CHAPTER 4

HIERARCHICAL FUZZY HIDDEN MARKOV CHAIN FOR WEB APPLICATIONS

4.1 INTRODUCTION

Hierarchical hidden Markov model (HHMM) is the recursive hierarchical generalization of the hidden Markov model (HMM) (Fine et al 1998).

HHMM has been recently extended and widely applied in many areas. HHMMs as dynamic Bayesian networks (DBNs) for health-state and remaining-useful-life (RUL) estimation for monitoring drill-bits on a CNC machine (Camci and Chinnam 2010). Unsupervised discovery of structure from video sequences and modeled the class of dense, stochastic structures in video using HHMM (Xie et al 2002). HHMM is proposed as a classifier to classify multiple vehicles using wireless sensor network (Aljaafreh and Dong 2010), learning models for information extraction that is on the basis of HHMM to represent the grammatical structure of the sentences (Skounakis et al 2003). Bayesian HHMM that attempts to capture the stochastic syntactic rules of cis-regulatory modules (CRMs) organization (Lin et al. 2008). Standard HMMs and hierarchical HMMs as dynamic Bayesian networks for health state estimation and HHMM can be extended for prognostics in order to estimate remaining-useful-life (RUL) (Camci and Chinnam 2005). HHMM
for team-play in multiple agents (Noda 2003), through the HHMM anomaly
detection method (Zhang et al 2003).

Bursty has been considered as one of the main features of the Internet
traffic; A new model is introduced for generating burst web traffic based
on HHMM which includes two underlying Markov state processes (Xie et
al 2011). Anomaly detection approach to detect intrusions into computer
systems based on HHMM (Chunfu and Feng 2007). Based on HHMM, pre-
diction of the market trends with a horizon of few days ahead against the
Indian stock market index (Nifty) (Troiano and Kriplani 2010). Subparts of
the hierarchy form HMM which are used to determine decoding credibility
(Lam and Hui 1993). HHMM by considering the knowledge of hierarchical
structure of protein based on the 6-state HMM (Shi and Zhang 2010). Accu-
rate inductive inference of the malware family using HHMM (Muhaya et al
2011). Shared–structure HHMM to recognize people behaviors (Nguyen et al
2005). Incarnated generative models for learning and automatic recreation of
human motion as motion engine and with a new HHMM with non-parametric
output densities (Wang et al 2005).

In this chapter we have introduced, analyzed, and demonstrated a re-
cursive hierarchical generalization of the widely used fuzzy hidden Markov
chain, that is, we have extended our Fuzzy Hidden Markov Chain (FHMC),
to Hierarchical Fuzzy Hidden Markov Chain (HFHMC) on possibility space
and computed the three problems for our proposed HFHMC. We have pro-
vided the algorithms separately to solve each of the three problems. In this
HFHMM also our proposed algorithm namely generalized Viterbi algorithm
itself solves the likelihood of given observation sequence and most likelihood
state sequence (given observation sequence) which again shows the reason-
able decrease in computation time.
World Wide Web is a large, distributed hypertext repository of information which users navigate through links and view with browsers. These links again have some other new links. Users hit the particulars of their own interest. Due to the hierarchical structure of the website we have applied our proposed model to our institution's website www.ssn.edu.in and performed the simulation to analyze the accessibility and trustworthiness of the website among the users.

In Section 2, HFHMC and three problems of HFHMC are explained, in Section 3, illustration and simulation have been presented and finally concluded.

4.2 HIERARCHICAL FUZZY HIDDEN MARKOV CHAIN (HFHMC)

In this section we have defined HFHMC and discussed the solution for the three problems of HFHMC.

4.2.1 Model Description

HFHMC is visualized as a tree structure (see Figure 4.1) in which there are three types of states, root state, internal states which are hidden states forming a fuzzy Markov chain and production states which emit observations. Each production state is associated with an output possibility vector. Each internal state is associated with an horizontal transition possibility matrix and a vertical transition vector. The horizontal transition matrix of an internal state defines the transition possibilities among its children. The vertical transition vectors define the possibility of an internal state to activate any of its children. Each internal and production state is associated with a child called an end-state which returns control to its parent. The end-states do not produce observations and cannot be activated through a vertical transition from their
The HFHMC is formally defined as a 3 tuple,

$$\tilde{\lambda} = \left\{ \tilde{\lambda}^{X_d} \right\}_{d \in \{1, \ldots, D\}} = \left( \left\{ \tilde{P}^{X_d} \right\}_{d \in \{1, \ldots, D-1\}} ; \left\{ \tilde{B}^{X_d} \right\} ; \left\{ \tilde{p}^{(0)X_d} \right\}_{d \in \{1, \ldots, D-1\}} \right)$$

Figure 4.1 Structure of HFHMC

### 4.2.2 Elements of HFHMC

1. $X^d_i (d \in \{1, \ldots, D\})$ is the state of an HFHMC where $i$ is the state index and $d$ is the hierarchy index. The hierarchy index of the root is 1 and of the production states is $D$. The internal states need not have the same number of sub–states; therefore we denote the number of sub–states of an internal state $X^d_i$ by $|X^d_i|$. We omit the state index and denote a state at level $d$ by $X^d$.

2. For each internal state $X^d_i (d \in 1, \ldots, D - 1)$, there is a state transition possibility matrix denoted by $\tilde{P}^{X_d} = (\tilde{p}^{X_d}_{ij})$, $\tilde{p}^{X_d}_{ij} = \sigma(X^d_{j+1} | X^d_i)$. $X^d_{j+1}$ is the possibility of making a horizontal transition from the $i^{th}$ state to the $j^{th}$ both of which are sub–states of $X^d$. 
3. \( \tilde{p}^{(0)X^d} = (\tilde{p}^{(0)X^d}(X^{d+1}_i)) = \sigma(X^{d+1}_i|X^d) \) is the initial possibility vector over the sub-states of \( X^d \), which is the possibility that state \( X^d \) will initially activate the state \( X^{d+1}_i \).

4. Each production state \( X^D \) is solely parameterized by its output possibility vector \( \tilde{B}^{X^D} = \{\tilde{b}^{X^D}(k)\} \), where \( \tilde{b}^{X^D}(k) = \sigma(o_k|X^D) \) is the possibility that the production state \( X^D \) will output the symbol \( o_k \in V \), where \( V = \{v_1, v_2, \ldots, v_m\} \) and \( m \) the number of distinct observation symbols per state, that is, the discrete output of the system.

As in the case of FHMCs, three problems typically arise for HFHMCs which are given below:

**Calculating the likelihood of a observation sequence**: Given an HFHMC and its parameter set \( \tilde{\lambda} = \{\tilde{\lambda}^{X^d}\} \), find the possibility of a sequence \( O \) to be generated by the model \( \tilde{\lambda} \), \( \sigma(O|\tilde{\lambda}) \).

**Finding the most likelihood state sequence**: Given an HFHMC, its parameter set \( \tilde{\lambda} = \{\tilde{\lambda}^{X^d}\} \), find the single state activation sequence that is most likely to generate the observation sequence.

**Estimating the parameters of a model**: Given the structure of an HFHMC and observation sequence, find the most likelihood parameter set \( \tilde{\lambda} \).

### 4.2.3 Computation of Likelihood of Observation Sequence

Each of the internal states of an HFHMC can be viewed as an autonomous model which can generate a subsequence of the observation using its substates, an efficient likelihood evaluation procedure should be recursive. For each state \( X^d \) we can calculate the likelihood of generating a subsequence \( o \), denoted by \( \sigma(o|\tilde{\lambda}, X^d) \). Assume for the moment that these possibilities are provided except for the root state \( X^1 \). Let \( \tilde{i} = (i_1, i_2, \ldots, i_l) \) be the indices of the states at the second level that were visited during the generation of the observation sequence \( O = o_0o_1 \cdots o_{N-1} \) of length \( N \). Note that the last state
entered at the second level is $X^2_{end}$, thus $X^2_{i_1} = X^2_{end}$. Let $\tau_j$ be the temporal position of the first symbol generated by state $X^2_{ij}$, and let the entire list of these indices be denoted by $\bar{\tau} = (\tau_1, \tau_2, \ldots, \tau_l)$. Since $X^2_{i_1}$ was activated by $X^1$ at the first time step and $X^2_{end}$ was the last state from the second level that was activated, we have $\tau_1 = 0$ and $\tau_l = N - 1$. The likelihood of the entire sequence given the above information is,

$$
\sigma(O|\bar{\tau}, \bar{i}, \bar{\lambda}) = \min \left[ \tilde{p}^{(0)}^{X^1_i}(X^2_{i_1}, \sigma(o_1 \cdots o_{\tau_2-1}|X^2_{i_1}, \bar{\lambda}), \tilde{p}^{X^1}_{i_2i_2}, \sigma(o_{\tau_2} \cdots o_{\tau_3-1}|X^2_{i_2}, \bar{\lambda})\tilde{p}^{X^1}_{i_3i_3} \right. \\
\cdots \tilde{p}^{X^1}_{i_{l-2}i_{l-2}}, \sigma(o_{\tau_{l-1}} \cdots o_{\tau_{N-1}}|X^2_{i_{l-1}}, \bar{\lambda}), \tilde{p}^{X^1}_{i_{l-1}i_{l-1}end} \left. \right]
$$

In order to calculate the unconditioned likelihood we need to maximize over all possible switching times $\tau$ and state indices $I$. Clearly this is not feasible since there are exponentially many such combinations. Fortunately, the structure of HFHMCs enables us to devise a generalized version of the Baum–Welch algorithm. The generalized forward variable $\tilde{\alpha}(\cdot)$ are defined to be,

$$
\tilde{\alpha}(n, n + m, X^d_i, X^{d-1}) = \sigma(o_n \cdots o_{n+m}, X^d_i \text{ finished at time step } n + m \mid X^{d-1} \text{ started at time step } n)
$$

That is, $\tilde{\alpha}(n, n + m, X^d_i, X^{d-1})$ is the possibility that the partial observation sequence $o_n \cdots o_{n+m}$ was generated by state $X^{d-1}$ and that $X^d_i$ was the last state activated by $X^{d-1}$ during the generation of $o_n \cdots o_{n+m}$. One can notice that the operation of each substate $X^{d-1}$ does not necessarily end at time step $n + m$ and that $o_n \cdots o_{n+m}$ can be a prefix of a larger sequence generated by $X^{d-1}$. To calculate the possibility that the sequence $O = o_0o_1 \cdots o_{N-1}$ was generated by $X^{d-1}$ we need to maximize over all possible states at level $d$ ending at $X^{d-1}_{end}$,

$$
\sigma(o_n \cdots o_{n+m}|X^{d-1}) = \max_{1 \leq i \leq |X^{d-1}|} \min \left( \tilde{\alpha}(n, n + m, X^d_i, X^{d-1}), \tilde{p}^{X^{d-1}}_{i \text{ end}} \right)
$$
Finally, the likelihood of the whole observation sequence is obtained by maximizing over all possible starting states (called by the root state $X^1$),

$$
\sigma(O|\tilde{\lambda}) = \max_{1 \leq i \leq |X^1|} \tilde{\alpha}(0, N - 1, X^2_i, X^1) \quad (4.3)
$$

The definition of the generalized $\tilde{\alpha}$ variables for the states at level $D - 1$, $\tilde{\alpha}(n, n + k, X^D_i, X^{D-1})$, is equivalent to the definition of the forward variable $\tilde{\alpha}_{n+k}(i)$ of an FHMC that consists of only this level and whose observation possibility vectors are defined by the productions states $X^D_i$. The evaluation of the $\tilde{\alpha}$ variables is done in a recursive bottom-up manner such that the $\tilde{\alpha}$ values calculated for the substates of an internal state $X$ are used to determine the $\tilde{\alpha}$ values of $X$.

In summary, for each internal state $X$ we need to calculate its $\tilde{\alpha}$ value for each possible sub-sequence of the observation sequence using a recursive decomposition of each subsequence based on the $\tilde{\alpha}$ values of $X$’s substates. Therefore the time complexity of evaluating the $\tilde{\alpha}$ values for all states of an HFHMC is $O(sN^3)$, where $s$ is the total number of states and $N$ is the length of the observation sequence.

**Estimation:**

$$
\tilde{\alpha}(n, n, X^D_i, X^{D-1}) = \sigma(O_n, X^D_i \text{ finished at } n| X^{D-1} \text{ started at } n) \quad (4.4)
$$

R.H.S of the equation (4.4) is possible only when at time step $n$, possibility of activating $X^D_i$ by $X^{D-1}$ and subsequently looking the $n^{th}$ observation of $X^D_i$ which is clearly given in Equation (4.5):

$$
\tilde{\alpha}(n, n, X^D_i, X^{D-1}) = \min \left( \tilde{p}^{(0)}X^{D-1}(X^D_i), \tilde{b}X^{D-1}(o_n) \right) \quad (4.5)
$$

$$
\tilde{\alpha}(n, n + m, X^D_i, X^{D-1})
= \min \left\{ \max_{1 \leq j \leq |X^{D-1}|} \left[ \min \left( \tilde{\alpha}(n, n + m - 1, X^D_j, X^{D-1}), \tilde{p}_ji \right), \tilde{b}X^{D-1}(o_{n+m}) \right] \right\} \quad (4.6)
$$
\[
\tilde{\alpha}(n, n, X_i^d, X^{d-1}) = \min \left\{ \tilde{p}^{(0)}(X_i^{d-1}) X_i^d \right\} \left( \min_{1 \leq s \leq |X_i^d|} \left[ \min \left( \tilde{\alpha}(n, n, X_s^{d+1}, X_i^d, \tilde{p}_s^{X_i^d}) \right) \right] \right\}
\]

(4.7)

\[
\tilde{\alpha}(n, n + m, X_i^d, X^{d-1}) = \max \left\{ \min_{0 \leq l \leq m - 1} \left[ \left\{ \max_{1 \leq j \leq |X^d|} \left[ \min_{P_{ji}} \left( \tilde{\alpha}(n + l, X_j^d, X^{d-1}) \right) \right] \right\} \right\} \left( \min_{1 \leq s \leq |X_i^d|} \left[ \min \left( \tilde{\alpha}(n + l + 1, n + m, X_s^{d+1}, X_i^d, \tilde{p}_s^{X_i^d}) \right) \right] \right\}
\]

(4.8)

4.2.4 Computation of the Most Likelihood State Sequence

The most likelihood state sequence is a multi-scale list of states: if state \(X\) had generated the sequence \(o_e \cdots o_f\), then its parent state generated the sequence \(o_a \cdots o_b\), such that \(a \leq e\) and \(f \leq b\). The process which finds the most likelihood state sequence for FHMCs is known as the Modified Viterbi algorithm, we have generalized the algorithm for HFHMCs and named it as the generalized Viterbi algorithm.

Let \(\tilde{\delta}(n, n + m, X_i^d, X^{d-1})\) be the most likelihood (hierarchical) state sequence generating \(o_n \cdots o_{n+m}\) given that \(X^{d-1}\) was entered at time step \(n\), its substate \(X_i^d\) was the last state to be activated by \(X^{d-1}\), and control returned to \(X^{d-1}\) at time \(n + m\). Since we are interested in the actual hierarchical parsing of the sequence into states we also maintain two additional variables:

- \(\psi(n, n + m, X_i^d, X^{d-1})\) is the index of the most likelihood state to be activated by \(X^{d-1}\) before activating \(X_i^d\).
\( n' = \tau(n, n + m, X^d_i, X^{d-1}) \), \( n < n' < n + m \) is the time step when \( X^d_i \) was activated by \( X^{d-1} \).

Given these two variables the most likelihood hierarchical state sequence is obtained by scanning the lists \( \psi \) and \( \tau \) from the root state to the production states. If a breadth-first-search is used for scanning then the states are listed by their level index from top to bottom. If a depth-first-search is used then the states are listed by their activation time. The time complexity of the generalized Viterbi algorithm is the same as the time of the generalized Baum–Welch, namely \( O(sN^3) \).

**Generalized Viterbi Algorithm**

The generalized Viterbi algorithm starts from the production states and calculate \( \tilde{\delta}, \psi, \) and \( \tau \) in a bottom up manner as follows:

**Production states:**

**Initialization:**

\[
\tilde{\delta}(n, n, X^D_i, X^{D-1}) = \min \left( \tilde{p}^{(0)}(X^{D-1}), \tilde{b}^{X^D_i}(o_n) \right) \quad (4.9)
\]

\[
\psi(n, n, X^D_i, X^{D-1}) = 0 \quad (4.10)
\]

\[
\tau(n, n, X^D_i, X^{D-1}) = n \quad (4.11)
\]

**Recursion:**

\[
\left( \tilde{\delta}(n, n + m, X^D_i, X^{D-1}), \psi(n, n + m, X^D_i, X^{D-1}) \right)
\]

\[
= \mathcal{M} \mathcal{A} \mathcal{X}_{1 \leq j \leq |X^{D-1}|} \min \left[ \tilde{\delta}(n, n + m - 1, X^D_j, X^{D-1}), \tilde{p}^{X^{D-1}}_{ji}, \tilde{b}^{X^D_i}(o_{n+m}) \right] \quad (4.12)
\]

\[
\tau(n, n + m, X^D_i, X^{D-1}) = n + m \quad (4.13)
\]

where the functional \( \mathcal{M} \mathcal{A} \mathcal{X}_j \{ f(i) \} \) is defined as

\[
\mathcal{M} \mathcal{A} \mathcal{X}_{1 \leq j \leq s} = \max_{1 \leq j \leq s} \{ f(i) \}, \ \text{argmax}_{1 \leq j \leq s} \{ f(i) \}.
\]
Internal states:

Initialization:

\[ \tilde{\delta}(n, n, X_i^d, X_i^{d-1}) = \max_{1 \leq s \leq |X_i^d|} \left\{ \min \left[ \tilde{p}^{(0), X_i^{d-1}}(X_i^d), \tilde{\delta}(n, n, X_s^{d+1}, X_i^d), \tilde{p}_s^{X_i^d} \right] \right\} \]  
(4.14)

\[ \psi(n, n, X_i^d, X_i^{d-1}) = 0, \tau(n, n, X_i^d, X_i^{d-1}) = n \]  
(4.15)

Recursion:

1. For \( n' = n + 1, \cdots, n + m \) set:

\[ R = \max_{1 \leq s \leq |X_i^d|} \left\{ \min \left[ \tilde{\delta}(n', n + m, X_s^{d+1}, X_i^d), X_i^d \right] \right\} \]  
\( (\Delta(n'), \Psi(n')) \)
\[ = M \mathcal{A} \chi_{1 \leq j \leq |X_i^{d-1}|} \left\{ \min \left[ \tilde{\delta}(n', n' - 1, X_j^d, X_i^{d-1}), X_i^{d-1} \right] \right\} \]  
(4.16)

2. For \( n \) set:

\[ \Delta(n) = \min \left\{ p^{(0), X_i^{d-1}}(X_i^d), \max_{1 \leq s \leq |X_i^d|} \left\{ \min \left[ \tilde{\delta}(n, n + m, X_s^{d+1}, X_i^d), X_i^d \right] \right\} \right\} \]  
(4.17)

\[ \Psi(n) = 0. \]  
(4.18)

3. For the most likelihood switching step:

\[ \left( \tilde{\delta}(n, n + m, X_i^d, X_i^{d-1}), \tau(n, n + m, X_i^d, X_i^{d-1}) \right) = M \mathcal{A} \chi_{n \leq n' \leq n + m} \Delta(n') \]  
(4.19)

\[ \psi(n, n + m, X_i^d, X_i^{d-1}) = \Psi \left( \tau(n, n + m, X_i^d, X_i^{d-1}) \right) \]  
(4.20)

Finally, the possibility of the most likelihood state sequence is found as follows:

\[ (P^*, X_{last}^2) = M \mathcal{A} \chi_{X_i^2} \left[ \tilde{\delta}(0, N - 1, X_i^2, X_1^1) \right] \]  
(4.21)

and the most likelihood states sequences itself is found by scanning the lists \( \psi \) and \( \tau \) starting from \( \tau(0, N - 1, X_{last}^2, X_1^1) \) and \( \psi(0, N - 1, X_{last}^2, X_1^1) \).
4.2.5 Re-estimation of the Model Parameters

A generalized backward variable \( \tilde{\beta} \) is defined as,

\[
\tilde{\beta}(n, n + m, X_i^d, X^{d-1}) = \sigma(o_{n, \cdots, o_{n+m}} | X_i^d \text{ started at time step } n, X^{d-1} \text{ finished at time step } n + m)
\]

We have formulated the generalized \( \tilde{\alpha} \) and \( \tilde{\beta} \) for internal states and production states for single time step as well as for multiple steps which are given below:

**Estimation:**

\[
\tilde{\beta}(n, n, X_i^D, X^{D-1}) = \sigma(O_n | X_i^D \text{ started at } n, X^{D-1} \text{ finished at } n)
\]

R.H.S. of the Equation (4.24) is possible only after looking the \( n \)th observation of \( X_i^D \), the process will end and return to its parent state \( X^{D-1} \) which is clearly given in Equation (4.25):

\[
\tilde{\beta}(n, n + m, X_i^D, X^{D-1})
\]

\[
= \min \left\{ \tilde{b} X_i^D(o_n), \left[ \max_{1 \leq j \neq \text{end} \leq |X^{D-1}|} \left[ \min \left( \tilde{p} X_j^{D-1}, \tilde{\beta}(n + 1, n + m, X_j^D, X^{D-1}) \right) \right] \right\}
\]

\[
\tilde{\beta}(n, n, X_i^d, X^{d-1}) = \min \left\{ \max_{1 \leq s \leq |X_i^d|} \left[ \min \left( \tilde{p} X_s^{d+1}, \tilde{\beta}(n, n, X_i^{d+1}, X_i^d) \right) \right], \tilde{X}_i^{d-1} \right\}
\]

\[
\tilde{\beta}(n, n + m, X_i^d, X^{d-1})
\]

\[
= \max \left\{ \max_{0 \leq l \leq k-1} \left[ \min_{1 \leq s \leq |X_i^d|} \left[ \min \left( \tilde{p} X_s^{d+1}, \tilde{\beta}(n, n + l, X_i^{d+1}, X_i^d) \right) \right] \right] \right\}
\]

\[
= \max \left\{ \max_{0 \leq l \leq k-1} \left[ \min_{1 \leq s \leq |X_i^d|} \left[ \min \left( \tilde{p} X_s^{d+1}, \tilde{\beta}(n, n + l, X_i^{d+1}, X_i^d) \right) \right] \right] \right\}
\]

\[
(4.28)
\]
The maximum-likelihood parameter estimation procedure for HFHMCs is a generalization of the Baum–Welch algorithm since we also need to consider stochastic vertical transitions which recursively generate observations. Therefore, in addition to the path variables \( \tilde{\alpha} \) and \( \tilde{\beta} \) which correspond to “forward” and “backward” transitions, we have added additional path variables which correspond to “downward” and “upward” transitions. The variables used in the expectation step are as follows:

\[
\tilde{\xi}(n, X^d_i, X^d_j, X^{d-1}) \text{ is the possibility of performing a horizontal transition from } X^d_i \text{ to } X^d_j, \text{ both are substates of } X^{d-1}, \text{ at time step } n \text{ after the production of } o_n \text{ and before the production of } o_{n+1},
\]

\[
\tilde{\xi}(n, X^d_i, X^d_j, X^{d-1}) = \sigma(o_0 \cdots o_n, X^d_i \longrightarrow X^d_j, o_{n+1} \cdots o_{N-1|\Lambda}) \quad (4.29)
\]

On the basis of \( \tilde{\xi} \) we define two auxiliary variables \( \tilde{\zeta}_{in} \) and \( \tilde{\zeta}_{out} \) which simplify the re-estimation step: \( \tilde{\zeta}_{in}(n, X^d_i, X^{d-1}) \) is the possibility of performing a horizontal transition to state \( X^d_i \) before \( o_n \) was generated. \( \tilde{\zeta}_{in} \) has been calculated using \( \tilde{\xi} \) by maximizing over all substates of \( X^{d-1} \) which can perform a horizontal transition to \( X^d_i \),

\[
\tilde{\zeta}_{in}(n, X^d_i, X^{d-1}) = \max_{1 \leq k \leq |X^{d-1}|} \tilde{\xi}(n-1, X^d_k, X^d_i, X^{d-1}) \quad (4.30)
\]

\( \tilde{\zeta}_{out}(n, X^d_i, X^{d-1}) \) is the possibility of leaving state \( X^d_i \) by performing a horizontal transition to any of the states in the same level \( d \) after the generation of \( o_n \). Analogous to \( \tilde{\zeta}_{in} \), \( \tilde{\zeta}_{out} \) has been calculated using \( \tilde{\xi} \) by maximizing over all substates of \( X^{d-1} \) that can be reached from \( X^d_i \) by a single horizontal transition,

\[
\tilde{\zeta}_{out}(n, X^d_i, X^{d-1}) = \max_{1 \leq k \leq |X^{d-1}|} \tilde{\xi}(n, X^d_i, X^d_k, X^{d-1}) \quad (4.31)
\]

The path variable used to estimate the possibility of a vertical transition is \( \tilde{\chi}, \tilde{\chi}(n, X^d_i, X^{d-1}) \) is the possibility that state \( X^{d-1} \) was entered at time step
Based on the above path variables and given the current set of parameters, the expectations are calculated which is given in the following:

\[\begin{align*}
\tilde{\chi}(n, X^d_i, X^{d-1}) &= \sigma(X^d_i \text{ started at } n | \tilde{\lambda}, O) \\
&= \sigma(o_0 \cdots o_{n-1}, o_n \cdots o_{N-1} | \tilde{\lambda}, O) 
\end{align*}\] (4.32)

Based on the above path variables and given the current set of parameters, the expectations are calculated which is given in the following:

- \[\max_{0 \leq n \leq N-2} \tilde{\xi}(n, X^d_i, X^d_j, X^{d-1})\]: The expected number of horizontal transitions from \(X^d_i\) to \(X^d_j\) both are substates of \(X^{d-1}\).

- \[\max_{1 \leq n \leq N-3} \tilde{\xi}_{in}(n, X^d_i, X^{d-1}) = \max_{1 \leq k \leq |X^{d-1}|} \max_{1 \leq n \leq N-3} \tilde{\xi}(n - 1, X^d_k, X^d_i, X^{d-1})\]: The expected number of horizontal transitions to state \(X^d_i\) from any of its neighboring substates in level \(d\).

- \[\max_{0 \leq n \leq N-2} \tilde{\xi}_{out}(n, X^d_i, X^{d-1}) = \max_{1 \leq k \leq |X^{d-1}|} \max_{0 \leq n \leq N-2} \tilde{\xi}(n, X^d_i, X^d_k, X^{d-1})\]: The expected number of horizontal transitions out of state \(X^d_i\) to any of its neighboring substates in level \(d\).

- \[\max_{1 \leq n \leq N-1} \tilde{\chi}(n, X^d_i, X^{d-1})\]: The expected number of vertical transitions from \(X^{d-1}\) to \(X^d_i\).

- \[\max_{1 \leq n \leq N-1} \tilde{\chi}(n, X^d_i, X^{d-1})\]: The expected number of vertical transitions from \(X^{d-1}\) to any of its substates in level \(d\).

- \[\max_{0 \leq n \leq N-2} \tilde{\chi}(n, X^d_i, X^{d-1})\] = \[\max_{1 \leq n \leq N-1} \tilde{\chi}(n, X^d_i, X^{d-1})\] : The expected number of vertical transitions to the production state \(X^d_i\) from state \(X^{d-1}\).

After the above expectations are calculated from the current parameters, a new set of parameters are re-estimated as follows:

\[\tilde{\pi}^{(0)} X^1_i \] (4.34)

\[\tilde{\pi}^{(0)} X^{d-1}_i = \max_{1 \leq n \leq N-1} \frac{n}{|X^{d-1}|} \tilde{\chi}(n, X^d_i, X^{d-1})\] (4.35)
In order to find a good set of parameters we have to iterate the expectation step that calculates all the variables and then we have to use above four equations to find a new estimation of the parameters and this verifies that the above steps in this iterative procedure correspond to the expectation and maximization steps of the EM algorithm. Hence, this procedure is guaranteed to converge to a stationary point. Division in the above formulas can be done by generalized division of TFN. Various formulas those needed to calculate the above re-estimated Equations “(4.34)–(4.37)” are given in the following:

**Definition:**

\[
\hat{\eta}_{in}(n, X_i^d, X_i^{d-1}) = \sigma(o_0 o_1 \cdots o_{n-1}, X_i^d \text{ started at } n|\lambda) \quad (4.38)
\]

**Estimation:**

\[
\hat{\eta}_{in}(0, X_i^2, X^1) \quad (4.39)
\]

for \( n \geq 1, \hat{\eta}_{in}(n, X_i^2, X^1) = \max_{1 \leq j \leq |X_i^1|} \left\{ \min \left[ \tilde{\alpha}(0, n - 1, X_j^2, X^1), \tilde{p}_{ji}^{X^1} \right] \right\} , \quad (4.40)
\]

\[
\hat{\eta}_{in}(0, X_i^d, X_i^{d-1}) = \min \left[ \hat{\eta}_{in}(0, X_i^{d-1}, X_i^{d-2}), \tilde{p}_{ji}^{X_{i}^{d-1}}(X_i^d) \right] . \quad (4.41)
\]
for $n \geq 1$, $\tilde{\eta}_{in}(n, X^d_i, X^{d-1}_i)$

$$
\tilde{\eta}_{in}(n, X^d_i, X^{d-1}_i) = \max \left\{ \min \left\{ \max_{0 \leq n' \leq n-1} \left[ \tilde{\eta}_{in}(n', X^{d-1}_i, X^{d-2}_i) \right] \right\}, \left( n < N \right) \right\} (4.42)
$$

Definition:

\[ \tilde{\eta}_{out}(n, X^d, X^{d-1}) = \sigma(X^d_i \text{ finished at } n, o_{n+1} o_{n+2} \cdots o_{N-1} \tilde{\lambda}) \] (4.43)

Estimation:

\[ \tilde{\eta}_{out}(n, X^2, X^1) = \max_{1 \leq j \leq |X^1|} \left\{ \min \left[ \tilde{\eta}_{ij}^{X^1}, \tilde{\eta}^{X^1}(n+1, N, X^2, X^1) \right] \right\}, (n < N) \] (4.44)

\[ \tilde{\eta}_{out}(n, X^d_i, X^{d-1}_i) \text{ (where } (n < N)) \]

\[ \tilde{\eta}_{out}(n, X^d_i, X^{d-1}_i) = \max \left\{ \min \left\{ \max_{n+1 \leq k \leq N} \left[ \tilde{\eta}_{out}(k, X^d_i, X^{d-2}_i) \right] \right\}, \tilde{\eta}_{out}(n, X^d_i, X^{d-1}_i) \right\} \] (4.45)

\[ \tilde{\eta}_{out}(N, X^d_i, X^{d-1}_i) = \min \left[ \tilde{p}_{i_{\text{end}}}^{X^d_i}, \tilde{\eta}_{out}(N, X^d_i, X^{d-2}_i) \right] \] (4.46)

Definition:

\[ \tilde{\xi}(n, X^d_i, X^d_j, X^{d-1}_i) = \sigma(X^d_i \text{ finished at } n, X^d_j \text{ started at } n+1 | \tilde{\lambda}, O) \]

\[ = \sigma(o_0 \cdots o_n, X^d_i \cdots X^d_j, o_{n+1} \cdots o_{N-1} | \tilde{\lambda}, O) \] (4.47)
Estimation:
\[ \tilde{\xi}(n, X^2_i, X^2_j, X^1) = \frac{\min \left[ \tilde{\alpha}(0, n, X^2_i, X^1), \tilde{\beta}(n+1, N-1, X^2_j, X^1) \right]}{\sigma(O | \tilde{\lambda})} \]

(4.48)

\[ \tilde{\xi}(N-1, X^2_i, X^2_j, X^1) = \frac{\min \left[ \tilde{\alpha}(0, N-1, X^2_i, X^1), \tilde{\beta}(0, n, X^1, X^2_j) \right]}{\sigma(O | \tilde{\lambda})} \]

(4.49)

Definition:
\[ \tilde{\chi}(n, X^2_i, X^1) = \sigma(X^d_i \text{ started at } n | \tilde{\lambda}, O) \]

(4.52)

Estimation:
\[ \tilde{\chi}(0, X^2_i, X^1) = \frac{\min \left[ \tilde{\beta}(0, N-1, X^2_i, X^1), \tilde{\beta}(0, n, X^1, X^2_j) \right]}{\sigma(O | \tilde{\lambda})} \]

(4.53)
\[ \tilde{\chi}(n, X^d_i, X^{d-1}_l) = \min \left\{ \begin{array}{c} \min \left[ \tilde{\eta}(n, X^d_i, X^{d-2}_l), \tilde{p}^{(0)} X^{d-2}_l \right] \sigma(O|\tilde{\lambda}) \\ \max_{n \leq e \leq N-1} \left[ \min \left( \tilde{\beta}(n, e, X^d_i, X^{d-1}_l), \tilde{\eta}_{\text{out}}(e, X^{d-1}_l, X^{d-2}_l) \right) \right] \end{array} \right\} \] (2 < d) \]

(4.54)

### 4.3 ILLUSTRATION

Our institution’s website appearance is depicted in the Figure 4.2

![Figure 4.2 Model of the website](image)

In this section, we have applied our proposed model to our institution’s website (www.ssn.edu.in) by assuming home page (H) as a root state, the web pages Department (D), Program (P) as internal states and EEE (E), MECH (M), BME (B) (Department attributes (DA)) and UG(undergraduate programme) (U) (Program attribute (PA)) as production states. Observations of DA have been taken as About the Department (A), Faculty (F) and News (N) and for PA, B.E.EEE (B.E. E.), B.E.MECH (B.E. M.) and B.E.BME (B.E. B.) Given observation sequence is

\[ o_0^{B.E. B.}, o_1^N, o_2^A, o_3^{B.E. B.}, o_4^{B.E. M.}, o_5^F, o_6^F, o_7^N, o_8^A, o_9^F, o_{10}^{B.E. B.}, o_{11}^{A}, o_{12}^F, o_{13}^{B.E. E.}, o_{14}^F, o_{15}^{B.E. E.}, o_{16}^F \]
This sequence shows that the observation is \( B.E.\ BME \ (B.E.\ B.) \) at time step 0, and the observation is the \( News \ (N) \) at time step 1, etc. From the extraction of user’s navigation paths we have computed the initial, transition \( \tilde{P} \) and the observation possibility \( \tilde{B} \) values for each state which in turn converted into type 2 fuzzy set whose grade of membership values are TFN on \([0, 1]\).

For the internal states (i.e., for the web pages department and program) the above mentioned values are given below:

\[
\begin{align*}
\text{Initial possibility} \\
\tilde{p}^H(D) &\approx \begin{bmatrix} 0.48, 0.53, 0.57 \end{bmatrix} \\
\tilde{p}^H(P) &\approx \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix}
\end{align*}
\]

\[
\begin{array}{ccc}
D & P & \text{end} \\
\begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} & \begin{bmatrix} 0.78, 0.81, 0.83 \end{bmatrix} & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} \\
\begin{bmatrix} 0.57, 0.60, 0.63 \end{bmatrix} & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix}
\end{array}
\]

Initial possibility, transition possibility matrix and end possibility values for the states (i.e., for the web pages \( EEE,\ MECH,\ BME,\ U.G.\)) are given below:

\[
\begin{align*}
\text{Initial possibility} \\
\tilde{p}^D(EEE) &\approx \begin{bmatrix} 0.30, 0.33, 0.37 \end{bmatrix} \\
\tilde{p}^D(MECH) &\approx \begin{bmatrix} 0.32, 0.35, 0.39 \end{bmatrix} \\
\tilde{p}^D(BME) &\approx \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} \\
\tilde{p}^D(U.G) &\approx \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix}
\end{align*}
\]

\[
\begin{array}{ccc}
EEE & MECH & BME \\
EEE & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} & \begin{bmatrix} 0.71, 0.74, 0.76 \end{bmatrix} & \begin{bmatrix} 0.75, 0.78, 0.80 \end{bmatrix} \\
MECH & \begin{bmatrix} 0.73, 0.76, 0.78 \end{bmatrix} & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix} & \begin{bmatrix} 0.69, 0.72, 0.74 \end{bmatrix} \\
BME & \begin{bmatrix} 0.87, 0.89, 0.90 \end{bmatrix} & \begin{bmatrix} 0.79, 0.82, 0.84 \end{bmatrix} & \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix}
\end{array}
\]

\[U.G. \approx \begin{bmatrix} 1.00, 1.00, 1.00 \end{bmatrix}\]
end possibility

\[
\begin{align*}
EEE & \begin{pmatrix} 1.00, 1.00, 1.00 \end{pmatrix} \\
MECH & (1.00, 1.00, 1.00) \\
BME & (0.92, 0.95, 0.98) \\
U.G. & \begin{pmatrix} 1.00, 1.00, 1.00 \end{pmatrix}
\end{align*}
\]

Observation possibility values for production states are given below:

\[
\begin{align*}
A & \begin{pmatrix} (1.00, 1.00, 1.00) & (0.82, 0.85, 0.87) & (0.70, 0.73, 0.75) \end{pmatrix} \\
F & \begin{pmatrix} (1.00, 1.00, 1.00) & (0.74, 0.77, 0.79) & (0.64, 0.68, 0.71) \end{pmatrix} \\
N & \begin{pmatrix} (0.46, 0.50, 0.53) & (1.00, 1.00, 1.00) & (0.61, 0.65, 0.68) \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
B.E. E. & (0.80, 0.84, 0.86) \\
B.E. M. & (1.00, 1.00, 1.00) \\
B.E. B. & (0.76, 0.79, 0.81)
\end{align*}
\]

To calculate the likelihood of a given observation sequence (given the model) by Equation (4.3).

\[
\sigma(o_0, o_1, \ldots, o_{16}|H) = \max \left\{ \tilde{\alpha}(0, 16, D, H), \tilde{\alpha}(0, 16, P, H) \right\} \tag{4.55}
\]

Since the given observation sequence follows from the R.H.S of the first expression of Equation (4.55), that is, Home page started at time step 0 and Department page finished at time step 16. By the definition of \(\tilde{\alpha}\), it is enough to find the first expression of the R.H.S., to calculate \(\tilde{\alpha}(0, 16, D, H)\), by Equation (4.8), we have
\[ \widetilde{\alpha}(0, 16, D, H) \]

\[
\begin{align*}
&= \max \left\{ \min \left[ \min \left[ \alpha(0, 16, E, D), \widetilde{P}_{E\text{ end}}^D \right], \min \left[ \alpha(0, 16, M, D), \widetilde{P}_{M\text{ end}}^D \right] \right], \min \left[ \alpha(0, 16, B, D), \widetilde{P}_{B\text{ end}}^D \right] \right\} \\
&\quad \quad \vdots \\
&\quad \quad \min \left\{ \widetilde{p}^{(0)H}(D), \max \left[ \min \left[ \alpha(0, 16, E, D), \widetilde{P}_{E\text{ end}}^D \right], \min \left[ \alpha(0, 16, M, D), \widetilde{P}_{M\text{ end}}^D \right], \min \left[ \alpha(0, 16, B, D), \widetilde{P}_{B\text{ end}}^D \right] \right] \right\}
\end{align*}
\]

(4.56)

One can notice from the given observation sequence that the program page finished at time step 15 and department page started at time step 16, therefore, it is enough to calculate,

\[
\widetilde{\alpha}(0, 16, D, H) = \min \left\{ \min \left[ \alpha(0, 16, E, D), \widetilde{P}_{E\text{ end}}^D \right], \min \left[ \alpha(0, 16, M, D), \widetilde{P}_{M\text{ end}}^D \right], \min \left[ \alpha(0, 16, B, D), \widetilde{P}_{B\text{ end}}^D \right] \right\}
\]

(4.57)

The R.H.S of expression is the only one which coincides with the given observation sequence. Where

\[
\widetilde{\alpha}(16, 16, E, D) = \min \left[ \widetilde{p}^{(0)D}(E), \widetilde{b}^{D_E}(o_{16} = F) \right] \text{ by Equation (4.5)}
\]

similarly \( \widetilde{\alpha}(16, 16, M, D) \) and \( \widetilde{\alpha}(16, 16, B, D) \).
Table 4.1 Forward possibility values to the observation sequence of HFHMC

<table>
<thead>
<tr>
<th>Forward Variable</th>
<th>Possibility Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\alpha}(0, 0, U, P) )</td>
<td>(0.76, 0.79, 0.81)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(0, 0, P, H) )</td>
<td>(0.76, 0.79, 0.81)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 1, E, D) )</td>
<td>(0.30, 0.33, 0.37)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 1, M, D) )</td>
<td>(0.32, 0.35, 0.39)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 1, B, D) )</td>
<td>(0.61, 0.65, 0.68)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 2, E, D) )</td>
<td>(0.61, 0.65, 0.68)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 2, M, D) )</td>
<td>(0.61, 0.65, 0.68)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(1, 2, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(0, 2, D, H) )</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>\vdots \</td>
<td>\vdots \</td>
</tr>
<tr>
<td>( \tilde{\alpha}(16, 16, E, D) )</td>
<td>(0.30, 0.33, 0.37)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(16, 16, M, D) )</td>
<td>(0.32, 0.35, 0.39)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(16, 16, B, D) )</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>( \tilde{\alpha}(0, 16, D, H) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
</tbody>
</table>

\( \tilde{\alpha}(0, 15, P, H) \)

\[
\tilde{\alpha}(0, 15, P, H) = \max \left\{ \min \left[ \tilde{\alpha}(0, 0, D, H), \tilde{p}_P^{H} \right], \min \left[ \tilde{\alpha}(1, 15, U, P), \tilde{p}_U^{P} \right] \right\} \text{ by Equation (4.8)}
\]

\[
\min \left\{ \tilde{p}(0)^{H}(P), \min \left[ \tilde{\alpha}(1, 15, U, P), \tilde{p}_U^{P} \right] \right\}
\]
In the above formula, the only term needed to proceed further is given below:

\[
\tilde{\alpha}(0, 15, P, H) = \min \left\{ \min \left[ \tilde{\alpha}(0, 14, D, H), \tilde{P}_D^H \right], \min \left[ \tilde{\alpha}(15, 15, U, P), \tilde{P}_U^P \right] \right\} \tag{4.58}
\]

By proceeding like this, that is, selecting the expressions which coincides with the given observation sequence finally we get,

\[
\tilde{\alpha}(0, 2, D, H) = \min \left\{ \min \left[ \tilde{\alpha}(0, 0, P, H), \tilde{P}_D^H \right], \max \left[ \min \left[ \tilde{\alpha}(1, 2, E, D), \tilde{P}_E^D \right], \min \left[ \tilde{\alpha}(1, 2, M, D), \tilde{P}_M^D \right], \min \left[ \tilde{\alpha}(1, 2, B, D), \tilde{P}_B^D \right] \right] \right\} \tag{4.59}
\]

by Equation (4.6),

\[
\tilde{\alpha}(1, 2, E, D) = \min \left\{ \max \left[ \min \left[ \tilde{\alpha}(1, 1, E, D), \tilde{P}_E^D \right], \min \left[ \tilde{\alpha}(1, 1, M, D), \tilde{P}_M^D \right], \min \left[ \tilde{\alpha}(1, 1, B, D), \tilde{P}_B^D \right] \right], \tilde{b}^E(0_2 = A) \right\} \tag{4.60}
\]

and by Equation (4.7)

\[
\tilde{\alpha}(0, 0, P, H) = \min \left\{ \tilde{p}^{(0)}(P), \min \left[ \tilde{\alpha}(0, 0, U, P), \tilde{p}_U^P \right] \right\} \tag{4.61}
\]

then by Equation (4.5)

\[
\tilde{\alpha}(0, 0, U, P) = \min \left[ \tilde{p}^{(0)}(U), \tilde{b}^{P_U}(0_0 = B.E.B) \right] \tag{4.62}
\]

Thus, the likelihood of the given observation sequence is obtained as

\[
\sigma(o_0, o_1, \ldots, o_{16}|H) = (0.46, 0.50, 0.53)
\]

Computed \(\tilde{\alpha}(\cdot)\) values are in Table 4.1 and remaining are given in the Table A.2.1 of Appendix 2.
To find the most likelihood state sequence:

\[
\tilde{\delta}(0, 0, P, H) = \max \left\{ \min \left[ \tilde{\delta}^{(0)H}(P), \tilde{\delta}(0, 0, U, P), \tilde{p}^{P}_{U \text{ end}} \right] \right\} \\
= (0.76, 0.79, 0.81)
\]

\[
\psi(0, 0, P, H) = 0,
\]

\[
\tau(0, 0, P, H) = 0,
\]

\[n' = 1,\]

\[
R = \max \left\{ \min \left[ \tilde{\delta}(1, 2, E, D), \tilde{p}^{P}_{E \text{ end}} \right] \right\} \\
= (0.46, 0.50, 0.53)
\]

“\(n'\)” is not in our case since there is no observation sequence starting with \(n\):

\[
(\Delta(1), \Psi(1)) = \max \left\{ \min \left[ \tilde{\delta}(0, 0, P, H), \tilde{p}^{H}_{PD}, R \right] \right\} \\
= \max \left\{ \min \left[ (0.76, 0.79, 0.81), \right. \right. \\
(0.57, 0.60, 0.63), \right. \\
(0.46, 0.50, 0.53) \left. \right\} \\
= ((0.46, 0.50, 0.53), D)
\]

\[
(\tilde{\delta}(0, 2, D, H), \tau(0, 2, D, H)) = \max_{0 \leq 1 \leq 2} \Delta(1) \\
= ((0.46, 0.50, 0.53), 1)
\]

\[
\psi(0, 2, D, H) = \Psi(\tau(0, 2, D, H)) \\
= \Psi(1) = D
\]

\[n' = 3,\]

\[
R = \min \left\{ \tilde{\delta}(3, 4, U, P), \tilde{p}^{P}_{U \text{ end}} \right\}
\]

\[
(\Delta(3), \Psi(3)) = \max \left\{ \min \left[ \tilde{\delta}(0, 2, D, H), \tilde{p}^{H}_{DP}, R \right] \right\} \\
= \max \left\{ \min \left[ \left. \right. \right. \right. \\
(0.57, 0.60, 0.63), \right. \right. \\
(0.78, 0.81, 0.83), \right. \right. \\
(0.76, 0.79, 0.81) \left. \right\} \\
= ((0.57, 0.60, 0.63), D)
\]
\[
\begin{align*}
(\tilde{\delta}(0, 4, D, H), \tau(0, 4, D, H)) &= \mathcal{M}\mathcal{A}\chi_{0 \leq 3 \leq 4}\Delta(3) \\
&= ((0.57, 0.60, 0.63), 3) \\
\psi(0, 4, P, H) &= \Psi(\tau(0, 4, D, H)) \\
&= \Psi(3) = D
\end{align*}
\]

By proceeding like this for \( n' = 16 \),
\[
R = \max \left\{ \min \left[ \tilde{\delta}(16, 16, E, D), \tilde{p}_{E_{end}}^D \right], \min \left[ \tilde{\delta}(16, 16, M, D), \tilde{p}_{M_{end}}^D \right], \min \left[ \tilde{\delta}(16, 16, B, D), \tilde{p}_{B_{end}}^D \right] \right\}
= (0.92, 0.95, 0.98)
\]

\[
(\Delta(16), \Psi(16)) = \mathcal{M}\mathcal{A}\chi \left\{ \min \left[ \tilde{\delta}(0, 15, P, H), \tilde{p}_{P_{end}}^H \right] \right\}
= \mathcal{M}\mathcal{A}\chi \left\{ \min \left[ (0.46, 0.50, 0.53), (0.57, 0.60, 0.63), (0.92, 0.95, 0.98) \right] \right\}
= ((0.46, 0.50, 0.53), D)
\]

\[
(\tilde{\delta}(0, 16, D, H), \tau(0, 16, D, H)) = \mathcal{M}\mathcal{A}\chi_{0 \leq 16 \leq 16}\Delta(16)
= ((0.46, 0.50, 0.53), 16)
\]

\[
\psi(0, 16, D, H) = \Psi(\tau(0, 16, D, H))
= \Psi(16) = D
\]

In this way,
\[
(\tilde{P}^*, X_{last}^2) = ((0.46, 0.50, 0.53), D)
\]

Computed optimal path is given in Table 4.2.

The result shows that the webpage “Department (D)” is the most likelihood path for all time steps which shows that the user’s navigated the webpage \( D \) more than the webpage “program (P)”. One can notice that the generalized Viterbi algorithm itself calculates the likelihood of the given
Table 4.2 Computation of the most likelihood path of HFHMC

<table>
<thead>
<tr>
<th>Time index</th>
<th>Value of time</th>
<th>state index</th>
<th>most possible state</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau(0, 16, P, H))</td>
<td>16</td>
<td>(\psi(0, 16, P, H))</td>
<td>(\Psi(16) = D)</td>
</tr>
<tr>
<td>(\tau(0, 15, P, H))</td>
<td>15</td>
<td>(\psi(0, 15, P, H))</td>
<td>(\Psi(15) = D)</td>
</tr>
<tr>
<td>(\tau(0, 14, D, H))</td>
<td>14</td>
<td>(\psi(0, 14, D, H))</td>
<td>(\Psi(14) = D)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(\tau(0, 0, P, H))</td>
<td>0</td>
<td>(\psi(0, 0, P, H))</td>
<td>(\Psi(0) = D)</td>
</tr>
</tbody>
</table>

observation sequence and most likelihood path. Thus the algorithm reduces our time consumption.

To re-estimate the model parameters, we have to calculate \(\tilde{\beta}(\cdot)\) values and remaining parameters. The various formulas needed to re-estimate the model parameters are calculated and the initial iterative TFN values of those variables are presented in Tables 4.3–4.9.

Table 4.3 Backward possibility values to the observation sequence of HFHMC

<table>
<thead>
<tr>
<th>Backward Variable</th>
<th>Possibility Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{\beta}(16, 16, E, D))</td>
<td>(0.82, 0.85, 0.87)</td>
</tr>
<tr>
<td>(\tilde{\beta}(16, 16, M, D))</td>
<td>(0.74, 0.77, 0.79)</td>
</tr>
<tr>
<td>(\tilde{\beta}(16, 16, B, D))</td>
<td>(0.92, 0.95, 0.98)</td>
</tr>
<tr>
<td>(\tilde{\beta}(16, 16, D, H))</td>
<td>(0.92, 0.95, 0.98)</td>
</tr>
<tr>
<td>(\tilde{\beta}(15, 15, U, P))</td>
<td>(0.80, 0.84, 0.86)</td>
</tr>
<tr>
<td>(\tilde{\beta}(15, 16, P, H))</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>(\tilde{\beta}(14, 14, E, D))</td>
<td>(0.82, 0.85, 0.87)</td>
</tr>
<tr>
<td>(\tilde{\beta}(14, 14, M, D))</td>
<td>(0.74, 0.77, 0.79)</td>
</tr>
<tr>
<td>(\tilde{\beta}(14, 14, B, D))</td>
<td>(0.92, 0.95, 0.98)</td>
</tr>
<tr>
<td>(\tilde{\beta}(14, 16, D, H))</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(\tilde{\beta}(0, 0, U, P))</td>
<td>(0.76, 0.79, 0.81)</td>
</tr>
<tr>
<td>(\tilde{\beta}(0, 16, P, H))</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
</tbody>
</table>

where \(E, B\) and \(B\) are \(EEE, MECH\) and \(BME\) respectively. Remaining values of \(\tilde{\beta}(\cdot)\) are given in Table A.2.3 of Appendix 2.
Table 4.4 \( \tilde{\eta}_{\text{in}} \) and \( \tilde{\eta}_{\text{out}} \) values

<table>
<thead>
<tr>
<th>( \tilde{\eta}_{\text{in}}(\cdot) )</th>
<th>Values</th>
<th>( \tilde{\eta}_{\text{out}}(\cdot) )</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\eta}_{\text{in}}(0, P, H) )</td>
<td>(1, 1, 1)</td>
<td>( \tilde{\eta}_{\text{out}}(15, P, H) )</td>
<td>(.57, .6, .63)</td>
</tr>
<tr>
<td>( \tilde{\eta}_{\text{in}}(1, D, H) )</td>
<td>(0.57, 0.60, 0.63)</td>
<td>( \tilde{\eta}_{\text{out}}(14, D, H) )</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>( \tilde{\eta}_{\text{in}}(3, P, H) )</td>
<td>(0.57, 0.60, 0.63)</td>
<td>( \tilde{\eta}_{\text{out}}(13, P, H) )</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \tilde{\eta}_{\text{in}}(16, D, H) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \tilde{\eta}_{\text{out}}(0, P, H) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
</tbody>
</table>

Table 4.5 Transition from the state \( EEE \) (\( E \)) to all other states

<table>
<thead>
<tr>
<th>( E - E )</th>
<th>Values of ( E - E )</th>
<th>( E - M )</th>
<th>Values of ( E - M )</th>
<th>( E - B )</th>
<th>Values of ( E - B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi(1, E, E, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(1, E, M, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(1, E, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \xi(5, E, E, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(5, E, M, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(5, E, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \xi(6, E, E, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(6, E, M, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(6, E, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \xi(7, E, E, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(7, E, M, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(7, E, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \xi(8, E, E, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(8, E, M, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
<td>( \xi(8, E, B, D) )</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>( \xi(11, E, E, D) )</td>
<td>(0.57, 0.60, 0.63)</td>
<td>( \xi(11, E, M, D) )</td>
<td>(0.57, 0.60, 0.63)</td>
<td>( \xi(11, E, B, D) )</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
</tbody>
</table>
Table 4.6 Transition from the state $MECH$ ($M$) to all other states

<table>
<thead>
<tr>
<th>$M - E$</th>
<th>Values of $M - E$</th>
<th>$M - M$</th>
<th>Values of $M - M$</th>
<th>$M - B$</th>
<th>Values of $M - B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi(1, M, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(1, M, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(1, M, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(5, M, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(5, M, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(5, M, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(6, M, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(6, M, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(6, M, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(7, M, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(7, M, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(7, M, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(8, M, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(8, M, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(8, M, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(11, M, E, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
<td>$\xi(11, M, M, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
<td>$\xi(11, M, B, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
</tbody>
</table>

Table 4.7 Transition from the state $BME$ ($B$) to all other states

<table>
<thead>
<tr>
<th>$B - E$</th>
<th>Values of $B - E$</th>
<th>$B - M$</th>
<th>Values of $B - M$</th>
<th>$B - B$</th>
<th>Values of $B - B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi(1, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(1, B, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(1, B, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(5, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(5, B, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(5, B, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(6, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(6, B, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(6, B, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(7, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(7, B, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(7, B, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(8, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(8, B, M, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(8, B, B, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\xi(11, B, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\xi(11, B, M, D)$</td>
<td>(0.57, 0.6, 0.63)</td>
<td>$\xi(11, B, B, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
</tbody>
</table>
### Table 4.8 $\tilde{\chi}$ values for the production states $E, M,$ and $B$

<table>
<thead>
<tr>
<th>Values</th>
<th>Values</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\chi}(1, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\tilde{\chi}(1, M, D)$</td>
</tr>
<tr>
<td>$\tilde{\chi}(5, E, D)$</td>
<td>(0.46, 0.50, 0.53)</td>
<td>$\tilde{\chi}(5, M, D)$</td>
</tr>
<tr>
<td>$\tilde{\chi}(11, E, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
<td>$\tilde{\chi}(11, M, D)$</td>
</tr>
<tr>
<td>$\tilde{\chi}(14, E, D)$</td>
<td>(0.57, 0.60, 0.63)</td>
<td>$\tilde{\chi}(14, M, D)$</td>
</tr>
</tbody>
</table>

### Table 4.9 $\tilde{\chi}$ values for the production states $B.E. E., B.E. M.,$ and $B.E. B.$

<table>
<thead>
<tr>
<th>$\tilde{\chi}(\cdot)$</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\chi}(0, U, P)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\tilde{\chi}(3, U, P)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\tilde{\chi}(10, U, P)$</td>
<td>(0.46, 0.50, 0.53)</td>
</tr>
<tr>
<td>$\tilde{\chi}(13, U, P)$</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
<tr>
<td>$\tilde{\chi}(15, U, P)$</td>
<td>(0.57, 0.60, 0.63)</td>
</tr>
</tbody>
</table>
4.3.1 Simulation

To evaluate the performance of the parameter estimation on HFHMC, we have executed the experiment on iterative use of $\tilde{\lambda}$ in the place of $\hat{\lambda}$. Final iteration values of $\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)$ are given in Table 4.10. Remaining values are given in Table A.2.2 and A.2.4.

In Figure 4.3 initial iteration of the forward possibility values $\tilde{\alpha}(\cdot)$ are given and final iteration values of $\tilde{\alpha}(\cdot)$ are given in Figure 4.4. By comparing both of the figure one can easily notice that the TFN values are improved from $[0.3, 0.87]$ to $[0.8, 0.84]$.

<table>
<thead>
<tr>
<th>Forward variable</th>
<th>Possibility values</th>
<th>Backward variable</th>
<th>Possibility values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\alpha}(0, 0, U, P)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(16, 16, E, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(0, 0, P, H)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(16, 16, M, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 1, E, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(16, 16, B, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 1, M, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(16, 16, D, H)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 1, B, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(15, 15, U, P)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 2, E, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(15, 16, P, H)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 2, M, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(14, 14, E, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(1, 2, B, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(14, 14, M, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(0, 2, D, H)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(14, 14, M, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\tilde{\alpha}(16, 16, E, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
<td>$\hat{\beta}(1, 2, B, D)$</td>
<td>(0.80, 0.83, 0.84)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(16, 16, M, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
<td>$\hat{\beta}(1, 16, D, H)$</td>
<td>(0.80, 0.83, 0.84)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(16, 16, B, D)$</td>
<td>(1.00, 1.00, 1.00)</td>
<td>$\hat{\beta}(0, 0, U, P)$</td>
<td>(0.80, 0.83, 0.84)</td>
</tr>
<tr>
<td>$\tilde{\alpha}(0, 16, D, H)$</td>
<td>(0.80, 0.83, 0.84)</td>
<td>$\hat{\beta}(0, 16, P, H)$</td>
<td>(0.80, 0.83, 0.84)</td>
</tr>
</tbody>
</table>
Figure 4.3 Schematic representation of initial iteration of $\tilde{\alpha}$ values

Figure 4.4 Schematic representation of final iteration of $\tilde{\alpha}$ values
Backward possibility values $\tilde{\beta}(\cdot)$ of the initial iteration are given in Figure 4.5 and in Figure 4.6 final iteration values of the $\tilde{\beta}(\cdot)$ are given. Here also we can notice that the possibility values are improved from $[0.46, 1]$ into $[0.8, 1]$. 
From the simulation results, we have obtained the maximum likelihood value of the model as \( \hat{P} = (0.8, 0.83, 0.84) \), that is, the possibility of given observation sequence is maximized from \((0.46, 0.5, 0.53)\) to \((0.8, 0.83, 0.84)\) which shows that website is more authentic among the users and the re–estimated values are given below:

For the internal states the re–estimated values are given below:

\[
\begin{align*}
\text{Initial possibility} \\
\hat{p}^H(D) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
\hat{p}^H(P) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
D \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] (1.00, 1.00, 1.00) (1.00, 1.00, 1.00) \ 
end \end{align*}
\]

Re–estimated initial, transition possibility matrix, and end possibility values for the production states are given below:

\[
\begin{align*}
\text{Initial possibility} \\
\hat{p}^D(EEE) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
\hat{p}^D(MECH) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
\hat{p}^D(BME) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
\hat{p}^P(U.G.) \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
EEE \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] (1.00, 1.00, 1.00) (1.00, 1.00, 1.00) \ 
MECH \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] (1.00, 1.00, 1.00) (1.00, 1.00, 1.00) \ 
BME \left[ \begin{smallmatrix} 0.80, 0.83, 0.84 \end{smallmatrix} \right] (1.00, 1.00, 1.00) (1.00, 1.00, 1.00) \ 
U.G. \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
\text{end possibility} \\
EEE \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
MECH \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
BME \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right] \\
U.G. \left[ \begin{smallmatrix} 1.00, 1.00, 1.00 \end{smallmatrix} \right]
\end{align*}
\]
Re-estimated observation possibility values for the production states are given below:

\[
\begin{align*}
A & \\
EEE & \begin{bmatrix} (1.00, 1.00, 1.00) & (1.00, 1.00, 1.00) & (0.80, 0.83, 0.84) \end{bmatrix} \\
MECH & \begin{bmatrix} (1.00, 1.00, 1.00) & (1.00, 1.00, 1.00) & (0.80, 0.83, 0.84) \end{bmatrix} \\
BME & \begin{bmatrix} (0.80, 0.83, 0.84) & (1.00, 1.00, 1.00) & (0.80, 0.83, 0.84) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
B.E. E. & \\
B.E. M. & \\
B.E. B. & \\
U.G. & \begin{bmatrix} (1.00, 1.00, 1.00) & (0.80, 0.83, 0.84) & (0.80, 0.83, 0.84) \end{bmatrix}
\end{align*}
\]

4.4 SUMMARY

To enable the different stochastic levels we have proposed hierarchical fuzzy hidden Markov chain on possibility space and solved three problems of the model. The algorithm which we have adapted namely generalized Viterbi algorithm itself solves the likelihood of the observations sequence and most likelihood path which shows that the algorithm reduces our time consumption. We have trained the model to maximize the likelihood of the observation sequence. Finally we have viewed our institution’s website as HFHMC and performed simulation. Simulation results show that our model has more legitimacy and more dependability toward web users.