5. CHARACTERIZATIONS FOR THE
LAPLACE-MARCHI-ZGRABLICH INTEGRAL TRANSFORM OF
DISTRIBUTIONS

5.1 Introduction

Gesztelyi, E. [24] defined two classes of transformations for ordinary function $f(u)$ viz. dilatation $\mathcal{U}_a$, where $a = 1, 2, 3, \ldots$ and exponential shift $\mathcal{T}^{-\nu}$, where $\nu$ is a complex number, as follows,

\begin{align*}
\mathcal{U}_a f(u) &= a f(au), \\
\mathcal{T}^{-\nu} f(u) &= e^{-\nu u} f(u).
\end{align*}

(5.1.1) (5.1.2)

Gesztelyi shows that whenever the sequence $\{\mathcal{U}_a f(u)\}$ converges (in the sense of Mikusinski convergence [35]), the limit is necessarily a complex number. In addition, he proves that if $f(u)$ is a function which has a Laplace transform at $\nu$, then the sequence of functions $\{\mathcal{U}_a \mathcal{T}^{-\nu} f(u)\}$ converges (in the Mikusinski sense) as $a \to \infty$ to the classical Laplace transform of $f(u)$ at $\nu$. He then defines the Laplace transform of Mikusinski operator $\nu$ as the limit (whenever it exists in the sense of Mikusinski convergence) of the sequence $\{\mathcal{U}_a \mathcal{T}^{-\nu} \nu\}$, and shows that this definition generalizes the previous formulations of the Laplace transform for Mikusinski operators of Doetsch, G. [17] and Ditkin, V.A. [15], [16]. Working on the same line, Price, D.B. [39], introduced the Laplace transform of a distribution $f(u)$ using the
sequence of the form \( \{ U_n \mathcal{T}^-^p f(u) \} \) and showed that the new definition is equivalent to the Schwartz’s extension of the transform to distributions.

For complex numbers \( p \in \mathbb{C}^n \) and positive integers \( a > 0 \) in \( \mathbb{R}^n \), Price, D.B. [39] considered transformations; dilatation \( U_n \) and exponential shift \( \mathcal{T}^-^p \) on the space \( \mathcal{D}'(\mathbb{R}^n) \) for every distribution \( f \in \mathcal{D}'(\mathbb{R}^n) \), as follows:

\[
\langle U_a f(u), \phi(u) \rangle = \langle \{ a_1, a_2, \ldots, a_n \} f(a_1 u_1, a_2 u_2, \ldots, a_n u_n), \phi(u_1, u_2, \ldots, u_n) \rangle
\]

(5.1.3)

and

\[
\langle \mathcal{T}^-^p f(u), \phi(u) \rangle = \langle e^{-p u} f(u), \phi(u) \rangle = \left\langle \exp \left( - \sum_i p_i u_i \right) f(u_1, u_2, \ldots, u_n), \phi(u_1, u_2, \ldots, u_n) \right\rangle
\]

(5.1.4)

for every test function \( \phi(u) \). Here \( a > 0 \) means \( \{ a_i > 0, \; i = 1, 2, \ldots, n \} \). Both the transformations \( U_n \) and \( \mathcal{T}^-^p \) are continuous and linear on \( \mathcal{D}'(\mathbb{R}^n) \).


In our paper [48], we have introduced Laplace-Marchi-Zgrablich integral transform \( F(s, \mu_\nu) \) of generalized functions \( f(x, t) \in \mathcal{L}M\mathcal{Z}'_{a,b,m,\nu}(I) \) for \( \nu \geq (-1/2) \). Now we rearrange Laplace-Marchi-Zgrablich integral transform as,

\[
F(s, \mu_\nu) = \mathcal{L}M \{ f(x, t) \} = \langle f(x, t), x e^{-\mu x} S_\nu(k_1, k_2, \mu x) \rangle
\]

(5.1.5)

where,
1. \( s = \sigma + i\omega \in \Omega_f = \{ s/ \rho_f < \text{Re}(s) < \sigma_f \} \).

2. \( S_r(k_1, k_2, \mu_n x) \) as defined in equation (1.5.3), where \( \mu_n \) are positive roots of the transcendental equation (1.5.4).

3. \( \mathcal{LM}^\prime_{a,b,m,v}(I) \) to be the dual space of the testing function space \( \mathcal{LM}_{a,b,m,v}(I) \) as defined in [48].

In this chapter we give two new characterizations of the Laplace-Marchi-Zgrablich transform for generalized functions with the help of dilatation \( U_a \) and exponential shift \( T^{-p} \). For each distribution \( f(x,t) \in D(I) \) and test function \( \phi(x,t) \in D(I) \) defined over \( I = \{(x,t)/ x \in (p,q), t \in (0,\infty)\} \subset \mathbb{R}^2 \), we define these two transformations on \( D(I) \) as follows:

If \( a = (a_1, a_2) > 0 \) in \( \mathbb{R}^2 \), the linear transformation dilatation \( U_a \) on \( D(I) \) is defined as,

\[
\langle U_a f(x,t), \phi(x,t) \rangle = \langle \{a_1, a_2\} f(a_1 x, a_2 t), \phi(x,t) \rangle = \bigg\langle f(x,t), \phi\left(\frac{x}{a_1}, \frac{t}{a_2}\right)\bigg\rangle. \quad (5.1.6)
\]

Here \( a > 0 \) implies \( a_i > 0, \ i = 1, 2 \).

For each \( p = (p_1, p_2) \in \mathbb{C}^2 \), let \( T^{-p} \) be defined by

\[
\langle T^{-p} f(x,t), \phi(x,t) \rangle = \bigg\langle e^{-(p_1 x + p_2 t)} f(x,t), \phi(x,t) \bigg\rangle. \quad (5.1.7)
\]

These two transformations are linear and continuous on \( D(I) \).

For, we consider space \( \mathcal{B}_0(I) \) which is dual of \( \mathcal{B}_0(I) \), a subspace of \( D(I) \) as defined in [39, Price, D. B.]. The distributions in \( \mathcal{B}_0(I) \) are frequently called integrable distributions. Now for \( a \) and \( b \) in \( I \), \( a < b \) means \( a_i < b_i, \ i = 1, 2 \) and \( e^{au} \) is the function \( e^{a_1 x + a_2 t} \), where \( au = a_1 x + a_2 t \) and \( u = (x,t) \). Let \( \mathbb{C}^2 \) denote the two dimensional
linear space of complex numbers.

**Theorem 5.1.1.** For \( u = (x, t) \in \mathbb{R}^2 \), let \( f(u) \in \mathcal{D}'(I) \). Let \( a = (a_1, a_2) \), \( b = (b_1, b_2) \) be in \( I \) such that \( a_1 \) and \( b_1 \) are in \( (p, q) \), \( a_2 \) and \( b_2 \) are in \( (0, \infty) \) and \( a_1 < b_1, a_2 < b_2 \). If for such \( a \) and \( b \), \( e^{-au}f(u) \) and \( e^{-bu}f(u) \) are both in \( \mathcal{S}' \) (the space of distribution of slow growth), then for every \( \nu \in \mathbb{C}^2 \) with \( a < Re(\nu) < b \), \( e^{-\nu u}f(u) \) is in \( \mathcal{B}_0(I) \), where

\[
\nu = \nu_1 + \nu_2 = -\left[ \frac{1}{x} \log(xS_v(k_1, k_2, \mu_n x)) \right] - s
\]

for \( x \neq 0 \) and \( \nu_1 = -\left[ \frac{1}{x} \log(xS_v(k_1, k_2, \mu_n x)) \right], \nu_2 = -s \).

**Proof:** Let \( \nu \) be in \( \mathbb{C}^2 \) with \( a < Re(\nu) < b \), and let \( \epsilon \) be in \( I \) with \( \epsilon > 0 \) such that

\[
\epsilon_i < \min \{ Re(\nu_i) - a_i, b_i - Re(\nu_i) : i = 1, 2 \}.
\]

If \( \lambda(u) = e^{eu} + e^{-eu} \), then \( \lambda(u)e^{-\nu u}f(u) \) is in \( \mathcal{S}' \) and \( 1/\lambda(u) \) is in \( \mathcal{S} \) (the space of testing functions of rapid descent on \( I \)). Also, for every \( \phi(u) \in \mathcal{B}_0(I) \), \( \left( \frac{\phi}{\lambda} \right)(u) \) is in \( \mathcal{S} \), so we may write

\[
\langle e^{-(\nu_1 x + \nu_2 t)} f(x, t), \phi(x, t) \rangle = \lambda(x, t)e^{-(\nu_1 x + \nu_2 t)} f(x, t), \left( \frac{\phi}{\lambda} \right)(x, t) \quad \text{as} (x, t) \in \mathbb{R}^2.
\]

This expression clearly identifies \( e^{-\nu u}f(u) \) as a continuous linear transformation on \( \mathcal{B}_0(I) \) as long as \( a < Re(\nu) < b \), so the theorem is proved.

**Theorem 5.1.2.** If \( f(x, t) \) and \( g(x, t) \) are in \( \mathcal{B}_0(I) \), then their convolution can be defined as

\[
\langle f(x, t) \ast g(x, t), \phi(x, t) \rangle = \langle f(x, t), (\check{g} \ast \phi)(x, t) \rangle,
\]

and it is in \( \mathcal{B}_0(I) \), where \( \langle \check{g}(x, t), \phi(x, t) \rangle = \langle g(x, t), \phi(-x, -t) \rangle \).
Proof: If \( f(x, t) \) is in \( \mathcal{D}(I) \), then \( \hat{f}(x, t) \) is the distribution defined for every \( \phi(x, t) \in \mathcal{D}(I) \) as,
\[
\langle \hat{f}(x, t), \phi(x, t) \rangle = \langle f(x, t), \phi(-x, -t) \rangle.
\]

Using the tensor product \( \otimes \) we formally defined \( f(x, t) \ast g(x, t) \) as,
\[
\langle f(x, t) \ast g(x, t), \phi(x, t) \rangle = \langle f(x, t) \otimes g(x, t), \phi(\zeta + x, t + \tau) \rangle
\]
\[
= \langle f(x, t), \langle g(\zeta, \tau), \phi(\zeta + x, t + \tau) \rangle \rangle
\]
\[
= \langle f(x, t), \langle \hat{g}(\zeta, \tau), \phi(x - \zeta, t - \tau) \rangle \rangle
\]
\[
= \langle f(x, t), (\hat{g} \ast \phi)(x, t) \rangle.
\]

We will define the convolution \( f(x, t) \ast g(x, t) \) as a distribution in \( \beta_0(I) \) if we can show that \( \hat{g}(x, t) \ast \phi(x, t) \) is in \( \beta(I) \) when \( \phi(x, t) \) is in \( \beta_0(I) \) and \( g(x, t) \) is in \( \beta_0(I) \), and that \( \hat{g}(x, t) \ast \phi(x, t) \) converges to zero in \( \beta(I) \) whenever \( \phi_k(x, t) \) converges to zero in \( \beta_0(I) \). To do this, consider,
\[
\sup_I |\partial^j (\hat{g}(x, t) \ast \phi(x, t))| = \sup_I |\partial^j \langle \hat{g}(\zeta, \tau), \phi(x - \zeta, t - \tau) \rangle|
\]
\[
= \sup_I |\partial^j \langle \hat{g}(\zeta, \tau), \phi(x + \zeta, t + \tau) \rangle|
\]
\[
= \sup_I \left| \langle \hat{g}(\zeta, \tau), \phi^{(j)}(x + \zeta, t + \tau) \rangle \right|
\]
\[
\leq \sup_I K \max_{|\alpha| \leq K} \sup_{(\zeta, \tau)} |\partial^j \phi^{(j)}(x + \zeta, t + \tau)|
\]
\[
= K \sup_I \max_{|\alpha| \leq K} |\phi^{(j)}(x, t)|
\]
\[
= B_{K,j},
\]

where \( K \) is the constant defined for \( g(x, t) \) as per Theorem 2.1 [39, Price, D.B.].
Therefore $\tilde{g}(x, t) \ast \phi(x, t)$ is in $\beta_0(I)$ and if $\phi_k(x, t)$ converges to zero in $\beta_0(I)$, then
\[
\sup_I |\phi_k^{(i+j)}(x, t)|
\]
converges to zero, and $\tilde{g}(x, t) \ast \phi_k(x, t)$ must converge to zero in $\beta(I)$. Thus
\[
\langle f(x, t) \ast g(x, t), \phi(x, t) \rangle = \langle f(x, t), (\tilde{g} \ast \phi)(x, t) \rangle.
\]
defines $f(x, t) \ast g(x, t)$ as a distribution in $\beta'_0(I)$, and the theorem is proved. □

We know that $\mathcal{B}_0(I)$ is a subset of $\mathcal{S}'$ since $\mathcal{S} \subset \mathcal{B}_0(I)$ and $\mathcal{S}$ is dense in $\mathcal{B}_0(I)$ with respect to the topology of $\mathcal{B}_0(I)$.

5.2 The transformation $\mathcal{U}_a$ and $\mathcal{T}^{-\nu}$

In this section we are concerned with the limit of the sequence of distributions $\{\mathcal{U}_a \mathcal{T}^{-\nu} f(x, t)\}$, where ‘$j$’ represents multi-index of order two, $j = (j_1, j_2)$. Let $j \to \infty$ means that both $j_1 \to \infty$, $j_2 \to \infty$ independent of each other. If $\{f_j(x, t)\}$ is a “sequence” of distribution in $\mathcal{D}'(\mathbb{R}^2)$ then the statement
\[
\lim_{j \to \infty} f_j(x, t) = h(x, t)
\]
means that if $\phi(x, t)$ is in $\mathcal{D}(\mathbb{R}^2)$ and $\epsilon > 0$, then there is a positive integer $N$ such that whenever $j_k \geq N$ for every $k = 1, 2$, then
\[
\left| \langle f_j(x, t), \phi(x, t) \rangle - \langle h(x, t), \phi(x, t) \rangle \right| < \epsilon.
\]

On the lines of [39, Price, D.B.], we state following theorems, corollaries and remarks for our case:
Theorem 5.2.1. If \( f(x, t) \) is in \( \mathcal{B}_0'(I) \), then

\[
\lim_{j \to \infty} U_j f(x, t) = \langle f(x, t), 1 \rangle \delta
\]

where the limit is taken in \( \mathcal{D}'(I) \).

Theorem 5.2.2. If \( h(x, t) \) is in \( \mathcal{D}'(I) \), then \( U_j h(x, t) = h(x, t) \) for every positive multi-index \( j \) if and only if

\[
h(x, t) = \sum_{\nu=1}^{4} c_\nu \left( \bigotimes_{i \in I_\nu} p.v. \frac{1}{\theta_i} \right) \otimes (\bigotimes_{i \in I_\nu} \delta(\theta_i))
\]

for some constants \( c_\nu, 1 \leq \nu \leq 4; i = 1, 2; \theta_1 = x, \theta_2 = t \).

Remark 5.2.1. The theorem says that any distribution \( h(x, t) \) in \( \mathcal{D}'(I) \) which is invariant under each \( U_j \) is a linear combination of four terms, each of which is a tensor product of two one-dimensional distributions of the form \( \delta(\theta_i) \) or \( p.v.(1/\theta_i) \).

In particular,

\[
h(x, t) = c_1 p.v.\frac{1}{x} \otimes p.v.\frac{1}{t} + c_2 p.v.\frac{1}{x} \delta(t) + c_3 \delta(x)p.v.\frac{1}{t} + c_4 \delta(x) \otimes \delta(t).
\]

By observing that \( h(x, t) = \lim_{j \to \infty} U_j f(x, t) \) for some distribution \( f(x, t) \) in \( \mathcal{D}'(I) \) if and only if \( U_j h(x, t) = h(x, t) \) for every \( j = (j_1, j_2) \), we get an important corollary to Theorem (5.2.2).

Corollary 5.2.1. If \( h(x, t) \) is in \( \mathcal{D}'(I) \), then

\[
h(x, t) = \lim_{j \to \infty} U_j f(x, t)
\]

for some distribution \( f(x, t) \) in \( \mathcal{D}'(I) \) if and only if there exists constants \( c_\nu, 1 \leq \nu \leq 4 \).
4, such that
\[ \Phi(t) = \sum_{i=1}^{4} c_i \left( \otimes_{\nu \in \mathcal{I}} \left( p \nu \frac{1}{\theta} \right) \otimes (\otimes_{\nu \not\in \mathcal{I}} \delta(\theta)) \right). \]

**Theorem 5.2.3.** If \( f(x, t) \) is in \( \mathcal{D}'(I) \) and there are two complex numbers \( \omega_1, \omega_2 \) with \( \text{Re}(\omega_1) \neq \text{Re}(\omega_2) \) and a positive integer \( i = 1, 2 \) such that \( \{ \mathcal{U}_j e^{-\omega_1 t} f(x, t) \} \) and \( \{ \mathcal{U}_j e^{-\omega_2 t} f(x, t) \} \) both converges in \( \mathcal{D}'(I) \) as \( j \to \infty \), then for every complex number \( q \) for which the sequence converges, there is a distribution \( h(q) \) in \( \mathcal{D}'[a, b] \) or \( \mathcal{D}'[0, \infty] \) such that
\[
\lim_{j \to \infty} \mathcal{U}_j e^{-q t} f(x, t) = \delta(\theta) \otimes h(q). \tag{5.2.2}
\]

**Corollary 5.2.2.** If \( f(x, t) \in \mathcal{D}'(I) \) is such that \( \lim_{j \to \infty} \mathcal{U}_j f(x, t) = h(0, 0) \) and for each \( i = 1, 2 \), there is a complex \( \nu_i \) such that \( \text{Re}(\nu_i) \neq 0 \) and \( \lim_{j \to \infty} \mathcal{U}_j e^{-\nu_i t} f(x, t) = h(\nu_i) \) then there is a constant \( c \) such that \( h(0, 0) = c \delta(x, t) \).

**Corollary 5.2.3.** Let \( \Omega \) be an open set in \( \mathbb{C}^2 \) with the property that if \( p \) is in \( \Omega \) then the sequence \( \{ \mathcal{U}_j T^{-p} f(x, t) \} \) converges in \( \mathcal{D}'(I) \) to a distribution \( h(p) \) as \( j \to \infty \). Then for every \( p \) in \( \Omega \) there is a constant \( c(p) \) such that \( h(p) = c(p) \delta(x, t) \).

**Theorem 5.2.4.** If \( f(x, t) \) is distribution such that the sequence \( \{ \mathcal{U}_j f(x, t) \} \) converges in \( \mathcal{D}'(I) \) as \( j \to \infty \), then \( f(x, t) \) is in \( \mathcal{S}'(I) \).

**Corollary 5.2.4.** If \( \omega_1 \) and \( \omega_2 \) are in \( \mathbb{C}^2 \) with \( \text{Re}(\omega_1) < \text{Re}(\omega_2) \) and are such that \( \{ \mathcal{U}_j T^{-\omega_1} f(x, t) \} \) and \( \{ \mathcal{U}_j T^{-\omega_2} f(x, t) \} \) both converges in \( \mathcal{D}'(I) \) as the multi-index \( j \to \infty \) whenever \( p \) is in \( \mathbb{C}^2 \) with \( \text{Re}(\omega_1) < \text{Re}(p) < \text{Re}(\omega_2) \),
\[
\lim_{j \to \infty} \mathcal{U}_j T^{-p} f(x, t) = \langle T^{-p} f(x, t), 1 \rangle \delta.
\]
5.3 The Laplace-Marchi-Zgrablich Transform

We say that a distribution $f(x,t)$ in $D'(I)$ is Laplace-Marchi-Zgrablich transformable if there are two constants $\alpha$ and $\beta$ in $\mathbb{R}^2$ such that whenever

$$\nu = \nu_1 + \nu_2 = -\left[\frac{1}{x} \log(xS_\nu(k_1,k_2,\mu_nx))\right] - s$$

is in $C^2$ with $\alpha < \text{Re}(\nu) < \beta$, then $T^{-\nu}f(x,t)$ is in $\mathcal{B}_0(I)$. If for any other pair $\alpha'$ and $\beta'$ satisfying the same property, $\alpha' \geq \alpha$ and $\beta' \geq \beta$, then we call the subset of $C^2$

$$\Omega = \{\nu : \alpha \leq \text{Re}(\nu) \leq \beta\}$$

the domain of definition of the Laplace-Marchi-Zgrablich transform of the distributions $f(x,t)$. If $f(x,t)$ is a Laplace-Marchi-Zgrablich transformable distribution whose transform has domain of definition $\Omega$, then for any $\nu \in \Omega$, we define the Laplace-Marchi-Zgrablich transform of $f(x,t)$ at $\nu$ by

$$\mathcal{LM}\{f\}(\nu) = \mathcal{F}(\nu) = \frac{1}{\phi(0,0)} \lim_{j \to \infty} \langle \mathcal{U}_j T^{-\nu}f(x,t), \phi(x,t) \rangle$$  \hspace{1cm} (5.3.1)$$

where $\phi(x,t)$ is a test function in $D(I)$ with $\phi(0,0) \neq 0$. Theorem (5.2.1) guarantees the existence of the limit in equation (5.3.1). Thus we have one characterization as,

$$\mathcal{LM}\{f\}(\nu) = \langle T^{-\nu}f(x,t), 1 \rangle.$$  \hspace{1cm} (5.3.2)$$

By equation (5.3.2) we see that $\mathcal{LM}\{f\}$ is a complex valued function of the complex variable $\nu$ with domain $\Omega$. It also follows that the mapping $\mathcal{LM}$ is linear.
if \( f(x, t) \) and \( g(x, t) \) are distributions that are transformable at \( p \) and \( \alpha, \beta \) to be complex numbers, then \( \alpha f(x, t) + \beta g(x, t) \) is Laplace-Marchi-Zgrablich transformable at \( p \) and

\[
\mathcal{LM}\{\alpha f(x, t) + \beta g(x, t)\} = \langle T^{-p}[\alpha f(x, t) + \beta g(x, t)], 1 \rangle
\]

\[
= \alpha \langle T^{-p}f(x, t), 1 \rangle + \beta \langle T^{-p}g(x, t), 1 \rangle
\]

\[
= \alpha \mathcal{LM}\{f\} + \beta \mathcal{LM}\{g\}
\]

The next theorem shows that if \( f(x, t) \) is Laplace-Marchi-Zgrablich transformable in \( \Omega \) then \( \mathcal{LM}\{f\} \) is analytic function of \( p \) in \( \Omega \).

**Theorem 5.3.1.** If \( f(x, t) \in \mathcal{D}'(I) \) is Laplace-Marchi-Zgrablich transformable in \( \Omega \), then \( \mathcal{LM}\{f\} \) is analytic in \( \Omega \) and

\[
\frac{\partial}{\partial p_i} \mathcal{LM}\{f\}(p_i) = \mathcal{LM}\{-\theta_i f(x, t)\}(p_i),
\]

where \( \theta_1 = x \) and \( \theta_2 = t \).

**Proof:** Using Hartlog’s theorem [9, Bochner and Martin] which says that a complex valued function of \( n \) complex variables is analytic, if it is analytic in each variable separately with all other variables held constant. For, suppose \( \Omega = \{p_i : \alpha < \text{Re}(p_i) < \beta\} \), pick \( p_0 \) in \( \Omega \) and \( \epsilon \) in \( (0, 1) \) such that \( \epsilon < \min\{\text{Re}(p_0) - \alpha, \beta - \text{Re}(p_0)\} \).

If \( \zeta(t) = e^{\epsilon t} + e^{-\epsilon t} \), then \( 1/\zeta \) is in \( S \subset B_0(I) \) and \( \zeta T^{-p_0}f(x, t) \) is in \( B_0(I) \). Also, as long as \( |p_i - p_0| < \epsilon \), we have

\[
\frac{\mathcal{LM}\{f\}(p_i) - \mathcal{LM}\{g\}(p_i)}{p_i - p_0} = \left\langle \frac{e^{-\epsilon t} - e^{\epsilon t}}{p_i - p_0} f(x, t), 1(x, t) \right\rangle
\]
\[
\frac{t^2}{\zeta(t)} \sum_{j=2}^{\infty} \frac{[-(p_i - p_0)t]^{j-2}}{j!}
\]

is bounded in absolute value by the corresponding derivative of \(\frac{t^2}{\zeta(t)} e^{-(p_i - p_0)t}f(x, t)\), and is therefore in \(S'\). Thus as \(p_i \to p_0\),

\[
\lim_{p_i \to p_0} \frac{1}{\zeta(t)} e^{-t - p_0} f(x, t) - 1
\]

converge in \(B_0(I)\) to \(-t/\zeta(t)\), and we have,

\[
\frac{\partial}{\partial p_i} \mathcal{L}M\{f\}(p_0) = \lim_{p_i \to p_0} \frac{\mathcal{L}M\{f\}(p_i) - \mathcal{L}M\{g\}(p_i)}{p_i - p_0}
\]

\[
= \left\langle \zeta(t) e^{-(p_i - p_0)t}f(x, t), \frac{-t}{\zeta(t)} \right\rangle
\]

\[
= \left\langle T^{-p_0}[-t f(x, t)], 1(x, t) \right\rangle
\]

\[
= \mathcal{L}M\{-t f(x, t)\}(p_0).
\]

Similarly we can show that

\[
\frac{\partial}{\partial p_i} \mathcal{L}M\{f\}(p_0) = \mathcal{L}M\{-xf(x, t)\}(p_0)
\]

by keeping \(t\) constant. \(\square\)
Theorem 5.3.2. If \( f(x, t) \) and \( g(x, t) \) are Laplace-Marchi-Zgrablich transformable distributions in \( \mathcal{D}'(I) \) and the domains of their respective transforms has intersection \( \Omega \), then \( f(x, t) * g(x, t) \) is Laplace-Marchi-Zgrablich transformable in \( \Omega \) and for every \( p \) in \( \Omega \),

\[
\mathcal{L}M[f * g](p) = \mathcal{L}M[f](p)\mathcal{L}M[g](p).
\]

Proof: For \( p \) in \( \Omega \), \( T^{-p}f(x, t) \) and \( T^{-p}g(x, t) \) are both in \( \mathcal{B}_0'(I) \); so by theorem (5.1.2), \( T^{-p}f * T^{-p}g = T^{-p}(f * g) \). Therefore, \( f(x, t) * g(x, t) \) is Laplace-Marchi-Zgrablich transformable at \( p \) and from equation (5.3.2) and the definition of convolution, we get,

\[
\mathcal{L}M[f(x, t) * g(x, t)](p) = \langle T^{-p}(f(x, t) * g(x, t)), 1 \rangle
\]

\[
= \langle T^{-p}f(x, t) * T^{-p}g(x, t), 1 \rangle
\]

\[
= \langle T^{-p}f(x, t) \otimes T^{-p}g(\eta, \tau), 1(x + \eta, t + \tau) \rangle
\]

\[
= \langle T^{-p}f(x, t), 1 \rangle \langle T^{-p}g(x, t), 1 \rangle
\]

\[
= \mathcal{L}M[f](p)\mathcal{L}M[g](p).
\]

\[\square\]

Theorem 5.3.3 (Inversion theorem). If \( f(x, t) \) is Laplace-Marchi-Zgrablich transformable in \( \Omega = \{p : \alpha < \text{Re}(p) < \beta\} \), then for any fixed \( \sigma \in \mathbb{R}^2 \) such that \( \alpha < \sigma < \beta \), we have,

\[
f(x, t) = \lim_{r \to \infty} \frac{-1}{4\pi^2} \int_{\sigma - ir}^{\sigma + ir} e^{\sum p_i \theta_i} \mathcal{L}M[f](p)dp,
\]

where the limit is taken in \( \mathcal{D}'(I) \) as \( r \to \infty \) in \( \mathbb{R}^2 \). The integral in equation (5.3.3)
is taken over the subset of two-dimensional complex plane defined by

\[
\{ p : \text{Re}(p_i) = \sigma_i, |\text{Im}(p_i)| < r_i, i = 1, 2 \}
\]

\[5. CHARACTERIZATIONS FOR THE LAPLACE-MARCHI-ZGRABLICH INTEGRAL \ldots\]

\textbf{Proof:} This theorem can be proved on the same lines of Theorem 4.3 of \[39, \text{p.64}\] and Theorem 5.6 of \[39, \text{p.76}\]. \qed

\textbf{Theorem 5.3.4 (Uniqueness theorem).} If \( f(x, t) \) and \( g(x, t) \) are Laplace-Marchi-Zgrablich transformable distributions in \( \mathcal{D}'(I) \) such that domains of their transforms have intersection \( \Omega = \{ p : \alpha < \text{Re}(p) < \beta \} \), and there is a fixed \( \sigma \in \mathbb{R}^2 \) with \( \alpha < \sigma < \beta \) such that whenever \( \text{Re}(p) > \sigma \) we have

\[
\mathcal{LM}\{f\}(p) = \mathcal{LM}\{g\}(p);
\]

then \( f(x, t) = g(x, t) \) as distributions.

The next theorem gives sufficient conditions that an analytic function \( \mathcal{F}(p) \) be the Laplace-Marchi-Zgrablich transform of distribution \( f(x, t) \) and characterize the function \( f(x, t) \).

\textbf{Theorem 5.3.5.} If \( \mathcal{F}(p) \) is analytic for \( \Omega = \{ p : \alpha < \text{Re}(p) < \beta \} \) and is bounded in \( \Omega \) by a polynomial in \( \omega \) (or in \( |p| \)), then \( \mathcal{F}(p) = \mathcal{LM}\{f\}(p) \), where the distribution \( f(x, t) \) is defined as a limit in \( \mathcal{D}'(I) \) by

\[
f(x, t) = \lim_{r \to \infty} \frac{-1}{4\pi^2} \int_{\sigma - ir}^{\sigma + ir} e^{\sum_{\theta_i} \mathcal{F}(p)} dp
\]

(5.3.4)

for any fixed \( \sigma \in \mathbb{R}^2 \) such that \( \alpha < \sigma < \beta \).
5. CHARACTERIZATIONS FOR THE LAPLACE-MARCHI-ZGRABLICH INTEGRAL

5.4 Some Operation transform formulae for $\mathcal{L}M$

Now let $k$ is bi-index, $\eta$ and $\tau$ are in $\mathbb{R}^2$ and $p, q$ are in $\mathbb{C}^2$, define $T^k = x^{k_1}t^{k_2}$,

$$\partial^k = \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial t^{k_2}} \quad \text{and} \quad \frac{p}{k} = \frac{p_1}{k_1} - \frac{p_2}{k_2}$$

then we obtain following results,

\begin{align*}
\mathcal{L}M\{f^{(k)}\} (p) &= p^k \mathcal{L}M\{f\} (p) \quad (5.4.1) \\
\mathcal{L}M\{T^k f(x, t)\} (p) &= (-1)^{(k)} \partial^k \mathcal{L}M\{f\} (p) \quad (5.4.2) \\
\mathcal{L}M\{f(x - \eta, t - \tau)\} (p) &= e^{-(pqx+rt)} \mathcal{L}M\{f\} (p) \quad (5.4.3) \\
\mathcal{L}M\{e^{-\sum q_\theta f(x, t)}\} (p) &= \mathcal{L}M\{f\} (p + q) \quad (5.4.4) \\
\mathcal{L}M\{U_k f\} (p) &= \mathcal{L}M\{f\} (p/k). \quad (5.4.5)
\end{align*}

In the above formulae the terms have their own usual meanings.