Chapter 8

Parametric Correlations in Random Matrix Spectra

Parametric correlations are correlations associated with the motion of energy levels when a parameter of the system is varied. These are useful in the theoretical study of the response of complex systems under external perturbations. Applications of parametric correlations have been found in Hydrogen atom in magnetic field [Goldberg et al., 1991], conductance fluctuations of mesoscopic systems in magnetic field [Beenakker, 1997; Guhr et al., 1998], and in model quantum system such as quantum billiards [Simons & Altshuler, 1993] and quantum kicked rotors [Kumar, 2008]. For these systems, the level correlations are strong on the global scale but die out rapidly on the local scale.

We investigate the parametric number covariance which is defined as the covariance of the number of energy levels in intervals of fixed length between spectra for two values of the parameter. We derive the number covariance result for the Gaussian ensembles (GE) by using the binary correlation method [French et al., 1988a]. We also make conjectures about the Laguerre ensembles (LE) (which except for a scale factor is the uncorrelated Wishart ensembles). For comparison we use exact results of the parametric correlations, given by Simons & Altshuler [1993] for the Gaussian ensembles. We also briefly review the results of quantum kicked rotors [Kumar, 2008].
8.1 Random Matrix Models with Parametric Correlations

We consider Gaussian ensemble of matrices $H$ and Laguerre ensemble of matrices $\mathcal{H}$. Both the ensembles have been defined in Chapter 2. The parametric variations in these ensembles with respect to a parameter $\alpha$ is described by the ensembles of matrices $H_\alpha$ and $\mathcal{H}_\alpha$ defined below.

Matrices $H_\alpha$ are defined as

$$H_\alpha = \frac{H_0 + \alpha V}{\sqrt{1 + \alpha^2}}, \quad (8.1.1)$$

where $H_0$ and $V$ are members of the Gaussian ensembles of the same invariance class. Variance of each of the $\beta$ distinct classes of the off-diagonal matrix elements is $\nu_{\beta}^2$ (as in (2.3.1)); it is same for $H_0$, $H_\alpha$ and $V$. Thus $H_\alpha$ and $H_0$ are identically distributed Gaussian ensembles with correlation coefficient $(1 + \alpha^2)^{-1/2}$ between $H_{0;jk}$ and $H_{\alpha;jk}$ for all $j, k$.

For the Laguerre ensembles we define $\mathcal{H}_\alpha$ in terms of $N \times M$ matrices $A_\alpha$:

$$\mathcal{H}_\alpha = A_\alpha A_\alpha^\dagger. \quad (8.1.2)$$

Here

$$A_\alpha = \frac{A_0 + \alpha W}{\sqrt{1 + \alpha^2}}, \quad (8.1.3)$$

and $A_\alpha^\dagger$ is the transpose, hermitian conjugate and dual of $A_\alpha$ respectively for $\beta = 1, 2, 4$, as defined in Sec 2.1. Matrices $A_0$, $A_\alpha$ and $W$ correspond to the same $\beta$ and have the same variance $\nu_{\beta}^2$ (as in (2.1.5)) for each of the $\beta$ distinct classes of the matrix elements. Again $A_0$ and $A_\alpha$ are identically distributed with correlation coefficient $(1 + \alpha^2)^{-1/2}$ for the $j, k$ matrix elements.

For both types of ensembles, we fix the scale $\beta v_{\beta}^2 N = 1$. For the Gaussian ensem-
bles, the level density, $\bar{\rho}(x)$, is given by

$$\bar{\rho}(x) = \frac{\sqrt{4 - x^2}}{2\pi} = \frac{\sin \psi(x)}{\pi},$$

(8.1.4)

where $\psi(x) = \pi - \cos^{-1}(x/2)$ and the bar denotes ensemble averaging. Note that the same density is valid for all $\alpha$ and $\beta$. Similarly for the Laguerre ensembles, the ensemble averaged level density is given by

$$\bar{\rho}(x) = \frac{\sqrt{b^2 - (x - a)^2}}{2\pi x}.$$

(8.1.5)

Here $b = 2\sqrt{\kappa}$ is the half of the span of the density, $a = \kappa + 1$ is mean of the end points of the density and $\kappa = M/N$. This density differs from (2.5.10) only in scale. In (2.5.10) we have fixed $v_B^2 = \sigma^2/\beta$. With $\sigma^2 = M/N = \kappa$, (2.5.10) yields the above density.

We are interested in the number covariance $\Sigma_{11}^2(x, y; \alpha)$ which is the covariance of number of levels in interval $[x, y]$ for $H_0$ and $H_\alpha$ (similarly for $H_0$ and $H_\alpha$). For large $N$ we denote the covariances in intervals of length $r$ as $\Sigma_{11}^2(r; \Lambda)$ for the unfolded spectra. We expect that $\Sigma_{11}^2(r; \Lambda)$ is a universal function for each $\beta$. For large $N$, we find that the local correlations depend on $\alpha$ in terms of the perturbation parameter $\Lambda = \alpha^2 v_B^2 N^2 \bar{\rho}^2 = \beta^{-1} \alpha^2 N \bar{\rho}^2$ for the GEs. Thus the correlations die on the scale $\alpha \sim N^{-1/2}$. For such small values of $\alpha$, the global correlations may remain large, the correlation coefficient being close to unity. Note also that for $\Lambda = 0$, $\Sigma_{11}^2$ becomes $\Sigma^2(r)$. A similar analysis can be done for the LEs. In this case the definition of $\Lambda$ will be different but similar to the result found by Kumar & Pandey [2009] for the LOE to LUE transitions; see the discussion after (8.2.12).
8.2 Binary Correlation Approximation for the Number Covariance

To derive $\Sigma_{11}^2(x, y; \alpha)$ we use

\[
\Sigma_{11}^2(x, y; \alpha) = N^2 [S^F_\alpha(x, x) + S^F_\alpha(y, y) - 2S^F_\alpha(x, y)],
\]

(8.2.1)

where $S^F_\alpha$ is the doubly integrated form of the density correlation function $S^\rho_\alpha(x, y)$. See Sec. 2.2 for the definitions of $S^F$ and $S^\rho$. The index $\alpha$ has been added here to indicate the dependence on $\alpha$.

We calculate first the moments, $\Sigma_{p,q}^2(\alpha)$, defined as

\[
\Sigma_{p,q}^2(\alpha) = \int dx \int dy S^\rho(x, y)x^py^q
\]

\[
= \langle H^\rho_\alpha \rangle \langle H^\rho_\alpha \rangle - \langle H^\rho_\alpha \rangle \langle H^\rho_\alpha \rangle,
\]

(8.2.2)

where $\langle . \rangle$ denotes the spectral averaging. For the GEs, we have [Brody et al., 1981]

\[
\Sigma_{p,q}^2(\alpha) = \sum_{\zeta \geq 1} \zeta \mu^p_\zeta \mu^q_\zeta \langle H^\rho_\alpha \rangle \langle H^\rho_\alpha \rangle
\]

\[
= \sum_{\zeta \geq 1} \mu^p_\zeta \mu^q_\zeta \langle H^\rho_\alpha \rangle \langle H^\rho_\alpha \rangle
\]

\[
= \frac{2}{\beta N^2} \sum_{\zeta \geq 1} \zeta \mu^p_\zeta \mu^q_\zeta \eta^\zeta.
\]

(8.2.3)

Here the first equation is exact and denotes a decomposition of summation in terms of $\zeta H_\alpha$'s in the first trace which is correlated with $\zeta H_0$'s in the second trace. In the second equality (valid for large $N$) $\mu^p_\zeta$ gives the number of correlated pairs that can be put in the first trace with fixed positions of $\zeta H_\alpha$'s and similarly $\mu^q_\zeta$ is the number for the second trace. Also to the leading order in $N$ the $\zeta$ cross correlations appear
in a cyclic order. The cross correlated traces can be shown to be

\[ \frac{\langle H_a^\xi \rangle \langle H_b^\zeta \rangle}{\zeta} = \frac{2\zeta}{\beta \eta N^2} \eta^\zeta, \]  

(8.2.4)

where \( \eta = (1 + \alpha^2)^{-1/2} \) is the correlation coefficient mentioned above. Since there are \( \zeta \) pairs, we get \( \eta^\zeta N^{-\zeta} \) while sum over \( \zeta \) free indices gives \( N^\zeta \). \( \mu_\zeta^p \) is given by Brody et al. [1981]

\[ \mu_\zeta^p = \left( \frac{p}{p - \zeta} \right) = -\frac{1}{\zeta} \int x^p \frac{d}{dx} \{ \bar{p}(x) \nu_{\zeta-1}(x) \} dx. \]  

(8.2.5)

Here

\[ \nu_\zeta(x) = (-1)^\zeta \frac{\sin[(\zeta + 1)\psi(x)]}{\sin[\psi(x)]} = \sum_s (-1)^s \left( \frac{\zeta - s}{s} \right) x^{\zeta - 2s}, \]  

(8.2.6)

is the Chebyshev polynomial of second kind of order \( \zeta \) which is valid for \(-2 \leq x \leq 2\) with the weight function \( \bar{p}(x) \) given in (8.1.4). The summation in (8.2.3) is valid for \( p + q = \text{even} \) and restricted to \( \zeta \) such that \( p - \zeta = \text{even} \), as in (2.4.22). The set \( \eta \) carries the complete \( \alpha \) dependence of \( \Sigma_{p,q}^2 \).

Next by using (8.2.3, 8.2.5, 8.2.6) in (8.2.2) and the relation of \( S^p \) with \( S^F \), we get

\[ S^F(x, y) \simeq \frac{2}{\beta \pi^2 N^2} \sum_{\zeta=1}^{N} \eta^\zeta \zeta^{-1} \sin[\zeta \psi(x)] \sin[\zeta \psi(y)] \]

\[ \simeq \frac{1}{\beta \pi^2 N^2} \sum_{\zeta=1}^{N} \eta^\zeta \zeta^{-1} \{ \cos[\zeta (\psi(x) - \psi(y))] - \cos[\zeta (\psi(x) + \psi(y))] \} \].

(8.2.7)

We have a natural cut-off, \( \zeta_{\text{max}} \simeq N \). Now, as in [French et al., 1988a], in order to
extend the \( \zeta \) summation to \( \infty \), we introduce a cut-off in \( \eta \)

\[
\eta'(x) = \eta \exp \left[ -\frac{\varepsilon}{\Omega(x)} \right],
\]

(8.2.8)

where \( \Omega(x) = 2N \pi^2 \rho^2 \) for GEs. Then summation of the infinite series yields

\[
S^F(x, y) \simeq \frac{1}{2\beta \pi^2 N^2} \ln \left[ \frac{1 - 2\eta' \cos(\psi(x) + \psi(y)) + \eta'^2}{1 - 2\eta' \cos(\psi(x) - \psi(y)) + \eta'^2} \right],
\]

(8.2.9)

\[
\simeq \frac{1}{2\beta \pi^2 N^2} \ln \left[ \frac{1 + \frac{4\eta'}{(1 - \eta')^2} \sin^2 \left( \frac{\psi(x) + \psi(y)}{2} \right)}{1 + \frac{4\eta'}{(1 - \eta')^2} \sin^2 \left( \frac{\psi(x) - \psi(y)}{2} \right)} \right].
\]

Using the relation (8.2.1) for short range, \( x \sim y \), we get

\[
\Sigma_{11}^2(x, y; \alpha) \simeq \frac{1}{\beta \pi^2} \ln \left[ 1 + \frac{4\eta'}{(1 - \eta')^2} \sin^2 \left( \frac{\psi(x) - \psi(y)}{2} \right) \right]
\]

\[
\simeq \frac{1}{\beta \pi^2} \ln \left[ 1 + \frac{2\eta'}{(1 - \eta')^2} \frac{(\cos[\psi(x)] - \cos[\psi(y)])^2}{1 - \cos[\psi(x) + \psi(y)]} \right].
\]

(8.2.10)

By renormalizing the density to unit local spacing we obtain the universal result for \( r = \bar{\rho} N|x - y| \),

\[
\Sigma_{11}^2(r; \Lambda) = \frac{1}{\beta \pi^2} \ln \left[ 1 + \frac{r^2 \pi^2}{(\beta \pi^2 \Lambda + \varepsilon)^2} \right],
\]

(8.2.11)

where \( \Lambda = N \alpha^2 \bar{\rho}^2 / \beta \).

The binary correlation method can also be worked out for the LEs by using (2.5.2 - 2.5.5). For large \( N \) we expect a result similar to (8.2.10) for the matrices \((\mathcal{H}_a - a)/b\)
Pandey et al., 2005. We conjecture that in this case the result will be

\[ \Sigma_{11}(x, y; \alpha) = \frac{1}{\beta \pi^2} \ln \left[ 1 + \frac{2\eta'}{(1 - \eta')^2} \right] \times \frac{(x - y)^2}{b^2 - (x - a)(y - a) + \sqrt{b^2 - (x - a)^2} \sqrt{b^2 - (y - a)^2}}, \]

(8.2.12)

with \( a \) and \( b \) as in (8.1.5) and \( \eta' \) in (8.2.8). Here \( \eta = (1 + \alpha^2)^{-1} \), \( \Omega = 2N\pi^2 \bar{\rho}^2 \) and \( \Lambda = 2xN\alpha^2 \bar{\rho}^2/\beta \). We then obtain (8.2.11) for LEs also.

In our result the cut-off parameter \( \varepsilon \) has to be fixed. Here \( \varepsilon \) depends on \( \beta \) and \( r \). In order to fix \( \varepsilon \) we use exact results of the number variance given in (2.3.23) for \( \beta = 1, 2 \) and 4. Since \( \varepsilon \) varies very slowly with \( r \), the \( r \) dependence of \( \varepsilon \) can be ignored for \( r \geq 1 \). For \( \beta = 1, 2 \) and 4, we obtain \( \varepsilon = 0.3570, 0.1053 \) and 0.01538 respectively. Our result is valid for \( r \geq 1 \).

### 8.3 Exact results

Exact results for the parametric correlations have been derived by Simons & Altshuler [1993] by using the Efetov’s supersymmetric non-linear \( \sigma \)-model [Efetov, 1997], for the Gaussian ensembles. We consider the unfolded density-density correlation function,

\[ R_{11}^{(p)}(r; \Lambda) = 1 + \frac{S_p(x, y)}{\bar{\rho}(x)\bar{\rho}(y)}, \]

(8.3.1)

where \( N(x - y)\bar{\rho} = r \) and \( N \to \infty \). For \( \Lambda = 0 \), we find \( R_{11}^{(p)}(r; \Lambda) = 1 + \delta(r) - Y_2(r) \).

The results for \( R_{11}^{(p)}(r; \Lambda) \) are given by the integrals
\[ R_{11}^{(1)}(r; \Lambda) = 1 + \Re \int_1^\infty dx \int_1^\infty dy \int_{-1}^1 dz \frac{(xy - z)^2(1 - z^2)}{(x^2 + y^2 + z^2 - 2xyz - 1)^2} \times \exp \left[ \frac{\pi^2 \Lambda}{2} (x^2 + y^2 + z^2 - 2x^2y^2 - 1) + i(\pi r + i\delta)(xy - z) \right], \]  
(8.3.2)

\[ R_{11}^{(2)}(r; \Lambda) = 1 + \int_0^1 dx \int_1^\infty dy \cos(\pi xr) \cos(\pi yr) \exp \left[ \pi^2 \Lambda(x^2 - y^2) \right], \]  
(8.3.3)

and

\[ R_{11}^{(4)}(r; \Lambda) = 1 + \Re \int_{-1}^1 dx \int_0^1 dy \int_0^\infty dz \frac{(xy - z)^2(z^2 - 1)}{(x^2 + y^2 + z^2 - 2xyz - 1)^2} \times \exp \left[ -2\pi^2 \Lambda(x^2 + y^2 + z^2 - 2(xy)^2 - 1) - i(2\pi r + i\delta)(xy - z) \right]. \]  
(8.3.4)

It is convenient to deal with the parametric spectral form factor \( K(k; \Lambda) \). This is defined as

\[ K^{(\beta)}(k; \Lambda) = \int_{-\infty}^\infty e^{2i\pi kr} \left[ R_{11}^{(\beta)}(r; \Lambda) - 1 \right]. \]  
(8.3.5)

For \( \beta = 1 \), from (8.3.2, 8.3.5) we obtain the form factor in terms of double-integrals of the variables \( u = xy \) and \( v = x^2 \):

\[ K^{(1)}(k; \Lambda) = 2k^2 \int_{(1,2|k|)}^{2|k|+1} du \left(1 - (u - 2|k|)^2\right) \exp(-4\pi^2 \Lambda u|k|) \times \int_1^{u^2} dv \frac{\exp \left[ -\pi^2 \Lambda(u^2 - 4k^2 + 1 - v - u^2/v)/2 \right]}{v(u^2 - 4k^2 + 1 - v - u^2/v)^2}, \]  
(8.3.6)
For $\beta = 2$, we obtain a compact result for the form factor

$$
\mathcal{K}^{(2)}(k; \Lambda) = \int_{0}^{(2|k|^{-1},1)} dx \exp[4\pi^2 |k| \Lambda(x - |k|)]
+ \int_{(1-2|k|,1)}^{1} dx \exp[-4\pi^2 |k| \Lambda(x + |k|)]
= \begin{cases} 
\exp(-4\pi^2 \Lambda |k|) \frac{\sinh(4\pi^2 \Lambda k^2)}{4\pi^2 \Lambda |k|}, & |k| \leq 1, \\
\exp(-4\pi^2 \Lambda k^2) \frac{\sinh(4\pi^2 \Lambda |k|)}{4\pi^2 \Lambda |k|}, & |k| \geq 1,
\end{cases}
$$

(8.3.7)

Finally, for $\beta = 4$ we get

$$
\mathcal{K}^{(4)}(k; \Lambda) = \frac{k^2}{4} \int_{(-1,1)}^{1} du \left((u + |k|)^2 - 1\right) \exp(-2\pi^2 \Lambda u |k|)
\int_{v^2}^{1} dv \frac{\exp(2\pi^2 \Lambda (u^2 - k^2 + 1 - v - u^2/v))}{v(u^2 - k^2 + 1 - v - u^2/v)^2}.
$$

(8.3.8)

The number covariance is related with the form factor as

$$
\Sigma_{11}^2(r; \Lambda) = \int_{-\infty}^{\infty} dk \mathcal{K}^{(\beta)}(k; \Lambda) \left[ \frac{\sin^2(\pi kr)}{(\pi k)^2} \right].
$$

(8.3.9)

We use this relation to calculate numerically the exact number covariance for the three $\beta$ cases.

Small $k$ behavior of $\mathcal{K}^{(\beta)}(k; \Lambda)$ has been useful in semiclassical study of quantum chaotic systems [Kuipers & Sieber, 2006]. For small $k$, $\mathcal{K}^{(\beta)}(k; \Lambda)$ is given by

$$
\mathcal{K}^{(\beta)}(k; \Lambda) = \frac{2|k|}{\beta} \exp[-2|k|\beta^2 \Lambda].
$$

(8.3.10)

This matches with the small $k$ result, derived from (8.2.11, 8.3.9). In fact it can be shown that the binary correlation approximation is equivalent to (8.3.10).
Figure 8.1: \( \Sigma^2_{11}(r; \Lambda) \) vs \( \Lambda \) for \( \beta = 1 \) where \( r = 1, 5 \) and 10. Solid lines represent the exact results obtained by the numerical integration of (8.3.6, 8.3.9). Dashed lines represent our approximate result. Circles and stars represent respectively GOE and kicked rotor data.

8.4 Numerical Verification

In this section we compare the above results with numerical simulations of the Gaussian ensembles and quantum kicked rotors.

We have considered a 500 member Gaussian ensemble of 1000-dimensional \( H_\alpha \) matrices for the three \( \beta \) values and many values of \( \alpha \). We have chosen \( v_\beta^2 = (\beta N)^{-1} \) so that the radius of the semicircle is 2 in each case. For the analysis of \( \Sigma^2_{11}(r; \Lambda) \) we have considered middle 200 levels of each spectrum to ensure that for a given \( \alpha \) the \( \Lambda \) does not vary appreciably due the variation in the density. The results for \( \beta = 1, 2 \) and 4 are shown in Fig. 8.1, Fig. 8.2 and Fig. 8.3 respectively for a few values of \( r \). Also shown in the same figure are the exact analytic result obtained by the numerical integration of (8.3.9) and the binary correlation result given in (8.2.11). We find that the binary correlation result gives a good approximation for the ensembles.

Also shown in the Fig. 8.1 and Fig. 8.2 are the results of quantum kicked rotors as calculated by Kumar [2008] (we refer to Sec. 7.2 for the details of quantum kicked rotors). These calculations have been done for the 200 matrices of the dimension 1001 where \( \theta_0 = \pi/2N \) and \( \gamma = 0 \) and 0.1 respectively for \( \beta = 1 \) and 2. Initially \( K \) is 10000 and then varied in steps of 0.005. This represents one member of the ensemble. The other independent members of the ensemble are obtained by increasing the initial
Figure 8.2: $\Sigma_{11}^2(r; \Lambda)$ vs $\Lambda$ for $\beta = 2$ where $r = 1, 5$ and 10. Solid lines represent the exact results obtained by the numerical integration of (8.3.7, 8.3.9). Dashed lines represent our approximate result. Circles and stars represent respectively GUE and kicked rotor data.

Figure 8.3: $\Sigma_{11}^2(r; \Lambda)$ vs $\Lambda$ for $\beta = 4$ where $r = 10$. Solid lines represent the exact results obtained by the numerical integration of (8.3.8, 8.3.9). Dashed lines represent our approximate result. Circles represent GSE data.
value of $K$ in steps of 10000. The analysis has been done for 50 such members of the ensembles. Eigenangle density for the system is constant ($R_1(\phi) = N/(2\pi)$). The $\Lambda$ parameter is given by [Pandey et al., 1993]

$$\Lambda = \frac{N(\delta K)^2}{8\beta \pi^2}, \quad (8.4.1)$$

where $\delta K$ is variation in the initial $K$.

## 8.5 Discussion

Using the parametric number covariance we have shown that the local spectral fluctuations become rapidly independent as the parameter $\alpha$ of the system is varied. Smooth statistical variations are found as a function of a rescaled parameter $\Lambda$. For the spectra with finite span, we have $\Lambda = \mathcal{O}(1)$ when $\alpha = \mathcal{O}(N^{-1/2})$ for the GEs. For such small $\alpha$ the global correlations between the spectra is very large, close to one.

We have dealt with the three $\beta$ cases and derived the number covariance for the Gaussian and Laguerre ensembles in binary correlation approximation. We have found the universality in our result. One knows that the spectral statistics of quantum kicked rotor, is described by COE or CUE depending on the time-reversal-breaking parameter $\varepsilon$. We have verified numerically that the number covariance results are also valid for spectra of quantum kicked rotor with or without time-reversal symmetry, and hence for COE and CUE.

It has been pointed out by Simons and Altshuler that $R_{11}^{(2)}(r; \Lambda)$, which has been derived by using the non-linear $\sigma$ model, can also be derived by using a random matrix model. For $\beta = 2$, this problem has been solved by using Dyson's Brownian motion model [Beenakker & Rejaei, 1994; Macedo, 1994; Pandey, 2004] for the Gaussian, Laguerre and circular ensemble. For $\beta = 1$ and 4, this is still an open problem in the context of Dyson's Brownian motion model. The small $k$ expansion of $K^{(\beta)}(k; \Lambda)$ has been useful in the semiclassical study of the spectra of quantum chaotic system [Kuipers & Sieber, 2006]. Our binary correlation result is valid up to the first order of $k$. 

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