n) \) of \( Y^c \) such that \( x_n \notin X \).

Now define \( r : X \to Y \) as \( r(x) =\) \[
\begin{cases} 
  x & \text{if } x \in Y \\
  a_n & \text{if } a_n < x < x_n \\
  b_n & \text{if } x_n < x < b_n 
\end{cases}
\]

This \( r \) is continuous. Hence \( Y \) is retract of \( X \).
**Corollary 3.1:** Let $X$ be a closed subset of $\mathbb{R}$ with empty interior. Then every closed subset of $X$ is a retract of $X$.

**Corollary 3.2:** Let $X$ be a zero dimensional metric space. Then every closed subset of $X$ is a retract of $X$. (Since $X$ is homeomorphic to a subspace of $\mathbb{R}$)

**Corollary 3.3:** A compact metric space $X$ is zero dimensional if and only if every closed subset of $X$ is a retract of $X$.

**Theorem 2:** Let $X$ be a compact subset of $\mathbb{R}$ such that its boundary is uncountable. Then $X$ does not have p.p.p.

**Proof:** The proof is divided into five major steps.

**Step 1:** First, we fix some notations. $dX$ is the boundary of $X$. It is equal to $X \setminus$ (interior of $X$).

$Y = \{ a \in dX \setminus$ every neighbourhood of $a$ contains uncountably many elements of $\partial X \}.$

$Z = Y \cup$ (union of all those components of $X$ whose boundary is contained in $Y$).

$A = \{ a \in Y \setminus a$ has an immediate successor $a^+$ in $Y$ and $[a, a^+] \subseteq Z \}.$

$B = \{ b \in Y \setminus b$ has an immediate successor $b^+$ in $Y$ and $[b, b^+] \cap Z$ is empty \}.$

The following can be proved easily.

**Proposition 3.1:**

1. $X$, $dX$, $Y$ and $Z$ are compact and uncountable.

2. $\partial Z = Y.$
Chapter 2. Periodic Point Property

3 A and B are countable.

4 Between any two non-adjacent elements of A lies an element of B.

5 If \( y_1 < y_2 \) in \( Y \), \( \exists c \in A \cup B \) such that \( y_1 < c < c' < y_2 \).

Step 2: Now we introduce some more notations. \( A = \{a_1, a_2, \cdots, a_n, \cdots\} \) is either finite or countable.

\( B = \{b_1, b_2, \cdots, b_n, \cdots\} \) is countably infinite.

\( J_1 = \{0, 1\} \).

\( B_0 = \{y \in Y \mid y \leq b_1\} \).

\( B_1 = \{y \in Y \mid y > b_1\} \).

\( \alpha_1 = \) that element of \( J_1 \) such that \( a_1 \in B_{\alpha_1} \).

\( J_1 = J_1 \cup \{\alpha_1^*\} \) where \( \alpha_1^* \) is just a symbol corresponding to \( \alpha_1 \). Suppose we have defined for some positive integer \( n \), the index set \( J_n \), the clopen subsets \( B_\alpha \) of \( X \) for each \( \alpha \in J_n \) the special element \( \alpha_n \) of \( J_n \) determined by \( a_n \) and the index set \( J_n \). Then we define

\( J_{n+1} = \{\alpha_0 \mid \alpha \in J_n\} \)

\( B_{\alpha_0} = \{y \in B_\alpha \mid y < b_{n(\alpha)}\} \).

\( B_{\alpha_1} = \{y \in B_\alpha \mid y > b_{n(\alpha)}\} \).

\( n(\alpha) = \) the least positive integer \( \exists b_{n(\alpha)} \) and \( b_{n(\alpha)}^+ \) are both in \( B_\alpha \).

\( \alpha_{n+1} = \) that element of \( J_{n+1} \) such that \( a_{n+1} \in B_{\alpha_{n+1}} \).

\( J_{n+1} = J_{n+1} \cup \{\alpha_{n+1}^*\} \) We let \( J_{n+1} = J_{n+1} \) if \( |A| < n \).

The following can be proved without difficulty

Proposition 3.2:

6 \( |J_n| = 2^{n+1} - 1 \) for all \( n = 1, 2, \cdots \) if \( |\mathbb{A}| > n \)
7 Each $B_{\alpha}$ is a clopen subset of $Y$; it is called a block of $n$th level if $\alpha \in J_n$.

8 For every fixed $n$

$$\{B_{\alpha}|\alpha \in J_n\}$$

is a partition of $Y$.

9 Every block of $n + 1$th level is contained in some block of $n$th level;

10 $a_n$ and $a^+\!$ belong to different blocks on $n$th level; similarly, $b_n$ and $b^+\!$ belong to different blocks of $n$th level.

11 Each block $B_{\alpha}$ is of the form $Y \cap I$ for some closed interval $I$.

12 Let $J = \bigcup_{n=1}^{\infty} J_n$. Then $\{B_{\alpha}|\alpha \in J\}$ is a base for the topology of $Y$.

**Step 3:** In this step we define for each $n = 1, 2, \cdots$, a function $\sigma_n : J_n \rightarrow J_n$, which will be used in the next step to define a function $f : Y \rightarrow Y$.

Initially, $\sigma_1 : J_1 \rightarrow J_1$ is defined by letting $\sigma_1(0) = 1$ and $\sigma_1(1) = 0$. Next, it is extended to $\sigma_1 : J \rightarrow J_1$, by letting $\tilde{\sigma}_1(\alpha_1^+) = \sigma_1(\alpha_1)$.

Suppose we have defined for some positive integer $n$, the functions $\sigma_n : J_n \rightarrow J_n$ and $\sigma_n : J_n \rightarrow J_n$. Then we define $\sigma_{n+1} : J_{n+1} \rightarrow J_{n+1}$ by defining

$$\sigma_{n+1}(\alpha 0) = \begin{cases} \beta 0 & \text{if } \beta = \sigma_n(\alpha) \neq 0 \cdots 0 \\ \beta 1 & \text{if } \beta = \sigma_n(\alpha) = 0 \cdots 0 \end{cases}$$

$$\sigma_{n+1}(\alpha 1) = \begin{cases} \beta 1 & \text{if } \beta = \sigma_n(\alpha) \neq 0 \cdots 0 \\ \beta 0 & \text{if } \beta = \sigma_n(\alpha) = 0 \cdots 0 \end{cases}$$

This defines $\sigma_{n+1}$ on the first $2^{n+1}$ elements of $J_{n+1}$. We define it at the other elements suitably so that the following two conditions are satisfied:

$$\{\sigma_{n+1}(\alpha 0), \sigma_{n+1}(\alpha 1)\} \subset \{\beta 0, \beta 1\} \text{where } f_i = \sigma_n(\alpha)$$

$$\sigma_{n+1}(\alpha_m^* 1 1 \cdots 1) = \sigma_{n+1}(\alpha_m 0 0 \cdots 0) \text{ for every } m < n$$
Finally we extend \( \sigma_{n+1} \) to \( \tilde{\sigma}_{n+1} : \tilde{J}_{n+1} \to \tilde{J}_{n+1} \) by defining

\[
\tilde{\sigma}_{n+1}(\alpha_{n+1}^n) = \sigma_{n+1}(\alpha_{n+1})
\]

The following can be proved without much difficulty.

**Proposition 3.3:**

13 The range of \( \sigma_n \) has exactly \( 2^n \) elements. These elements are precisely those words of \( J_n \) that do not involve the *-symbol.

14 Every element in the range of \( \sigma_n \) is a periodic point of period \( 2^n \).

15 These \( \sigma_n \)'s are mutually compatible in the sense that the following diagram commutes for each \( m < n \):

\[
\begin{array}{ccc}
\tilde{\sigma}_n & : & \tilde{J}_n \\
\downarrow & & \downarrow \tilde{r} \\
J_m & \to & J_m \\
\sigma_m & : & \sigma_m
\end{array}
\]

Here \( \tilde{r} = J_n \to J_m \) is the restriction map that associates to each word \( \alpha \) in \( J_n \), the truncated word in \( J_m \) omitting all but the first \( m \) letters of \( \alpha \).

**Step 4:** In this step, we define a function \( f : Y \to Y \) using the \( \tilde{\sigma}_n \)'s constructed in step 3. But this requires some preparatory result:

16 Any strictly decreasing sequence of blocks \( \{B_{\alpha_n}\} \) has the property that the intersection \( \bigcap_{n=1}^{\infty} B_{\alpha_n} \) is a singleton. [To prove this, we use the results (5) and (10)].
We now fix one more notation. For each $x$ in $Y$ and for each positive integer $n$, let $\alpha(x,n)$ denote the unique $a$ in $\sigma_n$ such that $y \in B_{\alpha(x,n)}$. [Here we use the result (8)].

We define $f : Y \to Y$ by the rule

$$f(x) = \text{the unique element of } \bigcap_{n=1}^{\infty} B_{\alpha(x,n)}.$$

**Proposition 3.4:**

17. $f(B_\alpha) \subseteq B_{\tilde{\sigma}(n)}$ for each $\alpha$ in $J_n$, by definition of $f$.

18. $f^{-1}(B_{\tilde{\sigma}}) \cup \{B_\beta | \beta \in J_n, \tilde{\sigma}_n(\beta) = \alpha\}$ for each $\alpha$ in $J_n$.
   - a finite union of blocks.
   - a clopen set.

19. $f$ is continuous. [Here, we use the result (12)].

20. If $x$ is a periodic point of $f$ and if $n$ is a positive integer, then $\alpha(x,n)$ is a periodic point of $\sigma_n$; moreover the $f$-period of $x$ divides the $\tilde{\sigma}_n$-period of $\alpha(x,n)$.

21. $f$ has no periodic points. [Here we use the results (20) and (14)].

22. $f(a_n) = f(a^+) \text{ for } n = 1, 2, \ldots$

**Step 5:** In this step, we extend the function $f$ defined in step 4. First, we note that $Z = Y \cup \left( \bigcup_{n=1}^{\infty} [a_n, a_n^+] \right)$

We define $\hat{f} : Z \to Z$ by the rule

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ f(a_n) & \text{if } a_n \leq x \leq a_n^+ \end{cases}$$
[We use the result (22) to prove that this definition does give a function]

**Proposition 3.5:**

23 $/ \text{ is continuous.}$

24 $\text{Range of } / \text{ is same as the range of } / \text{ and is contained in } Y.$

25 $/ \text{ has no periodic points. [Here, we use the result (21)].}$

26 $Z \text{ has the property that}$

$$\begin{align*}
  x, y & \in Z \\
  [x, y] & \subset Z.
\end{align*}$$

27 $Z \text{ is a retract of } X. \text{ [Here we use the Lemma 3].}$

[Note however that $Y$ is not in general a retract of $Z$ or $X].$

As the last step in the proof, we extend $/$ to a continuous function $f^* : X \to Z$
(using the result (27)). Indeed the range of $f^* = \text{range of } / = \text{range of } /.$

One can prove:

28 $\text{This } f^* : X \to X \text{ has no periodic points at all.}$

[Here, we use the result (25)].