Chapter 2

Wishart Ensemble

In this chapter we describe the Wishart ensembles. As mentioned in the Chapter 1, Wishart ensembles are also known as the Laguerre ensembles. These are useful in the study of uncorrelated stochastic and chaotic time series and several other contexts. We will take up the correlated case in the next chapter.

We review the binary correlation method and the results obtained in binary approximation [Brody et al., 1981] for the Gaussian ensembles. We also discuss exact asymptotic results for the eigenvalue correlation functions of the Gaussian ensembles [Mehta, 2004]. By using the binary correlation method we derive the level density and two-point correlation function for the (uncorrelated) Wishart ensembles. This will be useful in the next chapter where we discuss the correlated case.

The chapter is organized as follows. In Sec. 2.1 we describe the ensembles of Wishart matrices. In Sec. 2.2 we give definition of the correlation functions of the eigenvalues. We also define some other quantities which are useful in the later chapters. In Sec. 2.3 we review the exact results for the Gaussian ensemble. In Sec. 2.4 we describe the binary correlation method for the Gaussian ensembles. In Sec. 2.5, 2.6 we use this method to obtain the level density and the two-point function for the Wishart ensembles. In Sec. 2.7 we briefly discuss limitations of the binary correlation method.
2.1 Ensembles of Wishart Matrices

Suppose we have some dataset obtained from \( M \) independent realizations of \( N \) independent stochastic variables. For instance, consider \( X_1^{(j)} \), for \( j = 1, \ldots, N \), to be the price of a share in the stock market for \( N \) distinct companies recorded at time \( t_1 \). Then \( M \) repetitions of this experiment, each after an interval of unit time step, generate a time series given by \( X_k^{(j)} \) of the prices of the share of \( j \)th company. Here \( k = 1, \ldots, M \) and we have \( N \) distinct time series, each of length \( M \). The \( X_k^{(j)} \) could also denote \( M \) realizations of \( N \) independent initial conditions in a chaotic time series, such as \( M \) iterations of the momentum variable in the standard map. For another example, one could consider \( M \) realizations of \( N \) neighboring spins in a kinetic Ising model.

At time \( t_k = t_1 + k - 1 \), we define a column vector \( \Gamma_k \)

\[
\Gamma_k = \begin{pmatrix} X_k^{(1)} \\ \vdots \\ X_k^{(N)} \end{pmatrix}
\]

(2.1.1)

Now the covariance matrix \( H \), defined as

\[
H = \frac{1}{M} \sum_{k=1}^{M} \Gamma_k \Gamma_k^t,
\]

(2.1.2)

is the sample covariance matrix if \( X_k^{(j)} \) have zero sample mean. In this thesis, we consider the variables \( X_k^{(j)} \) to have zero ensemble mean (\( \overline{X_k^{(j)}} = 0 \), where the bar denotes the ensemble average). Here \( H \) is \( N \times N \) matrix by construction and contains the statistical information about the time series of the variables. An alternative definition of \( H \) is given in terms of matrix \( A \),

\[
H = \frac{1}{M} AA^t,
\]

(2.1.3)
where $A \equiv \{\Gamma_1 \Gamma_2 \ldots \Gamma_M\}$ and the matrix elements $A_{jk} = X_k^{(j)}$.

In the above definition each column of $A$ represents the time whereas each row represents a distinct variable. The matrix element $(AA^T)_{jk}/M$ gives the time-averaged covariance between the $j$th and $k$th variables. By construction $H$ is a real symmetric matrix with non-negative eigenvalues. Similarly one can define an $M \times M$ matrix $H = A^T A/N$ as the space-averaged sample covariance matrix, where space denotes the company, initial condition or location of the spin in above examples.

One can also consider an example where the matrix $A$ has complex entries, such as the wind velocity which varies with space $x$ and time $t$ and represented by a complex number $s \exp(it\theta)$ where $s$ is the wind speed and $\theta$ is the direction [Santhanam & Patra, 2001]. For such cases $H$ is a complex-hermitian matrix again with non-negative eigenvalues. Apart from being covariance matrix, $H$ is an a priori good model for systems where non-negative definite eigenvalues are encountered. For example, in communication engineering $A$ serves as a model for the channel matrix and $H$ describes the inter-carrier interface [Muller, 2003]. Similarly in the study of vibrational spectra of amorphous nanosystems, the Hessian matrix also has a similar structure [Sarkar et al., 2004; Matharoo et al., 2005].

We consider the three $\beta$ cases, viz. $\beta = 1, 2$ and $4$, where the parameter $\beta$ classifies the entries of the matrix $A$. For $\beta = 1$, $A_{jk}$ are real Gaussian variables. For $\beta = 2$, $A_{jk} = A_{jk}^{(0)} + iA_{jk}^{(1)}$ where $A_{jk}^{(0)}$ and $A_{jk}^{(1)}$ are real Gaussian variables. For $\beta = 4$ we consider

$$A_{jk} = A_{jk}^{(0)} I + \sum_{\gamma=1}^{3} A_{jk}^{(\gamma)} \tau_\gamma, \quad (2.1.4)$$

where $A_{jk}^{(\gamma)}$ for $\gamma = 0, \ldots, 3$, are the Gaussian variables, $I$ is the two-dimensional identity matrix and the $\tau_\gamma$, for $\gamma = 1, 2, 3$, are the two-dimensional matrix representative of the quaternion units given in terms of Pauli spin matrices $\sigma_\gamma$, as $\tau_\gamma = -i\sigma_\gamma$. For $\beta = 1$, $A$ is real so that $A^T = A^T$, $A^T$ being the transpose of $A$. $H$ is then real-symmetric matrix for $\beta = 1$. For $\beta = 2$, representing the hermitian conjugate of $A$ by $A^\dagger$ we obtain the complex hermitian matrices $H = H^{(0)} + iH^{(1)}$ where $H^{(0)} = A^{(0)}[A^{(0)}]^T + A^{(1)}[A^{(1)}]^T$ is
a symmetric matrix and \( H^{(1)} = A^{(1)}[A^{(0)}]^T - A^{(0)}[A^{(1)}]^T \) is an anti-symmetric matrix. Thus \( H = H^\dagger \) is a complex-hermitian matrix. For \( \beta = 4 \) we deal with quaternion-real matrices \( A \) (i.e., real \( A^\dagger \)). Then \( A^\dagger \) is the dual of \( A \); see Appendix A.1. Again \( H = H^\dagger \), so that \( H \) is quaternion-real self-dual matrix.

Without loss of generality, we consider \( N \leq M \). Let \( y_j \) be the singular values of \( A \) where \( j = 1, ..., N \). Then for all the three \( \beta \) cases, the eigenvalues \( x_j \) of \( H \) is given by \( x_j = y_j^2 \), for \( j = 1, ..., N \). Thus the \( x_j \) are always non-negative as mentioned above, in the examples given for \( \beta = 1 \) and 2 case. Moreover, \( N - M \) eigenvalues of \( H \) are precisely zero when \( M < N \).

We consider ensemble of \( H \), invariant under the orthogonal, unitary and symplectic transformations respectively for \( \beta = 1, 2 \) and 4. Let the \( A_{jk}^{(\gamma)} \) be independent Gaussian variables with zero mean and variances \( \sigma_{\beta}^2 \). The joint probability distribution (jpd) of \( A \) can be given by Gaussian probability measure

\[
P(A) dA = C_{N,\beta} \exp \left[ -\frac{\text{Tr} AA^\dagger}{2\sigma_{\beta}^2} \right] \prod_{j=1}^{N} \prod_{k=1}^{M} dA_{jk}^{(\gamma)},
\]

where \( C_{N,\beta} \) is the normalization constant given by

\[
C_{N,\beta} = (2\pi\sigma_{\beta}^2 N)^{-NM\beta/2}.
\]

Variance supplies the scale. We fix the scale \( \sigma_{\beta}^2 = \sigma^2 \beta^{-1} \). The jpd of \( H \) has been given by Wishart [1928] for \( \beta = 1 \). Generalizing this to \( \beta = 2 \) and 4, we write [Wilks, 1962; Beenakker, 1997]

\[
P(H) dH \propto (\det H)^{(\beta N(\kappa-1)+1-2\beta)/2} \exp \left[ -\frac{M\beta}{2\sigma^2} \text{Tr} H \right] dH,
\]

where \( dH = \prod_{j>k}^{N} \prod_{j=0}^{\beta-1} dH_{jk}^{(\gamma)} \prod_{j=1}^{N} dH_{jj} \) is infinitesimal volume in \( N + \beta N(N - 1)/2 \) dimensional matrix space and \( \kappa = M/N \). The jpd in \( N \)-dimensional eigenvalue space
is given by

\[ P(x_1, \ldots, x_N) \propto \prod_{j=1}^{N} w(x_j) \prod_{j>k}^{N} |x_j - x_k|^\beta, \quad (2.1.8) \]

where

\[ w(x) = x^{\beta N(\kappa - 1) + 1 - 2/\beta/2} \exp \left[ -\frac{\kappa N \beta}{2\sigma^2} x \right]. \quad (2.1.9) \]

The jpd (2.1.8) is obtained from (2.1.7) in two steps. First we use the Jacobian of the transformation from matrix element space to eigenvalue-eigenvector space. Next we perform integrations over the eigenvectors and obtain (2.1.8); see [Mehta, 2004].

The jpd (2.1.8) corresponds to the Laguerre ensembles (LE). Since the matrix elements of $A$ are independent, we may also refer to this ensemble as uncorrelated Wishart ensemble. These are invariant respectively under the orthogonal, unitary and symplectic transformation for $\beta = 1, 2$ and 4. The corresponding invariant ensembles are LOE, LUE and LSE respectively for $\beta = 1, 2$ and 4. For further discussions we need the correlation functions which are defined in the next section.

### 2.2 Correlation Functions

The $n$-point correlation function $R_n(x_1, \ldots, x_n)$ is defined as

\[ R_n(x_1, \ldots, x_n) = \frac{N!}{(N - n)!} \int dx_{n+1} \ldots dx_N P(x_1, \ldots, x_N). \quad (2.2.1) \]

$R_n(x_1, \ldots, x_n)$ gives the probability density of $n$ eigenvalues irrespective of the positions of remaining eigenvalues. For example, $R_1(x)$ is one-point function which gives statistics of one level, i.e., the level density. Similarly $R_2(x, y)$ is the two-point function which contains statistics of two levels.
The normalized eigenvalue density of $H$ is given by

$$\rho(x) = \langle \delta(x - H) \rangle = N^{-1} \sum_{j=1}^{N} \delta(x - x_j), \quad (2.2.2)$$

where $x_j$ are the eigenvalues of $H$, angular brackets denote the spectral averaging $\langle H \rangle = N^{-1} \text{Tr} H$ and the bar denotes ensemble averaging. (In Chapters 5, 6, the angular brackets and the bar will denote different averages.) It is immediate from (2.2.1)

$$R_1(x) = N\bar{\rho}(x). \quad (2.2.3)$$

The two-level correlation function is define as

$$S^p(x,y) = \overline{\rho(x)\rho(y)} - \overline{\rho(x)}\overline{\rho(y)}
= \frac{1}{N^2} \left[ R_2(x,y) - R_1(x)R_1(y) + N\delta(x-y)R_1(x) \right]
= \frac{1}{N^2} \left[ N\delta(x-y)R_1(x) - T_2(x,y) \right], \quad (2.2.4)$$

where $N\delta(x-y)R_1(x)$ comes from $N$ self-correlations. We have introduced here the two-level cluster function $T_2(x,y)$, defined as

$$T_2(x,y) = -R_2(x,y) + R_1(x)R_1(y). \quad (2.2.5)$$

In the limit $N \to \infty$ the mean density, $D(x)$, is given by

$$D(x) = [R_1(x)]^{-1}. \quad (2.2.6)$$

To describe the eigenlevel fluctuations in the neighborhood of $x$ we unfold the spectrum locally as

$$x_j = x + r_jD(x) \quad j = 1, 2, \ldots, n. \quad (2.2.7)$$
Now we define the n-point correlation function \( R_n(x_1, ..., x_n) \) for the unfolded spectrum as

\[
\mathbf{R}_n(r_1, ..., r_n; x) = \lim_{N \to \infty} [D(x)]^n R_n(x_1, ..., x_n).
\]  

(2.2.8)

The unfolded two-level cluster function \( Y_2(r; x) \) is defined as

\[
Y_2(r; x) = \lim_{N \to \infty} [D(x)]^2 T_2(x, y),
\]

(2.2.9)

where \( r = r_1 - r_2 \). Note that \( Y_2(r) = Y_2(-r) \). The unfolded spectral form factor \( b_2(k) \) is the Fourier transform of \( Y_2(r) \), given by

\[
b_2(k) = \int dr \exp(2\pi ikr) Y_2(r).
\]

(2.2.10)

For numerical considerations, we consider the fluctuation measures, e.g., the spacing distribution \( P(s) \), namely the density of nearest-neighbor spacings \( s \) normalized to unit average, and the number variance \( \Sigma^2(r) \), namely the variance of the number of eigenvalues in intervals containing on the average \( r \) eigenvalues, as defined in the Chapter 1. \( \Sigma^2(r) \) can be obtained from \( Y_2(r) \) and \( b_2(k) \) by using

\[
\Sigma^2(r) = r - \int_{-r}^r ds (r - s) Y_2(s) = \int_{-\infty}^{\infty} dk \left( \frac{\sin(\pi kr)}{\pi k} \right)^2 (1 - b_2(k)).
\]

(2.2.11)

In our derivations we use the Stieltjes transform, \( G(z) \), of the density which is defined as

\[
G(z) = \left( \frac{1}{z - H} \right) = \int_{-\infty}^{\infty} \rho(x) \frac{dx}{z - x}.
\]

(2.2.12)

\( G(z) \) may also be called the Green's function or the resolvent. Since \(-\pi \delta(x) = \Im(x + i\epsilon)^{-1}\), from (2.2.2, 2.2.12) we obtain the eigenvalue density \( \rho(x) \) uniquely via
the relation
\[
\rho(x) = \mp \frac{1}{\pi} \Im G(x \pm i\epsilon) = \frac{G(x \mp i\epsilon) - G(x \pm i\epsilon)}{2\pi i},
\]  
(2.2.13)

where \( \epsilon \) is positive infinitesimal. For the two-point function, we define the two-point Stieltjes transform \( S^G \):
\[
S^G(z_1, z_2) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{S^\rho(x, y)}{(z_1 - x)(z_2 - y)} = G(z_1)G(z_2) - \overline{G(z_1)} \overline{G(z_2)} \equiv \delta G(z_1)\delta G(z_2).
\]  
(2.2.14)

The inversion formula, relating \( S^\rho \) to \( S^G \), is obtained by using (2.2.13) in (2.2.4)
\[
S^\rho(x, y) = -\frac{1}{4\pi^2} \left[ S^G(z_1^+, z_2^+) + S^G(z_1^-, z_2^-) - S^G(z_1^+, z_2^-) - S^G(z_1^-, z_2^+) \right],
\]  
(2.2.15)

where we have used \( z^\pm = x \pm i\epsilon \).

Defining \( m_p \) as the \( p \)th moment of \( \rho(x) \), we write
\[
m_p = \langle H^p \rangle = \int dx \, x^p \rho(x),
\]  
(2.2.16)

for all integer \( p \geq 0 \). For large \( z \), \( G(z) \) can be expressed in terms of \( m_p \):
\[
G(z) = \sum_{p=0}^{\infty} z^{-p-1} m_p.
\]  
(2.2.17)

It should be noted that the series (2.2.17) is not convergent for all \( z \). However the sum of the convergent series can be extended, except for the branch cut on the real line, to all \( z \) by analytic continuation. Moments, \( \Sigma_{p,q}^{2} \), of \( S^\rho \) are defined as
\[
\Sigma_{p,q}^{2} = \langle H^p \rangle \langle H^q \rangle - \langle H^p \rangle \langle H^q \rangle = \overline{m_p m_q} - \overline{m_p} \overline{m_q} = \int dx dy \, x^p y^q S^\rho(x, y).
\]  
(2.2.18)
Writing \( m_\mu = \overline{m}_\mu + \delta m_\mu \) where \( \delta m_\mu = 0 \), we get
\[
\overline{\delta m}_p \overline{\delta m}_q = \overline{m}_p \overline{m}_q - \overline{m}_p \overline{m}_q.
\] (2.2.19)

Using these in (2.2.14), we obtain the moment expansion of \( S^G \)
\[
S^G(z_1, z_2) = \sum_{p, q=0}^{\infty} \frac{z_1^{-p-1} z_2^{-q-1}}{z_1^{p+1} z_2^{q+1}} \overline{\delta m}_p \overline{\delta m}_q
\]
\[
= \sum_{p, q=0}^{\infty} \frac{\Sigma^2_{p, q}}{z_1^{p+1} z_2^{q+1}}.
\] (2.2.20)

In our derivations, involving the Stieltjes transformation, we deal with \( G_L(z) \) which is defined as
\[
G_L(z) = \langle L \frac{1}{z - H} \rangle = \sum_{p=0}^{\infty} \frac{m^L_p}{z^{p+1}}.
\] (2.2.21)

Here \( L \) is an arbitrary operator and \( m^L_p = \langle LH^p \rangle \). The function \( G_L(z) \) is more general than \( G(z) \) and is also useful in dealing with the expectation value and strength distributions. Similarly we use \( L_1 \) and \( L_2 \) for the two-point Stieltjes transform \( S^G_{L_1, L_2} \):
\[
S^G_{L_1, L_2}(z_1, z_2) = \overline{G}_{L_1}(z_1) \overline{G}_{L_2}(z_2) - \overline{G}_{L_1}(z_1) \overline{G}_{L_2}(z_2)
\]
\[
= \sum_{p, q=0}^{\infty} \frac{z_1^{-p-1} z_2^{-q-1}}{z_1^{p+1} z_2^{q+1}} \left( \overline{m}^{L_1}_{p} \overline{m}^{L_2}_{q} - \overline{m}^{L_1}_{p} \overline{m}^{L_2}_{q} \right)
\]
\[
= \sum_{p, q=0}^{\infty} \frac{z_1^{-p-1} z_2^{-q-1}}{z_1^{p+1} z_2^{q+1}} \overline{\delta m}^{L_1}_{p} \overline{\delta m}^{L_2}_{q}.
\] (2.2.22)

We also deal with the distribution function \( F(x) = \int_{-\infty}^{x} dx' \rho(x') \). The two-point function \( S^F \) and the two-point Stieltjes transform \( S^g \) of \( F \) are related to \( S^p \) and \( S^G \)...
Note that, $S^g$ and $S^F$ are related in the same way as $S^G$ and $S^o$. Thus the inversion formula relating $S^g$ to $S^F$, is given by

$$S^F(x, y) = \frac{1}{4\pi^2} \left[ S^g(z_1^+, z_2^+) + S^g(z_1^-, z_2^-) - S^g(z_1^-, z_2^+) - S^g(z_1^+, z_2^-) \right].$$

(2.2.25)

### 2.3 Correlation Functions of Gaussian Ensembles

Gaussian ensembles (GE) are the ensembles of hermitian random matrices. There are three invariant GEs, viz, Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), and Gaussian symplectic ensemble (GSE). These are invariant respectively under the orthogonal, unitary and symplectic transformation. The matrix elements $H_{jk}$ are real ($\beta = 1$), complex ($\beta = 2$), and real quaternion ($\beta = 4$) respectively for the GOE, GUE and GSE. Matrix $H$ is real symmetric, complex hermitian and self-dual respectively for $\beta = 1, 2$ and $4$. Let the mean and variance for the $\beta$ distinct classes of the off-diagonal matrix elements $(H^{(\gamma)}_{jk}, j > k$, for $\gamma = 0, ..., \beta - 1)$ are respectively zero and $v^2_\beta$. The jpd of $N$-dimensional matrix $H$ is given by

$$P(H)dH = C_{N,\beta} \exp \left[ -\text{Tr}H^2/4v^2_\beta \right] dH,$$

(2.3.1)

in $N + \beta N(N-1)/2$ dimensional infinitesimal volume space $dH$ where prefactor $C_{N,\beta}$ is the normalization constant,

$$C_{N,\beta} = 2^{-N/2}(2\pi v^2_\beta)^{-1/2[N+\beta N(N-1)/2]}.$$
As in (2.1.7 - 2.1.8), from (2.3.1) the jpd of the eigenvalues $x_j$, for $j = 1, \ldots, N$, of $H$ is obtained to be

$$P(x_1, \ldots, x_N) \propto \prod_{j>k}^{N} |x_j - x_k|^{\beta} \exp \left[ - \sum_{j=1}^{N} x_j^2 / 4v_\beta^2 \right]. \quad (2.3.3)$$

For GEs of large matrices, the one-point function is the well-known semicircle,

$$R_1(x) = \begin{cases} \frac{2N}{\pi b^2} \sqrt{b^2 - x^2}, & |x| < b, \\ 0, & |x| > b, \end{cases} \quad (2.3.4)$$

where $b^2 = 4\beta v_\beta^2 N$. Two choices of $v_\beta^2$ have been often used by authors. For $v_\beta^2 = (2\beta)^{-1}$, the radius becomes $b = \sqrt{2N}$. For $v_\beta^2 = (\beta N)^{-1}$, from (2.2.3) we obtain the level density of radius 2, given ahead in (2.4.9).

Exact asymptotic result of the unfolded $n$-point correlation function is known in terms of quaternion determinants [Mehta, 2004]

$$R_n^{(\beta)}(r_1, \ldots, r_n) = Q\text{det} \left[ \sigma_\beta(r_j - r_k) \right]_{j,k=1,\ldots,n} = \left\{ \text{det} \left[ \sigma_\beta(r_j - r_k) \right] \right\}^{1/2}. \quad (2.3.5)$$

Here

$$\sigma_2 = \begin{pmatrix} S(r) & 0 \\ 0 & S(r) \end{pmatrix}, \quad (2.3.6)$$

$$\sigma_1 = \begin{pmatrix} S(r) & D(r) \\ I(r) - \epsilon(r) & S(r) \end{pmatrix}, \quad (2.3.7)$$

$$\sigma_4 = \begin{pmatrix} S(2r) & D(2r) \\ I(2r) & S(2r) \end{pmatrix}, \quad (2.3.8)$$

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\[ S(r) = \frac{\sin(\pi r)}{\pi r}, \quad (2.3.9) \]

\[ D(r) = \frac{dS(r)}{dr}, \quad (2.3.10) \]

\[ I(r) = \int_0^r S(r')dr', \quad (2.3.11) \]

\[ \epsilon(r) = \frac{r}{2|r|}. \quad (2.3.12) \]

The quaternion determinant (Qdet) in (2.3.5) has a determinant-like expansion in terms of quaternion matrix elements \( \sigma_{\beta}(r_j-r_k) \) of the \( n \)-dimensional self-dual quaternion matrix. In fact, it is square root of the \( 2n \)-dimensional ordinary determinant as given in (2.3.5).

The two-level cluster function for \( \beta = 2 \) is given by

\[ Y_2(r) = [S(r)]^2. \quad (2.3.13) \]

Thus the unfolded spectral form factor is obtained as

\[ b_2(k) = \begin{cases} 
0, & \text{for } |k| \geq 1, \\
1 - |k|, & \text{for } |k| \leq 1. 
\end{cases} \quad (2.3.14) \]

For \( \beta = 1 \), \( Y_2(r) \) is given by

\[ Y_2(r) = \left( \int_r^\infty dr' S(r') \right) \left( \frac{dS(r)}{dr} \right) + (S(r))^2. \quad (2.3.15) \]

For small \( r \) this has an expansion

\[ Y_2(r) = 1 - \frac{\pi^2 r^2}{6} + \frac{\pi^4 r^4}{60} - \frac{\pi^4 r^4}{135} + \ldots, \quad (2.3.16) \]
and for large $r$

$$Y_2(r) = \frac{1}{\pi^2 r^2} - \frac{1 + \cos^2(\pi r)}{\pi^4 r^4} + ...$$  \hspace{1cm} (2.3.17)$$

Then the unfolded spectral form factor is given by

$$b_2(k) = \begin{cases} 
-1 + |k| \ln \left( \frac{2|k| + 1}{2|k| - 1} \right), & \text{for } |k| \geq 1, \\
1 - 2|k| + |k| \ln(1 + 2|k|), & \text{for } |k| \leq 1.
\end{cases}$$  \hspace{1cm} (2.3.18)$$

For $\beta = 4$, $Y_2(r)$ is

$$Y_2(r) = (S(2r))^2 - \left( \int_0^r dr' S(2r') \right) \left( \frac{dS(2r)}{dr} \right).$$  \hspace{1cm} (2.3.19)$$

For small $r$ this gives

$$Y_2(r) = 1 - \frac{(2\pi r)^4}{135} + ...,$$  \hspace{1cm} (2.3.20)$$

Figure 2.1: $Y_2(r)$ vs $r$ for different $\beta$ values.
and for large $r$

$$Y_2(r) = -\frac{\pi}{2} \cos(2\pi r) + \frac{1 + (\pi/2) \sin(2\pi r)}{(2\pi r)^2} - \frac{1 + \cos^2(2\pi r)}{(2\pi r)^4} + \ldots \quad (2.3.21)$$

The unfolded spectral form factor for $\beta = 4$ is given by

$$b_2(k) = \begin{cases} 
0, & \text{for } |k| \geq 2, \\
1 - \frac{|k|}{2} + \frac{1}{4} |k| \ln|1 - |k||, & \text{for } |k| \leq 2.
\end{cases} \quad (2.3.22)$$

We illustrate these results in Fig. 2.1 and in Fig. 2.2. In Fig. 2.1 we show $Y_2(r)$ vs $r$ where we have obtained $Y_2(r)$ from the numerical integration of (2.3.15) and (2.3.19) respectively for $\beta = 1$ and 4. In Fig. 2.2 we show $b_2(k)$ vs $k$ obtained from (2.3.18, 2.3.14) and (2.3.22) respectively for $\beta = 1, 2$ and 4. Note that in Fig. 2.2, $b_2^{(\beta)}(k)$ is not the same for all three $\beta$ as it seems to be nearly at $k = 0.86$. In fact $b_2^{(1)}(k) = b_2^{(2)}(k)$ at $k = 0.8573$ and $b_2^{(2)}(k) = b_2^{(4)}(k)$ at $k = 0.8647$.

For small $k$, $b_2(k) = 1 - 2|k|/\beta$. This leads to rigidity in the spectrum, e.g., $\Sigma^2(r)$
Figure 2.3: $\Sigma^2(r)$ vs $r$ for different $\beta$ values. These are the exact results obtained from the numerical integration of (2.2.11). Oscillations for $\beta = 4$ are not contained in the logarithmic expansion of equation (2.3.23).

becomes logarithmic [Kumar & Pandey, 2008]

$$
\Sigma^2(r) = \frac{2}{\pi^2} \left[ \ln(\tilde{r}) + \gamma + 1 - \frac{\pi^2}{8} \right], \quad \beta = 1,
$$

$$
= \frac{1}{\pi^2} [\ln(\tilde{r}) + \gamma + 1], \quad \beta = 2,
$$

$$
= \frac{1}{2\pi^2} \left[ \ln(2\tilde{r}) + \gamma + 1 + \frac{\pi^2}{8} \right], \quad \beta = 4, \quad (2.3.23)
$$

valid for $r \gtrsim 1$, where $\tilde{r} = 2N \sin(\pi r/N)$, and $\gamma$ is the Euler constant. For $1 \lesssim r \ll N$, $\tilde{r} = 2\pi r$ giving thereby more familiar form of $\Sigma^2(r)$ [Brody et al., 1981]. For small $r$ ($r \lesssim 1$) one can directly integrate (2.3.15, 2.3.13, 2.3.19), respectively for $\beta = 1, 2$ and 4. In Fig. 2.3, we have shown the number variances for the three $\beta$ cases. These are obtained from the numerical integration of (2.2.11, 2.3.15), (2.2.11, 2.3.13) and (2.2.11, 2.3.19), respectively for $\beta = 1, 2$ and 4.

The nearest neighbor spacing distribution, $P(s)$, is described by the Wigner sur-
Figure 2.4: $P(s)$ vs $s$ for different $\beta$ values. These are the Wigner surmisises.

\[ P(s) = \frac{\pi}{2} s \exp(-\pi s^2/4), \quad \beta = 1, \]
\[ = \frac{32}{\pi^2} s^2 \exp(-4s^2/\pi), \quad \beta = 2, \]
\[ = \frac{(64)^3}{729\pi^3} s^4 \exp(-64s^2/9\pi), \quad \beta = 4. \quad (2.3.24) \]

Wigner surmise is close to the exact result of $P(s)$ [Mehta, 2004]. Fig. 2.4 shows the importance of the repulsion law for small spacings. As mentioned in the Chapter 1, these results have been useful in the analysis of the spectra of quantum chaotic systems.

### 2.4 Binary Correlation Method for Gaussian Ensembles

In this section derive the level density and two-point function of the GEs by the binary correlation method. We always consider large matrices. The binary correlation method gives approximate results for large matrices. We describe this method for the
Gaussian ensembles. This will be useful when we consider the Wishart ensembles later in this chapter and in the next chapter.

We begin with (2.3.1) which gives

$$H_{\gamma \gamma'}^{(\gamma)} H_{\gamma \gamma}^{(\gamma)} = v_\beta^2 \delta_{\gamma \gamma'} (\delta_{jm} \delta_{kn} + \varepsilon_{\gamma} \delta_{jn} \delta_{km}),$$

(2.4.1)

where $\gamma, \gamma' = 0, \ldots, \beta - 1$ and $\varepsilon_{\gamma} = 1, -1$ respectively for $\gamma = 0$ and $\gamma = 1, 2, 3$. As in [Mon & French, 1975], we fix the scale by choosing

$$\beta v_\beta^2 N = 1.$$  

(2.4.2)

With this choice, the variance of the ensemble averaged eigenvalue is unity:

$$\langle H^2 \rangle = \frac{1}{N} \sum_{j,k} \sum_{\gamma=0}^{\beta-1} H_{\gamma j}^{(\gamma)} H_{\gamma k}^{(\gamma)} = 1 + \beta^{-1} N^{-1} (2 - \beta) \to 1.$$  

(2.4.3)

Here we use arrow to denote equality to the leading order. Using (2.4.1) we obtain two exact results, valid for arbitrary fixed matrices $\phi$ and $\psi$,

$$\langle H\phi H\psi \rangle = \langle \phi \rangle \langle \psi \rangle + \frac{2 - \beta}{\beta N} \langle \phi \tilde{\psi} \rangle,$$

(2.4.4)

$$\langle H\phi \rangle \langle H\psi \rangle = \frac{1}{\beta N^2} \left[ \langle \phi \psi \rangle + \langle \phi \tilde{\psi} \rangle \right].$$  

(2.4.5)

Here $\tilde{\psi}$ should be interpreted as $\psi^T$, $\psi$ and $\psi^D$ respectively for $\beta = 1, 2$ and 4. These results may be derived by noting that in each case the average must be a linear combination of the invariants $\langle \phi \rangle \langle \psi \rangle$, $\langle \phi \psi \rangle$ and $\langle \phi \tilde{\psi} \rangle$ because of the invariance of $H$ under orthogonal, unitary and symplectic transformation respectively for $\beta = 1, 2$ and 4. Then the coefficients follow from some simple choices of $\phi$ and $\psi$ (e.g., $\phi_{jk} = \delta_{jp} \delta_{kq}$ and $\psi_{jk} = \delta_{jr} \delta_{ks}$ for arbitrary $p, q, r, s$). The only complication here arises for $\beta = 4$ in which the non-commutativity of $\tau$ matrices must be borne in mind, as well as the need for taking the traces over the internal quaternion space.

For the Gaussian ensembles, the odd-moments vanishes; we have $m_{2p+1} = 0$. Also,
\[ \bar{m}_0 = 1 = \bar{m}_2, \] which follows from the definition (2.2.16) and (2.4.3). For \( p \geq 2 \), using (2.4.4), we get

\[
\begin{align*}
\bar{m}_p &= \sum_{r=0}^{p-2} \langle HH^{p-r-2}H^r \rangle \\
&= \sum_{r=0}^{p-2} \bar{m}_{p-r-2}\bar{m}_r + \frac{(2 - \beta)(p - 1)}{\beta N} \bar{m}_{p-2} \\
&\to \sum_{r=0}^{p-2} \bar{m}_{p-r-2}\bar{m}_r.
\end{align*}
\] (2.4.6)

Here in the last term, where the equality is for the leading order, the binary association across the traces, as in (2.4.5), and the binary association with crossed correlation lines (inter-linked-pairs) produce lower order terms and hence are neglected. Using (2.4.6) in the moment expansion of \( \bar{G}(z) \), given in (2.2.17), we find

\[
\bar{G}(z) = \frac{1}{z} + \sum_{p=2}^{\infty} \sum_{r=0}^{p-2} \frac{\bar{m}_{p-r-2}\bar{m}_r}{z^{p+1}}
\]

\[
= \frac{1}{z} + \frac{1}{z} \sum_{r=0}^{\infty} \frac{\bar{m}_r}{z^{r+1}} \sum_{p=0}^{\infty} \frac{\bar{m}_p}{z^{p+1}}
\]

\[
= \frac{1}{z} \left[ 1 + \left( \bar{G}(z) \right)^2 \right].
\] (2.4.7)

Solving this for \( \bar{G}(z) \) we obtain

\[
\bar{G}(z) = \frac{1}{z - \bar{G}(z)}
\]

\[
= \frac{z - \sqrt{z^2 - 4}}{2}.
\] (2.4.8)

We take the branch cut \([-2, 2]\) and choose the the branch in which \( \bar{G}(z) \) goes to zero as \( z^{-1} \) for large \( |z| \). By using the inversion formula (2.2.13) we obtain,

\[
\bar{\rho}(x) = \frac{\sqrt{4 - x^2}}{2\pi},
\] (2.4.9)
the semicircular density of radius 2, same as obtained from the (2.3.4). In this case one can also work out the moments explicitly. The odd moments vanish and the even moments are given as the "Catalan numbers":

$$\bar{m}_{2p} = \frac{1}{p + 1} \binom{2p}{p}.$$  \hspace{1cm} (2.4.10)

As we have done for the density, to obtain the two-point function, we calculate the two-point Stieltjes transform first. We begin with the moments again. Using (2.4.4, 2.4.5) we obtain

$$\overline{m_p m_q} = \sum_{r=0}^{p-2} \overline{m_{p-r-2} m_r m_q} + \frac{(2 - \beta)(p - 1)}{\beta N} \overline{m_{p-2} m_q}$$

$$+ \sum_{r=0}^{q-1} \langle HH^{p-1} H^r H H^q H H^{q-r-1} \rangle,$$  \hspace{1cm} (2.4.11)

where the first two terms, in the right-hand side, are coming from the binary associations of type (2.4.4). The last term in (2.4.11) comes from binary associations across the trace (2.4.5), giving thereby $O(N^{-2})$ contribution. The Stieltjes transform of two-point function can be obtained in terms of moment expansion, as given in (2.2.20). Using (2.2.19) it is easy to prove that $\delta m_p \delta m_0 = \overline{\delta m_0 \delta m_q} = 0$ for $p, q \geq 0$. We use the expansion $m_{\mu} = \overline{m_{\mu}} + \delta m_{\mu}$ in (2.4.11) and (2.2.19). We find

$$\overline{\delta m_p \delta m_q} \rightarrow \sum_{r=0}^{p-2} \left[ \delta m_{p-r-2} \delta m_q \overline{m_r} + \delta m_r \delta m_q \overline{m_{p-r-2}} \right]$$

$$+ \sum_{r=0}^{q-1} \langle HH^{p-1} H^r H H^q H H^{q-r-1} \rangle,$$  \hspace{1cm} (2.4.12)

where other terms of $\overline{m_p m_q}$ cancel with the same expansion of $\overline{m_p m_q}$ in the leading order; terms of order $O(1)$ and $O(N^{-1})$ cancel exactly and rest of the terms are of higher order $O(N^{-3})$ and hence are ignored. Also, the first two terms of (2.4.12) are valid for $p \geq 2$, $q \geq 1$ whereas the last term includes $p \geq 1$ and $q \geq 1$. The latter includes the series $\overline{\delta m_1 \delta m_q}$ for $q \geq 1$.  

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Now by using (2.2.20), we find
\[
\delta G(z_1) \delta G(z_2) = \frac{1}{z_1} \sum_{p,q,r=0}^{\infty} \left[ \frac{\delta m_p \delta m_q \ m_r + \delta m_r \delta m_q \ m_p}{z_1^{p+1} z_2^{q+1} z_1^{r+1}} \right] \\
+ \frac{2}{z_1 z_2 \beta N^2} \sum_{p,q=0}^{\infty} (q+1) \frac{\delta_p + \delta_q}{z_1^{p+1} z_2^{q+1}}.
\]

(2.4.13)

Here in the intermediate step \( q \) appears in the third term because of the invariance of trace under the cyclic rotation. Next, we obtain (2.4.13) after rearrangement of the series. Summations over the indices \( p, q \) and \( r \) yield
\[
\delta G(z_1) \delta G(z_2) = \frac{2}{z_1} \left[ \frac{\delta G(z_1) \delta G(z_2)}{\delta G(z_1)} \right] - \frac{1}{\beta N^2} \frac{\partial}{\partial z_2} \left( \left( z_1 - H \right)^{-1} \left( z_2 - H \right)^{-1} \right) \\
= \frac{2}{\beta N^2 [z_1 - 2G(z_1)]} \frac{\partial}{\partial z_2} \left[ \frac{G(z_1) - G(z_2)}{z_1 - z_2} \right].
\]

(2.4.14)

One can further simplify (2.4.14) by observing that
\[
\frac{\partial^2}{\partial z_1 \partial z_2} \ln \left[ \frac{1 - G(z_1) - G(z_2)}{z_1 - z_2} \right] = \left( z_1 - 2G(z_1) \right)^{-1} \frac{\partial}{\partial z_2} \left[ \frac{G(z_1) - G(z_2)}{z_1 - z_2} \right].
\]

(2.4.15)

Thus we obtain
\[
\delta G(z_1) \delta G(z_2) = \frac{2}{\beta N^2} \frac{\partial^2}{\partial z_1 \partial z_2} S^g(z_1, z_2),
\]

(2.4.16)

where
\[
S^g(z_1, z_2) = \frac{2}{\beta N^2} \ln \left[ \frac{1 - G(z_1) - G(z_2)}{z_1 - z_2} \right].
\]

(2.4.17)

See Appendix A.2 for the details of (2.4.15).

Carrying out the differentiation in (2.4.14) and making use of (2.4.8), we obtain
\[
S^g(z_1, z_2) = -\frac{4 - z_1 z_2 + \sqrt{(z_1^2 - 4)(z_2^2 - 4)}}{\beta N^2 (z_1 - z_2)^2 \sqrt{(z_1^2 - 4)(z_2^2 - 4)}}.
\]

(2.4.18)
The inversion formula (2.2.15) relating $S^p$ to $S^G$ yields the two-point function
\[
S^p(x, y) = \frac{4 - xy}{\beta \pi^2 N^2 (x - y)^2 \sqrt{(4 - x^2)(4 - y^2)}}. \tag{2.4.19}
\]

Alternatively $S^p$ can be derived from (2.4.17) in two steps. First we obtain $S^F$ from (2.4.17) by using the inversion formula (2.2.25). Next we use the relation between $S^F$ and $S^p$ given in (2.2.23) and obtain (2.4.19).

It is known from the work of Pandey et al. [2005] that for a single-band density given with the support $[-b, b]$, $S^p$ can be written as
\[
S^p(x, y) = \frac{(b^2 - xy)}{\beta N^2 \pi^2 (x - y)^2 \sqrt{(b^2 - x^2)(b^2 - y^2)}}, \tag{2.4.20}
\]

where $x, y$ are many levels apart. For the Gaussian ensembles, with our choice of $v_2^2$, $b = 2$. This result has been established for a wide class of non-Gaussian ensembles with single-band spectra [Beenakker, 1997; Brezin & Zee, 1993]. It comes about from these works that the variance of $\text{tr} H$ is also a universal quantity and is a measure of long range statistics. With suitable rescaling it gives the universal value for the variance of the conductance fluctuations in mesoscopic systems:
\[
\frac{\text{var}(\text{tr} H)}{(2b)^2} = \frac{1}{8\beta}. \tag{2.4.21}
\]

The higher order moments are given by
\[
\Sigma_{p,q}^2 = \frac{2}{\beta N^2} \left( \frac{b}{2} \right)^{p+q} \sum_{\zeta > 0} \zeta \left( \frac{p - \zeta}{2} \right) \left( \frac{q - \zeta}{2} \right), \tag{2.4.22}
\]
valid for $p + q = \text{even}$ ($\Sigma_{pq}^2 = 0$ for $p + q = \text{odd}$). The sum is restricted to $\zeta$ such that $p - \zeta = \text{even}$. In analogy with the binary correlation treatment of Gaussian ensemble this can be interpreted as an expansion in number ($\zeta$) of $H$'s correlated pairwise between the traces of $H^p$ and $H^q$ [Brody et al., 1981; French et al., 1988a]. The universality in (2.4.21) is immediate from (2.4.22) where one needs to calculate $N^2 \Sigma_{1,1}^2/(2b)^2$. 

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Now by renormalizing the density to unit local spacing, we obtain

$$\frac{S^\rho(x, y)}{\rho(x)\rho(y)} = \frac{4(4 - xy)}{\beta N^2(x - y)^2(4 - x^2)(4 - y^2)}. \tag{2.4.23}$$

To compare this result with the asymptotic limit of the exact result, we consider $r \geq 1$ where $r = N|x - y|\bar{\rho}$ is the average number of levels separating the two points. We find

$$\frac{S^\rho(x, y)}{\rho(x)\rho(y)} \rightarrow \frac{1}{\beta \pi^2 r^2}, \tag{2.4.24}$$

for $r \geq 1$, agreeing with the leading (non-periodic) term in the asymptotic expansion of exact result (2.3.13, 2.3.17, 2.3.21). Using a small non-zero imaginary part in $z_1$ and $z_2$ in (2.4.18) one can get approximations in (2.4.24) valid for all $r$. This expression gives good estimates for $\Sigma^2(r)$ for $r \geq 1$ [Brody et al., 1981; Pandey, 1981].

### 2.5 Binary Correlation Method for Wishart Ensembles

We consider ensembles of uncorrelated Wishart matrices $H = AA^\dagger/M$ described in Sec. 2.1. In this section we obtain first the level density result by using the binary correlation method and then we obtain the same result by using a method given by Ghosh & Pandey [2002]. We first recall the scale, $v^2 = \sigma^2 \beta^{-1}$, that we have fixed in (2.1.8). Using (2.1.5) we can write

$$\overline{A_{jk}^{(\gamma)} A_{\gamma m}^{(\gamma')}} = \frac{\sigma^2}{\beta} \delta_{\gamma \gamma'} \delta_{j \gamma} \delta_{k m}. \tag{2.5.1}$$

We use the notation $\langle \cdot \rangle_N = \kappa \langle \cdot \rangle_M$, where $\kappa = M/N$ and subscripts $N$ and $M$ denote spectral averages of $N \times N$ and $M \times M$ matrices respectively. Using (2.5.1) we obtain
the following exact results valid for arbitrary fixed matrices $\phi$ and $\psi$,

$$\frac{1}{M} \langle A\phi A^\dagger \hat{\psi} \rangle_N = \sigma^2 \langle \phi \rangle_M \langle \psi \rangle_N,$$

(2.5.2)

$$\langle A\phi A\hat{\psi} \rangle_N = \frac{(2-\beta)\sigma^2}{\beta} \langle \hat{\psi}\phi \rangle_N,$$

(2.5.3)

$$\langle A\phi \rangle_N \langle \psi \rangle_N = \frac{\sigma^2}{N} \langle \psi \phi \rangle_N,$$

(2.5.4)

$$\langle A\phi \rangle_N \langle A\psi \rangle_N = \frac{(2-\beta)\sigma^2}{N\beta} \langle \hat{\psi}\phi \rangle_N.$$  

(2.5.5)

Here $\hat{\psi} = \psi^T$, $\check{\psi} = \psi$ and $\check{\psi} = -\tau_2 \psi^T \tau_2$ respectively for $\beta = 1, 2$ and 4. Also, $\phi$ is $M \times M$, $M \times N$, $M \times N$ and $\psi$ is $N \times N$, $M \times N$, $N \times M$, $M \times N$ dimensional matrix respectively in (2.5.2, 2.5.3, 2.5.4, 2.5.5). As in (2.4.4, 2.4.5), these results can also be derived by noting that in each case the average must be given in terms of the quadratic invariants, $\langle \phi \rangle \langle \psi \rangle$, $\langle \hat{\psi}\phi \rangle$ and $\langle \psi\phi \rangle$, since the ensemble of $A$ is invariant under orthogonal, unitary and symplectic transformation respectively for $\beta = 1, 2$ and 4 on the left as well as on the right side of $A$.

It is straight forward from (2.5.2) that $\bar{m}_0 = 1$ and $\bar{m}_1 = \sigma^2$. For $p \geq 2$, we have

$$\bar{m}_p = \frac{1}{M^p} \left[ \langle AA^\dagger (AA^\dagger)^{p-1} \rangle + \sum_{r=0}^{p-2} \langle AA^\dagger (AA^\dagger)^r AA^\dagger (AA^\dagger)^{p-r-2} \rangle \right]$$

$$+ \langle AA^\dagger (AA^\dagger)^r AA^\dagger (AA^\dagger)^{p-r-2} \rangle \right]$$

$$= \sigma^2 \bar{m}_{p-1} + \frac{\sigma^2}{\kappa} \sum_{r=0}^{p-2} \bar{m}_{p-r-2} \bar{m}_{r+1} + \frac{\sigma^2(2-\beta)(p-1)}{\beta \kappa N} \bar{m}_{p-1} \bar{m}_q$$

$$\rightarrow \sigma^2 \bar{m}_{p-1} + \frac{\sigma^2}{\kappa} \sum_{r=0}^{p-2} \bar{m}_{p-r-2} \bar{m}_{r+1},$$  

(2.5.6)

where in the last term the binary associations across the traces produce $O(N^{-1})$ terms.
and hence are ignored. Using these in (2.2.17), we obtain

\[
\bar{G}(z) = \frac{1}{z} + \sigma^2 \sum_{p=1}^{\infty} \frac{\bar{m}_{p-1}}{z^{p+1}} + \frac{\sigma^2}{\kappa} \sum_{p=2}^{\infty} \sum_{r=0}^{p-2} \frac{\bar{m}_{p-r-2}}{z^{p+1}} \bar{m}_{r+1} \\
= \frac{1}{z} + \sigma^2 \sum_{p=0}^{\infty} \frac{\bar{m}_{p}}{z^{p+1}} + \frac{\sigma^2}{\kappa} \sum_{r=0}^{\infty} \frac{\bar{m}_{r+1}}{z^{r+2}} \sum_{p=0}^{\infty} \frac{\bar{m}_{p}}{z^{p+1}}.
\]

These summations can be written in terms of the Stieltjes transform of the density. Thus in (2.5.7) we obtain

\[
\bar{G}(z) = \frac{1}{z} + \sigma^2 \frac{\bar{G}(z)}{z} + \frac{\sigma^2}{\kappa} \left( \bar{G}(z) - \frac{1}{z} \right) \bar{G}(z) \\
= \frac{1}{z - \frac{\sigma^2}{\kappa} (\kappa - 1 + z\bar{G}(z))} \\
= \frac{z - \sigma^2 (\kappa - 1)/\kappa - \sqrt{(z - \sigma^2 (\kappa - 1)/\kappa)^2 - 4z\sigma^2 /\kappa}}{2\sigma^2 /\kappa},
\]

where, again, we choose the branch of \( \bar{G} \) which behaves as \( z^{-1} \) for large \( |z| \). The inversion formula (2.2.13) yields the density

\[
\bar{\rho}(x) = \kappa \sqrt{4\sigma^4 \kappa^{-1} - (x - \sigma^2 \kappa^{-1} - \sigma^2)^2} \\
= \frac{\kappa \sqrt{b^2 - (x - \sigma)^2}}{2\pi x \sigma^2},
\]

where \( 2b = (x_{\text{max}} - x_{\text{min}}) = 4\sigma^2 /\sqrt{\kappa} \) is the span and \( a = (x_{\text{min}} + x_{\text{max}})/2 = \sigma^2 (\kappa + 1)/\kappa \) is the mean of the end points of the level density. The level density (2.5.9) is also written some times as

\[
\bar{\rho}(x) = \kappa \sqrt{(x_{\text{max}} - x)(x - x_{\text{min}})} / 2\pi x \sigma^2.
\]

An alternative proof comes from the method used by Ghosh & Pandey [2002]. We use the exact hierarchic equation linking \( R_1 \) to \( R_2 \) [Pandey & Shukla, 1991; Pandey,
\[ \frac{\partial R_1(x)}{\partial x} = \beta \int dy \frac{R_2(x, y)}{x - y} + \frac{w'(x)}{w(x)} R_1(x). \] (2.5.11)

For large \( N \), the integral on the right-hand side can be replaced by a principal-value integral involving \( R_2(x, y) \approx R_1(x)R_1(y) \) and on the left-hand side \( \partial R_1/\partial x \) can be dropped. These can be rigorously justified from the behavior of \( R_1 \) and \( R_2 \) for large \( N \). We thus find

\[ \beta N \int dy \frac{\bar{\rho}(y)}{x - y} = -\frac{w'(x)}{w(x)}, \] (2.5.12)

where we have used \( R_1 = N\bar{\rho} \). Using the weight function given in (2.1.9), we find

\[ \int dy \frac{\bar{\rho}(y)}{x - y} = -\frac{\kappa - 1}{2x} + \frac{\kappa}{2\sigma^2}. \] (2.5.13)

After multiplication by \( x\bar{\rho}/(z - x) \) and integration over \( x \), we get

\[ \int dx \int dy \frac{x\bar{\rho}(x)\bar{\rho}(y)}{(x - y)(z - x)} = \int dx \frac{\bar{\rho}(x)}{z - x} \left[ -\frac{\kappa - 1}{2} + \frac{x\kappa}{2\sigma^2} \right]. \] (2.5.14)

On the right-hand side we use

\[ \int dx \int dy \frac{x\bar{\rho}(x)\bar{\rho}(y)}{(x - y)(z - x)} = \int dx \int dy \frac{y\bar{\rho}(x)\bar{\rho}(y)}{(y - x)(z - y)} = \frac{z}{2} \int dx \int dy \frac{\bar{\rho}(x)\bar{\rho}(y)}{(z - x)(z - y)}. \] (2.5.15)

Then the integral form of \( \bar{G}(z) \) given in (2.2.12) yields

\[ \bar{G}(z) = \frac{1}{z - \sigma^2 (\kappa - 1 + z\bar{G}(z)) / \kappa} = \frac{z - \sigma^2 (\kappa - 1) / \kappa - \sqrt{(z - \sigma^2 (\kappa - 1) / \kappa)^2 - 4z\sigma^2 / \kappa}}{2z\sigma^2 / \kappa}. \] (2.5.16)

We see that the Stieltjes transform is the same as the one obtained by using binary
correlation method (2.5.8) and so also is the level density. The level density (2.5.9) has been obtained earlier by several authors [Marchenko & Pastur, 1967; Dyson, 1971; Brody et al., 1981]. See also Fox & Kahn [1964].

In Fig. 2.5 we have shown the density (2.5.9) for different \( \kappa \) values where \( \sigma^2 = 1 \). As shown in the figure, span of the density shrinks on increasing the value of \( \kappa \). The density peak shifts toward \( x = 1 \) as the value of \( \kappa \) increases. Now consider very large value of \( \kappa \). Then replacing \( x - \kappa^{-1} \) by \( y \) in (2.5.9), we obtain

\[
\bar{\rho}(y) = \kappa \frac{\sqrt{4\kappa^{-1} - (y - 1)^2}}{2\pi(y + \kappa^{-1})}
\approx \kappa \frac{\sqrt{4\kappa^{-1} - (y - 1)^2}}{2\pi},
\]

(2.5.17)

since the change in \( y \) is small in comparison to \( \sqrt{4\kappa - \kappa^2(y - 1)^2} \). The semicircular density is peaked around \( x = 1 \) with radius \( \sqrt{2/\kappa} \), where \( \bar{\rho}_{max} \approx \sqrt{\kappa}/\pi \). For instance, with \( \kappa = 10^5 \) we get semicircular density, \( 0.993675 \leq x \leq 1.0063245 \) and \( \bar{\rho}_{max} = 100.6587 \), as shown in the inset of Fig. 2.5.
2.6 Two-Point Function for Uncorrelated Wishart Ensembles by the Binary Correlation Method

To calculate the two-point function for the uncorrelated Wishart ensembles we closely follow the method given in Sec. 2.4. First we use (2.5.2-2.5.5) and obtain an equation analogous to (2.4.11),

\[ m_p m_q = \sigma^2 m_{p-1} m_q + \frac{\sigma^2}{\kappa} \sum_{r=0}^{p-2} m_{p-r-2} m_{r+1} m_q + \frac{\sigma^2 (2 - \beta)(p - 1)}{\beta \kappa N} m_{p-1} m_q. \]

\[ + \frac{1}{M^2} \sum_{r=0}^{q-1} \left[ \langle AA^\dagger H^{p-1} \rangle \langle H^r AA^\dagger H^{q-r-1} \rangle + \langle AA^\dagger H^{p-1} \rangle \langle H^r A A^\dagger H^{q-r-1} \rangle \right]. \] (2.6.1)

This is valid for \( p \geq 2 \) and \( q \geq 1 \). It is easy to prove that

\[ \overline{\delta m_0 \delta m_q} = \delta m_0 \delta m_0 = 0, \] (2.6.2)

where \( p, q \geq 0 \). Next by expanding \( m_\mu \) as \( m_\mu + \delta m_\mu \) in (2.6.1) and using (2.2.19), we obtain in the leading order

\[ \overline{\delta m_\mu \delta m_q} \rightarrow \sigma^2 \delta m_{p-1} \delta m_q + \frac{\sigma^2}{\kappa} \sum_{r=0}^{p-2} \left[ \delta m_{p-r-2} \delta m_q \overline{m_{r+1}} \right] \]

\[ + \frac{2q \sigma^2}{\beta \kappa N^2} \overline{m_{p+q-1}}. \] (2.6.3)

Here the first term is valid for \( p \geq 1 \) and \( q \geq 0 \), the second and third terms give contributions only for \( p \geq 2 \) and \( q \geq 0 \), and the last term is valid for \( p \geq 1 \) and \( q \geq 1 \). As in the Gaussian case, terms of order \( \mathcal{O}(N^{-1}) \) cancel exactly here also with the
expansion of $\overline{m}_p$, $\overline{m}_q$. Now using these in moment expansion of $S^G$, we find

$$\begin{align*}
\overline{\delta G(z_1)\delta G(z_2)} &= \frac{\sigma^2}{z_1} \delta G(z_1)\delta G(z_2) + \frac{\sigma^2}{\kappa} \left[ \delta G(z_1)\delta G(z_2) \left( \overline{G(z_1)} - \frac{1}{z_1} \right) \right] \\
&\quad + \frac{2\sigma^2}{\beta\kappa N^2} \frac{\partial}{\partial z_2} \left[ \left( (z_2 - H)^{-1} \right) \left( (z_1 - H)^{-1} - z_1^{-1} \right) \right].
\end{align*}$$

(2.6.4)

This can be simplified further as

$$\begin{align*}
\overline{\delta G(z_1)\delta G(z_2)} &= \frac{2\sigma^2}{\beta\kappa N^2} \left[ z_1 - \frac{\sigma^2}{\kappa} \left( \kappa - 1 + 2z_1 \overline{G(z_1)} \right) \right]^{-1} \\
&\quad \times \frac{\partial}{\partial z_2} \left[ z_1 \overline{G(z_1)} - z_2 \overline{G(z_2)} \right].
\end{align*}$$

(2.6.5)

We can write (2.6.5) as

$$\begin{align*}
\overline{\delta G(z_1)\delta G(z_2)} &= \frac{\partial^2}{\partial z_1 \partial z_2} S^g(z_1, z_2),
\end{align*}$$

(2.6.6)

where $S^g$ is given by

$$S^g(z_1, z_2) = \frac{2}{\beta N^2} \ln \left[ 1 - \frac{\sigma^2}{\kappa} \frac{z_1 \overline{G(z_1)} - z_2 \overline{G(z_2)}}{z_1 - z_2} \right].$$

(2.6.7)

Now, by carrying out the differentiation in (2.6.4), we get

$$S^G(z_1, z_2) = -\frac{b^2 - (z_1 - a)(z_2 - a) + \sqrt{(z_1 - a)^2 - b^2} \sqrt{(z_2 - a)^2 - b^2}}{\beta N^2(z_1 - z_2)^2 \sqrt{(z_1 - a)^2 - b^2} \sqrt{(z_2 - a)^2 - b^2}}.$$  

(2.6.8)

Using the inversion formula (2.2.15), we obtain the two-point function

$$S^g(x, y) = -\frac{b^2 - (x - a)(y - a)}{\beta \pi^2 N^2(x - y)^2 \sqrt{b^2 - (x - a)^2} \sqrt{b^2 - (y - a)^2}}.$$  

(2.6.9)

This is consistent with the universal form given in (2.4.20) for the density having
For unfolded spectra we obtain

\[
\frac{S^\beta(x, y)}{\bar{\rho}(x)\bar{\rho}(y)} = -\frac{b^2 - (x - a)(y - a)}{\beta \pi^2 N^2(x - y)^2(2\pi \sigma^2/\kappa)^2xy\bar{\rho}^2(x)\bar{\rho}^2(y)} \rightarrow -\frac{1}{\beta \pi^2 r^2},
\]

for \( r \gtrsim 1 \). This result coincides with the result obtained for the Gaussian ensembles (2.4.23).

It has been rigorously proved by Ghosh & Pandey [2002] that the exact \( R_n^{(\beta)} \) for given \( n \) and \( \beta \) is identical for the GEs and LEs.

### 2.7 Limitations of the Binary Correlation Method

The Binary correlation method gives approximate results. It always gives exact results for the level density for large dimensional matrices. For the two-point function, it produces results correct up to the leading order (without the periodic terms) and valid for \( (r \gtrsim 1) \). An advantage of this method is that the results can be derived, simultaneously, for all the three \( \beta \) cases. Nevertheless one may consider beyond binary associations, viz. tertiary or quaternary associations for the more accurate results. However, the method becomes tedious. On the other hand, by making reasonable assumptions about the very short range behavior \( (r \lesssim 1) \) one can obtain expression for \( \Sigma^2 \) with good accuracy. For example, by including a cutoff in the inversion formula \( \rho(x) = -\pi^{-1} \Im G(z) \) with \( z = x + i\epsilon \) and \( \epsilon = O(N^{-1}) \), instead of being 0, one may obtain results with more accuracy. In fact one such procedure, which takes account of the self-correlation term, gives the exact result for \( \beta = 2 \) [Brody et al., 1981].