CHAPTER 0
INTRODUCTION

The principal endeavour of this thesis is to introduce the notion of fuzzy ordered fuzzy topological space and to study its properties.

Lotfi A. Zadeh's classic paper of 1965 opened up a new area in modern mathematics, namely, Fuzzy Set Theory. The roots of fuzzy sets can be traced back to the second half of the nineteenth century, more precisely, to the well known controversy between G. Cantor and L. Kronecker on the mathematical meaning of infinite sets. Cantor was for infinite sets and Kronecker refused to accept the concept of infinite sets. R. Dedekind reacted in favour of Cantor. A compromise between Kronecker's and Dedekind's points of view could be described thus: A set S is completely determined if and only if there is a decision procedure specifying whether an element is a member of S or not. Using naive set theory, this approach leads to characteristic functions in the context of binary logic, whereas in the case of many-valued logic this leads to the concept of membership functions introduced by Zadeh. Therefore, Kronecker's rejection of infinite sets and Dedekind's defence of Cantor's set might have resulted in the advent of fuzzy set theory.
Another source of fuzzy sets lies in the inherent imprecision in human decision making, which was Zadeh's main motivation. Since the very inception of the theory, several people all over the world have explored its various facets and a large number of results have been generated.

Fuzzy set theory has now become a major area of interest for modern scientists. According to S. Mac Lane:

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"... The case of fuzzy sets is even more striking. The original idea was an attractive one .... Someone then recalled (pace Lowere) that all mathematics can be based on set theory; it followed at once that all mathematics could be rewritten so as to be based on fuzzy sets. Moreover, it could be based on fuzzy sets in more than one way, so this turned out to be a fine blue print for the publication of lots and lots of newly based mathematics."

Fuzzy set theory offers wider applications than ordinary set theory. Besides, it provides sufficient motivation to researchers to review various concepts and theorems of mathematics in the broader frame work of fuzzy setting. The manifold applications of fuzzy set theory have permeated almost all spheres of human activity like
artificial intelligence and robotics, image processing and speech recognition, biological and medical sciences, applied operations research, control, economics and geography, sociology, psychology, linguistics, semiotics and quantum mechanics.

Binary relations play a vital role in pure mathematics. The notions of equivalence and ordering relations are used practically in all fundamental mathematical constructions. They are being applied in modelling various concepts in the field of psychology, sociology, linguistics, art, etc. Many important models in decision making and measurement theories are based on binary relations.

The theory of Ordered Sets is a rapidly developing branch of mathematics, and ordered sets are abundant in all branches of mathematics. Axioms defining the concept of an ordered set are found in the work of C.S. Piere on the algebra of logic in 1880. In 1890 such axioms were studied systematically by Schroeder. These studies, also, were carried out from the point of view of the needs of logic.

It was R. Dedekind who first observed the frequent occurrence of ordered sets in mathematics. In 1897 he
suggested that the theory of ordered sets has to be treated as an independent autonomous subject. This suggestion was later backed up by many an eminent mathematician like, Hausdorff (Foundation of Set Theory), Emmy Noether (Algebra), L. Nachbin and S. Purish. Several others also studied ordered sets rigorously and the theory has been enriched further by introducing intrinsic topologies, i.e., topologies defined purely in terms of order relations.

Order topology has been extensively studied by Vaidyanathaswamy, G. Birkhoff, L. Nachbin, J. Van Dalen, David J. Lutzer, H.R. Bennet, S. Eilenberg, M.E. Rudin, S.A. Gaal, etc. Gaal has studied the continuity properties of functions whose domain and range are totally ordered and endowed with the order topologies. In [Ga] he discussed various properties of a totally ordered topological space with least upper bound property.

The approach of fuzzy sets provides a very natural basis for generalising the concept of order relations. The theory of fuzzy order relations was initiated by Zadeh [Z1]. In [Z2] he gave various aspects of fuzzy binary relations, particularly of similarity relations...
and fuzzy orderings. He defined the notion of similarity as a generalization of the notion of equivalence relation. Fuzzy ordering on a set $X$ was defined as a subset of $X \times X$ by generalizing the notions of reflexivity, antisymmetry and transitivity. Since then, several authors have studied fuzzy relations and orderings. Among them, Sergei Ovchinnikov [O1-O4], M.K. Chakraborty, M. Das [Ch-D1 - Ch-D3], S. Sarkar [Ch-S], P. Venugopalan [Ve], A.K. Katsaras [Kat], V. Murali [Mu1, Mu2], S.K.Bhagat, P. Das [Bh-D] and Marc Roubens [O-R] deserve special mention.

In order to introduce the properties of fuzzy binary relations as in the classical case, we have to model basic logical connectives as operations on the unit interval $[0,1]$. The general method is based on MAX. and MIN. operations as models for logical connectives OR and AND and the negation is represented by $x \mapsto 1-x$. Recently, logical connectives and operations on fuzzy sets have been defined by means of triangular norms and conorms and general negation functions. Fuzzy interval orderings and fuzzy orderings of fuzzy numbers are of special interest in fuzzy set theory and its applications.

The introduction of the idea of metric spaces by Frechét in 1906 marked the beginning of a new discipline
called Set Topology. The works of people like Hausdorff, Kuratowski, A. Tychonoff, A.H. Stone and Dieudonnée, were pioneering contributions to this area.

Fuzzy topology was initiated by C.L. Chang [Chan] in 1968. After he introduced fuzzy set theory into topology, C.K. Wong [WO₁-WO₃], R. Lowen [LO₁-LO₆], Bruce Hutton [Hut], Hu Cheng-Ming [Hu], Goguen J.A. [Go], Gottwald [Got], A.K. Srivastava [Sri-D, Sri-L] etc. have studied different aspects of fuzzy topology. Here a fuzzy topological space is defined as a crisp subset of the fuzzy power set of a non empty set (crisp), which is closed for finite intersection and arbitrary union operations and contains the largest and the smallest elements. This fuzzy topology is generally called Chang's topology. However, in [Lo-W] Lowen has defined fuzzy topology by including all constant functions to the subset considered in Chang's definition. This topology is termed as Lowen's topology. Recently Hazra R.N., Samanta, S.K. and Chattopadhyay [Ha] introduced the idea of gradation of openness (closedness) of fuzzy subsets and proposed a new definition of fuzzy topology.

In this study we combine the notions of fuzzy order
and fuzzy topology of Chang and define fuzzy ordered fuzzy topological space. Its various properties are analysed. Product, quotient, union and intersection of fuzzy orders are introduced. Besides, fuzzy order preserving maps and various fuzzy completeness are investigated. Finally an attempt is made to study the notion of generalized fuzzy ordered fuzzy topological space by considering fuzzy order defined on a fuzzy subset.

Our approach is distinct from those of the earlier authors like A.K. Katsaras [Kat], R. Lowen [Lo6] and P. Venugopalan [Ven]. Katsaras has defined a fuzzy topology on a crisp ordered set and investigated its various properties analogous to the work of Nachbin [N]. Lowen studied ordered fuzzy topology on the real line. He put forward a new definition of fuzzy real line and observed that it was the order of $\mathbb{R}$, and not the topology, which determined the fuzzy real line. Venugopalan's definition of fuzzy order was different from ours. He considered a special type of transitivity and introduced the fuzzy interval topology on a fuzzy ordered set $(P, \mu)$ generated by the fuzzy sets $P \downarrow e, P \uparrow d$ for $e, d$.
fuzzy points of $P$ as subbasic open sets, where $e = x_\lambda$
is a fuzzy point of $P$.

The notion of fuzzy ordered fuzzy topological space $(X, F, \mathcal{R})$ is introduced in the first section. It is proved that every fuzzy order $\mathcal{R}$ defined on a set $X$ determines a total order $\prec$ as $x \prec y$ iff

$$
\downarrow e(y) = [\mu(y, x) + \triangledown -1] \lor 0
$$

and

$$
\uparrow e(y) = [\mu(x, y) + \triangledown -1] \lor 0
$$

We now give the summary of each chapter.

The thesis comprises seven chapters and an introduction to the subject.

**Chapter 1**

Preliminary definitions of the terms, like fuzzy topology and fuzzy ordering, required for the later chapters are given in chapter 1. The valuation set of every fuzzy set is taken as the unit interval $[0,1]$ and the Chang's definition of fuzzy topology is followed throughout. Also we stick to strict partial ordering $\mathcal{R}$ on $X$ satisfying irreflexivity, i.e., for $x \neq y$;

$$
\mathcal{R}(x, y) \neq \mathcal{R}(y, x), \text{ and max-min transitive,}
$$

i.e., $\mathcal{R}(x, z) \supset \mathcal{V}[\mathcal{R}(x, y) \land \mathcal{R}(y, z)]$, $x, y, z \in X$. Besides, the algebra of fuzzy sets and various other definitions of reflexivity, antisymmetry and transitivity are also discussed.
Chapter 2

The notion of fuzzy ordered fuzzy topological space \((X,F_R)\) on a foset \((X,R)\) is introduced in the first section. It is proved that every fuzzy order \(R\) defined on a set \(X\) determines a total order \(\leq\) as \(x \leq y\) iff \(R(x,y) > R(y,x)\) and the corresponding order topology on \(X\) is denoted by \(T_\leq\). It is found that the associated topology \(\mathcal{L}(F_R)\) of \(F_R\) contains \(T_\leq\). An example in which this inclusion is strict is provided. Also the fuzzy topological space \((X,F)\) defined by taking all lower semi continuous functions and \((X,F_R)\), the fuzzy ordered fuzzy topological space are compared.

In section 2 we recall several definitions of interval topologies in the crisp sense and their fuzzy analogues are proposed.

Chapter 3

Chapter 3 begins with the product of strong fuzzy orders. An example showing that the product of fuzzy orders, if they are not strict, need not be a fuzzy order is given. It is shown that the product of crisp orders induced by fuzzy orders is the same as the induced crisp order of the product of fuzzy orders.
Quotient spaces are analysed in section 2. Certain maps from $X$ to $X/\sim$, $(X, \tau')$ to $(X/\sim, \tau')$, and $(X, \underline{l}(F_R))$ to $(X/\sim, \underline{l}(F_R))$ are proved to be quotient.

In section 3 union of fuzzy orders is considered and it is proved that the union of strong fuzzy orders $R_i, i \in \Lambda$ defined on $X$ is a strong fuzzy order iff $R_i(x, y) \wedge R_j(y, x) = 0, i \neq j, x \neq y$. Intersection of fuzzy orders is also mentioned.

Finally, certain aspects of fuzzy ordered subspaces are discussed in section 4. Let $Y$ be a subspace of a foset $(X, R)$. Then a fuzzy order $R_Y$ is defined on $Y$ and it is proved that $F_{R_Y} = F_R \wedge Y$ and $\underline{l}(F_{R_Y}) = \underline{l}(F_R) \cap Y$; where $F_R$ and $F_{R_Y}$ are the fuzzy ordered fuzzy topological spaces on $X$ and $Y$ respectively and $\underline{l}(F_R)$ and $\underline{l}(F_{R_Y})$ are their corresponding associated topologies.

Chapter 4

Certain separation properties of the fuzzy ordered fuzzy topological space are discussed in chapter 4. It is proved that the fuzzy ordered fuzzy topological space
(X,F_R) is fuzzy T_1. Also, when R is strong (X,F_R) is found to be fuzzy Hausdorff. If R is not strong the notion of weak fuzzy Hausdorffness is introduced and (X,F_R) is found to be weak fuzzy Hausdorff.

Chapter 5

This chapter consists of a brief analysis of fuzzy order preserving maps. Two types of fuzzy order preserving maps between fuzzy ordered fuzzy topological spaces are defined. It is proved that a bijective strict order preserving map is a fuzzy homeomorphism. Also the set Y^X of all maps from X to Y, when Y is a strong fuzzy ordered set is made a strong fuzzy ordered set. Natural monoid homomorphisms between Y, (X,Y) and between X x Y, (X,Y) are obtained.

Chapter 6

In this chapter various characterizations of fuzzy ordered completeness are considered. In particular, least upper bound property, greatest lower bound property, Didekind completeness and Cantor completeness are discussed.
Chapter 7

An extension of the definition of fuzzy order defined on a crisp set to fuzzy order defined on a fuzzy set is the chunk of the final chapter. The notion of the generalised fuzzy ordered fuzzy topological space, defined by means of a generalised fuzzy order is introduced as well.

The author does not claim that the study made in this thesis is a complete exposition in all respects—rather, there are various problems connected with the work done here, worth investigating. As is often found, any investigation opens up new areas for further exploration.

\[ x_A(x) = 1 \quad \text{if} \quad x \in A \]
\[ = 0 \quad \text{otherwise} \]

i.e., \[ A = \{(x, x_A(x)) \mid x \in X\} \]

Here \( \{0,1\} \) is called the valuation set or membership set. For a classical or crisp set the membership (non-membership) of element is abrupt and its boundary is rather rigid.

It is worth considering the membership of elements to be gradual rather than abrupt. This can be achieved by extending the valuation set \( \{0,1\} \) to the unit interval \( I = [0,1] \). This is the basic characteristic of a fuzzy set.