CHAPTER 3

METHODOLOGY

The thermal images considered from online database are subjected to various preprocessing methods such as BM3D and OWTSURELET which could simultaneously smoothen the image and improve the edges near lower breast boundaries and infra mammary folds. The performance of preprocessing algorithms is evaluated by estimation of SNR of raw and preprocessed images.

Figure 3.1 Flow diagram describing the pipeline of processes
The segmentation of breast tissues is attempted on denoised images using Reaction Diffusion Level Set Method (RDLSM) and Adaptive Level Set Method (ALSM). The intensity based edge maps are integrated into RDLSM and ALSM methods. In order to improve the segmentation accuracy, further experiments are carried out by incorporating phase based edge map.

The segmented breast tissues are analyzed by extracting transform based statistical texture features. The pipe line of the processes adopted in this work is shown in Figure 3.1.

3.1 IMAGE ACQUISITION AND EXPERIMENTAL PROTOCOL

3.1.1 Online Image Database

Images available in online database (http://visual.ic.uff.br/en/proeng/motta/images.php) are used in this work. These images were acquired using FLIR ThermaCam S45 by the Federal University of Pernambuco (UFPE). Patients older than 35 years were examined at the Clinical Hospital of UFPE. A consent form is obtained from them for allowing the use of their images. The patients were made to wait in a temperature controlled room for about 10 minutes for acclimatization and to stabilize their metabolism (Motta et al 2010a).

The most common position for image acquisition was frontal breast region. These images are gray scale of size 320 × 240, in which the maximum intensity values of pixels represent the relatively highest temperature. The original images and their corresponding Ground Truth (GT) images can be viewed or downloaded in the database website. A set of 42 images with 12 normal and 30 abnormal images that are diagnosed and confirmed with biopsy as carcinoma (10), nodule (10), fibroadenoma (6) and cyst (4) are considered in this study.
3.2 PREPROCESSING OF BREAST THERMAL IMAGES

The acquired noisy image \( I(m, n) \) is represented as

\[
I(m, n) = G(m, n) + N(m, n)
\]  \hspace{1cm} (3.1)

where, \( G(m, n) \) is the original noise free image and \( N(m, n) \) is the noise image. The aim of denoising technique is to estimate the noise free image \( G(m, n) \) according to the thermal image statistics.

For optimum performance of denoising techniques, the Poisson distribution is converted to approximately standard Gaussian distribution with constant standard deviation using variance stabilizing transformation such as Anscombe transformation (Kafieh & Rabbani 2011; Romualdo et al 2013). The transformation of \( I(m, n) \) is given as

\[
F(m, n) = 2\sqrt{I(m, n) + \left(\frac{3}{8}\right)}
\]  \hspace{1cm} (3.2)

These transformed images are further subjected to two different denoising algorithms which include BM3D and Orthonormal Wavelet Transform Stein’s Unbiased Risk Estimator with Linear Expansion of Thresholds (OWTSURLET).

After estimating the denoised version of \( F(m, n) \), i.e., \( \hat{F}(m, n) \), the inverse Anscombe transformation is applied to convert the Gaussian distributed data to Poisson process (Kafieh & Rabbani 2011; Romualdo et al 2013) which is given by

\[
\bar{G}(m, n) = \left(\frac{1}{4}\right) \hat{F}(m, n)^2 - \left(\frac{1}{8}\right)
\]  \hspace{1cm} (3.3)

where \( \bar{G}(m, n) \) is the denoised image that follows Poisson distribution.
3.2.1 Block Matching and 3D Filtering based Denoising

BM3D performs denoising based on grouping and collaborative filtering. The basic steps involved in this algorithm are 3D transformation of groups, shrinkage of transform spectrum and inverse 3D transformation. The square block of size 5x5 of noisy image $F(m,n)$ is considered. The blocks that are similar are stacked together to form 3D array such that there is maximum similarity between blocks and minimum overlap between blocks (Dabov 2010). Maximum of 16 blocks are stacked. The collaborative filtering is done on these blocks and the blocks are grouped using 3D transformation. The 2D discrete cosine transform is applied across the grouped blocks and 1D Harr transform is performed across the third dimension of a group along which blocks are stacked (Shao et al 2008). The block estimates are determined using element by element multiplication of 3D transformed coefficients of noisy data with Wiener shrinkage coefficients. The inverse transform produces group of estimates. The final estimate is obtained by weighted averaging of overlapping block estimate at all pixel position (Garcia 2010).

3.2.2 OWTSURELET based Denoising

The breast thermal images are also subjected to denoising using OWTSURELET. The inter-scale and linear expansions of thresholds principle are integrated in the framework of OWT to preserve the edge details. The orthonormal wavelet transform is applied to input image and subbands of $\psi_j$ wavelet coefficients are derived. These coefficients are subjected to soft thresholding with a level dependent threshold value $T$ (Benazza & Pesquet 2005). The noise free subbands $x_j$ are estimated using a pointwise denoising function $\theta$ as given by
The optimal threshold determined using SURE based objective function is given by

\[
\text{SURE}(T; y) = \sigma^2 - \frac{1}{N} \cdot (2 \sigma^2 \{n : |y_n| \leq T \} - \sum_{n=1}^{N} \min(|y_n|, T)^2) \tag{3.5}
\]

The noise variance \( \sigma \) is estimated by median absolute deviation estimator. Then the optimal threshold value estimated is

\[
T_{\text{opt}} = \arg \min_T \left( SURE(T; y) \right) \tag{3.6}
\]

SURE represents the image-domain minimization of an estimate of the mean squared error in denoising process (Luisier & Blu 2008). The sensitivity of SURE function in finding optimal threshold is improved by implementing a non linear search algorithm in which the denoising function is defined as

\[
\theta(y) = \sum_{k=1}^{K} a_k \varphi_k(y) \tag{3.7}
\]

where \( K \), the number of parameters is chosen as 2, \( a_k \) are linear parameters and \( \varphi_k \) are basis functions. To improve the denoising quality, the interscale predictor \( y_p \) obtained from filtered version of lowpass subband is integrated in pointwise denoising function (Luisier et al 2007) and is given as

\[
\theta(y, y_p) = f(y_p) \sum_{k=1}^{K} a_k \varphi_k(y) + \left(1 - f(y_p)\right) \sum_{k=1}^{K} b_k \varphi_k(y) \tag{3.8}
\]

where \( f(y_p) \) is the function of interscale predictor and \( b_k \) are linear parameters obtained by minimizing mean square error given as
\[ f(y_p) = e^{-\frac{y_p^2}{2r^2}} \]  

(3.9)

The denoising is applied independently in every subbands. The denoised image is reconstructed by applying inverse discrete wavelet transform from the processed wavelet subimages \( \mathbb{R} \).

The performance of denoising algorithms is evaluated by estimating the signal to noise ratio of both raw and denoised images (Garcia 2010) which is expressed as

\[
\text{SNR(dB)} = 10 \log_{10} \left( \frac{\text{Image variance}}{\text{Noise variance}} \right)
\]

(3.10)

3.2.3 Gradient Magnitude Similarity Deviation

The edge preserving capability is evaluated using Gradient Magnitude Similarity Deviation (GMSD). The gradient magnitudes of raw (r) and denoised (d) at location i, denoted by \( m_r(i) \) and \( m_d(i) \), are computed as follows

\[
m_r(i) = \sqrt{\left( r \otimes h_x \right)^2(i) + \left( r \otimes h_y \right)^2(i)}
\]

(3.11)

\[
m_d(i) = \sqrt{\left( d \otimes h_x \right)^2(i) + \left( d \otimes h_y \right)^2(i)}
\]

(3.12)

where symbol \( \otimes \) denotes the convolution operation, \( h_x \) and \( h_y \) are the Prewitt filters along horizontal and vertical directions. The Gradient Magnitude Similarity (GMS) map is defined as

\[
GMS(i) = \frac{2m_r(i)m_d(i)+c}{m_r^2(i)+m_d^2(i)+c}
\]

(3.13)
where $c$ is a positive constant that supplies numerical stability. Gradient Magnitude Similarity Deviation index is computed as standard deviation of GMS map and is given by

$$GMSD = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (GMS(i) - GMSM)^2} \quad (3.14)$$

where $GMSM = 1/N \sum_{i=1}^{N} GMS(i)$, $N$ is the total number of pixels in the image (Xue et al 2014). The value of GMSD is zero which reflects the range of deviation in the edges of an image.

3.3 ANALYSIS OF SEGMENTATION OF BREAST TISSUES USING LEVEL SETS

The segmentation of breast tissues is carried out using RDLSM and ALSM.

3.3.1 Reaction Diffusion Level Set Method

Active contour models are dynamic curves or surfaces that are represented as $C(s, t) \rightarrow \mathbb{R}^2$, $s \in [0, 1]$ and $t \in [0, \infty]$. The curve or surface under evolution is given by

$$\frac{\partial C(s, t)}{\partial t} = F \vec{N} \quad (3.15)$$

where $F$ is the parameter to influence motion of the contour and $\vec{N}$ is the inward normal vector to the curve $C$. The general form of rate equation can be written as

$$\frac{\partial \phi}{\partial t} = Reg(\phi) + F\delta(\phi) \quad (3.16)$$
where $\text{Reg}(\phi) = \alpha \text{div}[r(\phi)\nabla \phi]$, $r(\phi)$ can be any existing conventional diffusion rate equations, $\alpha$ is a small constant and $F$ is the force term. Although many diffusion rate equations exist, they do not give stable solution in the level set equation and thereby lead to boundary leakage problems. A level set evolution method is derived by adding diffusion term $``\varepsilon \Delta \phi''$ in the LSE equation to provide stable solutions (Zhang et al 2013). The constructed reaction diffusion rate equation is given by

$$\frac{\partial \phi}{\partial t} = \varepsilon \Delta \phi - \frac{1}{\varepsilon} L(\phi), \ x \in \Omega \mathbb{R}^n$$

(3.17)

where $\varepsilon$ is a small constant, $\varepsilon^{-1} L(\phi) = F \delta(\phi)$ is the reaction term, $\Delta$ is the Laplacian operator and force term $F$ in the equation $L(\phi) = -F \delta(\phi)$ is given by

$$F = \text{div} \left( \frac{\text{g}(\|
abla I_\sigma\|) \nabla \phi}{\|
abla I_\sigma\|} \right) + v g$$

(3.18)

where $v$ is constant and $g$ is an edge indicator function, which is given by

$$g(\|
abla I_\sigma\|) = \exp \left( -\frac{(\|
abla I_\sigma\|)}{k} \right)$$

(3.19)

where $I_\sigma$ is the smoothened image which is obtained by convolving the original image with the Gaussian kernel of standard deviation $\sigma$ and $k > 0$ is a contrast parameter. This edge indicator function, which represents the intensity gradient of Gaussian filtered image is used as the stopping criterion for level set function. The edge maps are also generated using coherence enhanced diffusion, total variation and Phase Congruency (PC) methods. As the accuracy of segmentation depends on the edge indicators and diffusion rate of level set method, improved gradient based edge indicators which include coherence enhanced diffusion, total variation and phase based edge indicators are adopted in this work.
In CED, the tensor product of gradient of Gaussian smoothed version of an image $I(x, t)$ is used as the structure descriptor and is given as

$$J_\rho(\nabla I_\sigma) = K_\rho \ast \nabla I_\sigma \otimes \nabla I_\sigma \quad (\rho \geq 0) \quad (3.20)$$

Where $K_\rho$ is the Gaussian kernel, $\nabla I_\sigma$ is the gradient smoothed version of the image $I(x, t)$ and the matrix $J_\rho(\nabla I_\sigma)$ is structure tensor or second-moment matrix.

The Eigen values of $J_\rho$ provide useful information on the coherence of a structure, i.e. the actual amount of anisotropy and tend to zero for isotropic structures (Weikert 1999). By integrating this structure tensor based image gradient in the reaction diffusion level set formulation, the force term $F$ is given by

$$F = \text{div} \left( \frac{J_\rho(\nabla I_\sigma) \nabla \phi}{|\nabla I_\sigma|} \right) + \nu J_\rho(\nabla I_\sigma) \quad (3.21)$$

where $\nu$ is constant.

Image gradient is also derived using total variational non-linear filter that uses variational approach of energy functional minimization. The classical TV minimization process of Rudin Osher Fatemi formulation is given by

$$E_{TV} = \int_{\Omega} \left( |\nabla I| + \frac{1}{2} \lambda (\hat{I} - I)^2 \right) dxdy \quad (3.22)$$

where $I$ is the noisy input image, $\hat{I}$ is the TV norm of the input image and $\lambda$ is the scalar fidelity term (Leonid et al 1992). The $\phi$ formulation is used to generalize this function and is given by

$$E_{\phi} = \int_{\Omega} \left( \phi |\nabla I| + \frac{1}{2} \lambda (\hat{I} - I)^2 \right) dxdy \quad (3.23)$$
An approximation of TV is obtained by standard discretization of the Euler Lagrange equation which is given by

$$ F = \text{div} \left( \phi'(\cdot) \frac{\nabla I}{|\nabla I|} \right) + \lambda (\hat{I} - I) = 0 \quad (3.24) $$

The processed image $I_{TV}$ is obtained by solving the above equation using steepest decent method (Gilboa et al 2003). The edge map $g$ is obtained by taking image gradient of the $I_{TV}$ image.

The phase based edge indicator that helps in robust convergence at edges is derived using PC function. PC image have maximally high magnitude values at features such as edges and lines and low values at all other points (Kovesi 1999). The peaks in local energy function correspond to the points of maximum PC and are related as

$$ E(x) = PC(x) \sum_n A_n \quad (3.25) $$

where PC represents Phase Congruency function, $A_n$ is the amplitude of frequency components and $E$ is the energy function given as $E(x) = \sqrt{P^2(x) + H^2(x)}$, where $P(x)$ is the intensity profile and that can be obtained by convolving the image with a band of even wavelet filters. $H(x)$ is the Hilbert transform of $P(x)$ and can be obtained by the convolving the image with band of odd wavelet filters but shifted by $90^0$ in phase (Kovesi 1999). For an image $I$, with $M^n_e$ and $M^n_o$ representing the even and odd symmetric quadrature at scale $n$, $P(x)$ and $H(x)$ are approximated as

$$ P(x) = \sum_n I(x) * M^n_e \quad (3.26) $$

$$ H(x) = \sum_n I(x) * M^n_o \quad (3.27) $$
The sum of amplitudes of frequency components of \( P(x) \) is given by

\[
\sum_n A_n(x) = \sum_n \sqrt{(I(x) * M_n^o)^2 + (I(x) * M_n^s)^2}
\]  

(3.28)

Thus the PC measure (Venkatesh & Owen 1989) is given by

\[
PC(x) = E(x)/(\text{eps} + \sum_n A_n(x))
\]  

(3.29)

where \( \text{eps} \) is a small constant used to prevent the equation from becoming unstable. In an image to detect features at all orientations (Kovesi 1999), filters were designed in multiple orientations and the PC is obtained by summing up the orientations as given by

\[
PC(x) = (\sum_o (W_o(x)(E_o(x) - T_o))) / (\sum_o \sum_n (A_{no}(x)) + \text{eps})
\]  

(3.30)

where, \( 'o' \) defines the orientation, \( 'W' \) is the weighting function that reduces PC at narrow filter response regions and \( T_o \) is the noise threshold along the orientation and is given by \( T_0 = e^{\log(\sqrt{A_x})} \). The obtained PC map is used as an edge indicator in Eq. (3.14) for level set contour evolution.

### 3.3.2 Adaptive Level Set Method

The denoised images are also subjected to adaptive level set method to segment the breast tissues. The adaptive level set evolution equation is defined as

\[
\frac{\partial \varphi}{\partial t} = -\mu \left[ \Delta \varphi - \text{div} \left( \frac{v \varphi}{|v \varphi|} \right) \right] - \lambda \delta(\varphi) \text{div} \left( g \frac{v \varphi}{|v \varphi|} \right) + v(x,y)g \delta(\varphi)
\]  

(3.31)

where \( \varphi \) is the initial contour, \( \delta \) is Dirac delta function, \( \mu, \lambda \) are constants that are used to control the movement of the contour and \( v(x,y) \) is an adaptive direction function which is described as

\[
\frac{\partial \varphi}{\partial t} = -\mu \left[ \Delta \varphi - \text{div} \left( \frac{v \varphi}{|v \varphi|} \right) \right] - \lambda \delta(\varphi) \text{div} \left( g \frac{v \varphi}{|v \varphi|} \right) + v(x,y)g \delta(\varphi)
\]  

(3.31)
\[ v(x,y) = k \left[ \frac{1}{1 + \exp(-\zeta A_d(x,y))} - 0.5 \right] \] (3.32)

where \( k \) controls the amplitude and \( \zeta \) denotes the nonlinear degree of the velocity, respectively (Wang et al 2014) and the direction term \( A_d(x,y) = P(\Omega_1 | I(x,y)) - T_p \), where \( P(\Omega_1 / I) \) is a posterior probability illustrating the probability that the pixels belong to the object and \( T_p \) is threshold chosen as 0.5. The posterior probability is determined by bayesian rule. When this direction term is positive, the closed curve expand to include the pixel \((x,y)\) that belongs to the object and the closed curve shrink to include the pixel to the background for negative values.

A edge and region based stopping force is formulated by combining the local and global features and is given as

\[ g = \frac{1}{1 + P(\Omega_1 / I) P(\Omega_2 / I) G} \] (3.33)

where \( P(\Omega_2 / I) \) is a posterior probability illustrating the probability that the pixels belong to the background (Wang et al 2014) and \( G \) is the edge indicator function that can be derived using Gaussian, CED, TV and PC filters.

### 3.3.3 Validation of Segmented Results

The efficiency of segmentation algorithms can be quantitatively analyzed by computing a metric that best describes similar or dissimilar regions of the segmented image to the expected image. The expected image is the GT image, which is a reference image that is manually segmented by a radiologist or a trained professional. The performance of proposed segmentation methods on breast thermal images is evaluated quantitatively using similarity metrics based on geometry and overlap measures. These measures take values between 0 and 1. High similarity measure indicates good agreement of segmented and GT images (Frounchi et al 2011).
3.3.3.1 Regional Statistics Measures

Regional statistics measures consider the number of pixels that are segmented as ROI and non-ROI. True Positive (TP) that counts the number of pixels that are correctly segmented as ROI, True Negative (TN) that counts the number of pixels that are correctly identified as non-ROI, False Positive (FP) that counts for the pixels that are incorrectly identified as non-ROI and False Negative (FN) that counts for the pixels that are incorrectly identified as in ROI are calculated. Based on these values, statistical evaluation of accuracy, sensitivity, specificity, Positive Predictive Rate (PPR) and Negative Predictive Rate (NPR) are calculated. Accuracy is defined as the ratio of number of correctly identified pixels to the total number of pixels that gives the degree of similarity between GT and segmented image. Sensitivity measures the number of positive results among all positives. Specificity measures the number of negative results among all negatives (Machado et al 2013). These measures are defined as the following

\[ \text{Accuracy} = \frac{(TP + TN)}{(FN + FP + TN + TP)} \] \hspace{1cm} (3.34)

\[ \text{Sensitivity} = \frac{TP}{(TP + FN)} \] \hspace{1cm} (3.35)

\[ \text{Specificity} = \frac{TN}{(TN + FP)} \] \hspace{1cm} (3.36)

\[ \text{PPR} = \frac{TP}{(TP + FP)} \] \hspace{1cm} (3.37)

\[ \text{NPR} = \frac{TN}{(TN + FN)} \] \hspace{1cm} (3.38)

3.3.3.2 Overlap Measures

Overlap measures are determined by identifying intersecting and non-intersecting regions of segmented results and GT images. The classic measures commonly used are Jaccard Coefficient (JC), TaniMoto coefficient (TM), Dice Similarity (DS) and Volume Similarity (VS) indices (Cardemes et
al 2009). If $X_V$ represents the segmented result and $Y_V$ represents the GT image, these measures are defined as following:

$$JC = \frac{|X_V \cap Y_V|}{|X_V \cup Y_V|} \quad (3.39)$$

$$DS = 2|X_V \cap Y_V|/(|X_V| + |Y_V|) \quad (3.40)$$

$$TM = \frac{|X_V \cap Y_V| + |X_V \cup Y_V|}{(|X_V \cup Y_V| + |X_V \cap Y_V|)} \quad (3.41)$$

$$VS = 1 - \frac{||X_V| - |Y_V||}{(|X_V| + |Y_V|)} \quad (3.42)$$

3.4 ANALYSIS OF BREAST TISSUES USING TRANSFORM BASED FEATURE EXTRACTION

The left and right regions of the segmented breast tissues are separated using midpoint of infra mammary folds. The separated breast regions are grouped as normal and abnormal breast tissues. Normal and abnormal breast tissues are grouped based on healthy and pathological conditions of the separated breast regions. These breast tissues are subjected to wavelet, radon transform and phase based transform which include QHT, Riesz transform, steerable Riesz transform and multi scale quadrature filter. The statistical features are extracted from the transformed coefficients and analysed.

3.4.1 Wavelet Transform

The separated normal and abnormal breast tissues are subjected to wavelet analysis. The wavelet analysis decomposes images into four sub-bands namely $A^1$, $H^1$, $V^1$ and $D^1$. The subband $A^1$ corresponds to coarse level or approximation coefficients and the subbands labelled $H^1$, $V^1$ and $D^1$ represent the finest level or detail coefficients. The approximate and detail coefficients are computed by convolving signals with the low-pass filter and
high-pass filter respectively. The convolved coefficients are down-sampled by keeping the even indexed elements. An image \( f(x, y) \) with a spatial resolution of \( 2^j \) decomposed at first level is represented as

\[
f(x, y) = A^1 \varphi_{2j}(X - 2^{-j}n, Y - 2^{-j}m) + H^1 \psi_{2j}(X - 2^{-j}n, Y - 2^{-j}m)
\]

\[
+ V^1 \varphi_{2j}(X - 2^{-j}n, Y - 2^{-j}m) + D^1 \psi_{2j}(X - 2^{-j}n, Y - 2^{-j}m)
\]

(3.43)

where integer \( j \) is a decomposition level, \( m, n \) are integers, \( \varphi(x) \) is a one-dimensional scaling function which provides low frequency information, and \( \psi(x) \) is a one-dimensional wavelet function which provides high frequency information. The wavelet decomposition can thus be interpreted as image decomposition in a set of independent, spatially oriented frequency channels (Mallat 1989), so that \( \varphi(t) \) and \( \psi(t) \) can be defined as:

Dilation equation

\[
\varphi(t) = \sqrt{2} \sum_k c(k) \varphi(2t - k)
\]

(3.44)

Wavelet equation

\[
\psi(t) = \sqrt{2} \sum_k d(k) \varphi(2t - k)
\]

(3.45)

Haar wavelet has coefficients: \( c(0) = c(1) = 1/\sqrt{2} \), \( d(0) = 1/\sqrt{2} \), and \( d(1) = -1/\sqrt{2} \). Thus its dilation equation and wavelet equation can be expressed as:

\[
\varphi(t) = \varphi(2t) + \varphi(2t-1)
\]

(3.46)

\[
\psi(t) = \varphi(2t) - \varphi(2t-1)
\]

(3.47)

The statistical features are derived from approximation coefficients \( A^1 \).
3.4.2 Radon Transform

The Radon transform of a 2D function is defined on the space of straight lines $L$ in $\mathbb{R}^2$ by the line integral along each such line which is expressed as

$$R(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - x \cos \alpha - y \sin \alpha) dx dy$$  \hspace{1cm} (3.48)

where $s$ is perpendicular distance of a line $L$ from the origin and $\alpha$ is angle formed by the distance vector to $L$ makes with the $x$ axis. Fourier slice theorem states that for a 2D function $f(x, y)$, the 1D Fourier transforms of the Radon transform along $s$, are the 1D radial samples of the 2D Fourier transform of $f(x, y)$ at the corresponding angles (Boehm et al 2009). The statistical features are extracted from these transformed coefficients.

3.4.3 Hilbert Transform

The Hilbert transform $f_{Hi}$ of a real 1D-signal $f$ is given by:

$$f_{Hi}(x) = f(x) \ast \frac{1}{\pi x}$$  \hspace{1cm} (3.49)

where * denotes convolution. The complex analytic signal representation of $f$ is given as

$$f_A(x) = f(x) + i f_{Hi}(x) = f(x) \ast (\delta(x) + i/\pi x)$$  \hspace{1cm} (3.50)

The real part of $f_A$ is identical to the input signal, while the imaginary part is a phase-shifted version (or the Hilbert transform $f_{Hi}$) of $f$. In the frequency domain, the Hilbert transform is defined by

$$F_{Hi}(u) = -\frac{iu}{|u|} F(u)$$  \hspace{1cm} (3.51)
where $F$ and $F_{Hi}$ are the Fourier transforms of $f$ and $f_{Hi}$, respectively.

The polar representation of analytic signal can be written as

$$f_A(x) = |f_A(x)| \exp(i\phi(x))$$ (3.52)

where $|f_A(x)|$ is called the local amplitude and $\phi(x)$ is the local phase of $f$ (Granlund & Knutsson 1995). Two dimensional analytic signal is designed by generalization of Hilbert transform using quaternion, Riesz and MQF analysis (Nunes et al 2005).

3.4.3.1 Quaternion Hilbert transform

A quaternion is represented in hypercomplex form (Bulow & Sommer 2001) as

$$q = a + bi + cj + dk$$ (3.53)

where $a$, $b$, $c$ and $d$ are real, $i$, $j$ and $k$ are complex operators which obey the following rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = i, kj = -i, ki = j, ik = -j$$ (3.54)

The modulus and conjugate of a quaternion are given by

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$$ (3.55)

$$q = a - bi - cj - dk$$ (3.56)

The quaternion in polar form is represented as $q = |q|e^{u\varphi}$, where $u$ and $\varphi$ are referred to as the Eigen-axis and Eigen-angle of the quaternion respectively. The $u$ identifies the direction of the vector part and may be
regarded as a true generalization of the complex operator $i$ and $\varphi$ is the argument of a complex number given as

$$\varphi = \arctan \frac{|V(q)|}{S(q)} \quad (3.57)$$

where $S(q)$ is the real part and $V(q) = bi + cj + dk$ is the imaginary part of quaternion.

Let $f$ be a real two-dimensional signal and $F^q$ be quaternionic Fourier transform (Ell & Sangwine 2007). In the quaternionic frequency domain, the quaternionic analytic signal of a 2D real signal is defined as

$$F^q_A(\mu) = (1 + sgn(u))(1 + sgn(v))F^q(\mu) \quad (3.58)$$

where $\mu = (u, v)$. This can be expressed in the spatial domain as follows

$$f^q_A(x) = f(x) + n^T f_{Hi}(x) \quad (3.59)$$

where $n^T = (i, j, k)^T$ and $f_{Hi}$ is a vector which consists of the total and the partial Hilbert transforms of $f$ (Xu 2012) and is given by

$$f_{Hi} = \begin{pmatrix} f_{Hi_1}(x) \\ f_{Hi_2}(x) \\ f_{Hi_3}(x) \end{pmatrix} \quad (3.60)$$

The instantaneous phase is defined as the triple of phase angles of the quaternionic value at each position. The polar representation of analytic signal of quaternion Hilbert transform (Qiao et al 2009) is represented as

$$f^q_A(x) = Ae^{u\varphi} \quad (3.61)$$

where $A$, amplitude and $\varphi$, phase is given by
\[ A = \sqrt{f(x)^2 + f_{H1}(x)^2 + f_{H2}(x)^2 + f_H(x)^2} \]  
(3.62)

\[ \varphi = \arctan \left( \frac{|f_{H1}(x)i + f_{H2}(x)j + f_H(x)k|}{f(x)} \right) \]  
(3.63)

\[ u = \left( (f_{H1}(x)i + f_{H2}(x)j + f_H(x)k) / |f_{H1}(x)i + f_{H2}(x)j + f_H(x)k| \right) \]  
(3.64)

The statistical features are extracted from amplitude and phase of quaternion Hilbert transformed coefficients.

### 3.4.3.2 Riesz transform

The monogenic signal, 2D generalization of Hilbert transform is derived using Riesz transform (Luo et al 2015; Felsberg & Sommer 2001). Riesz is the scalar-to-vector signal transformation \( R \) (Unser & Van De Ville 2010) whose frequency-domain definition is

\[ R_\nu f(\omega) = -j\frac{\omega \nu}{||\omega||} \hat{f}(\omega) \]  
(3.65)

where \( R_\nu f \) is a real-valued function and \( \hat{f}(\omega) \) is the Fourier transform of the input signal \( f(x) \). The monogenic signal represented by Riesz (Storath 2010) is defined as

\[ Mf = f - iR_1f - iR_2f \]  
(3.66)

The amplitude and phase of Riesz is given by

\[ |A| = \sqrt{f(x)^2 + R_1f(x)^2 + R_2f(x)^2} \]  
(3.67)

\[ \varphi = \arctan \left( \frac{|R_1f(x)^2 + R_2f(x)^2|}{f(x)} \right) \]  
(3.68)

The statistical features are extracted from amplitude and phase of Riesz transformed coefficients.
3.4.3.3 **Steerable Riesz transform**

For a two–dimensional signal \( f(x) \), the different components of the second \( N^{th} \)–order Riesz transform \( R \) is defined as

\[
R^{(n_1,n_2)}f(\omega) = \frac{\sqrt{n_1+n_2}}{n_1!n_2!} \frac{-j\omega_1^{n_1}-j\omega_2^{n_2}}{||\omega||^{n_1+n_2}} \hat{f}(\omega)
\]  
(3.69)

for all combinations of \((n_1, n_2)\) with \( n_1 + n_2 = N \) and \( n_{1,2} \in N \). \( \hat{f}(\omega) \) denotes the Fourier transform of \( f(x) \), where the vector \( \omega \) is composed by \( \omega_{1,2} \) corresponding to the frequencies in the two image axes. The multiplication by \( j\omega_{1,2} \) in the numerator corresponds to partial derivatives of \( f \) and the division by the norm of \( \omega \) in the denominator results in only phase information being retained (Depeursinge et al 2011).

The considered second order Riesz transform \( R \) corresponds to an all pass filter bank with directional kernels \( h \). Riesz transform with steerable filter bank allows to analyse the thermal patterns in any direction. Every Riesz component is mapped to a multiscale representation by convolving them with four Laplacian of Gaussian filters with a dyadic scale progression. The Riesz coefficients corresponding to first and second scale are considered for further analysis. The Riesz filters \( h \) is rotated to obtain maximal response with dominant orientation \( \theta_{dom} \) given by

\[
\theta_{dom}(x_p) = \arg\max_{\theta \in [0, \pi]} \left( (h^{(\theta)} * g) * f \right)(x_p)
\]  
(3.70)

where \( h^{(\theta)}(x) \) is \( h \) rotated by \( \theta \) at the position \( x_p \) and \( g(x) \) is a Gaussian kernel (Depeursinge et al 2014). 2nd–order Riesz coefficients are acquired after performing local orientation. The statistical features are extracted from 2nd–order Riesz coefficients.
3.4.3.4 Multiscale quadrature filtering

Quadrature filters transform a real-valued signal to an analytic signal with the addition of weighting the frequency components. The generalization of quadrature filters to 2D based on the directional formulation is defined with respect to half spaces \( w \) and is given as

\[
F_q(\omega) = \begin{cases} 
2F(\omega_x, \omega_y), & \omega \cdot \hat{n} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \omega = (\omega_x, \omega_y) \) denote the spatial frequency coordinates, \( \hat{n} \) is a 2D unit vector defining the half-space. The transfer function of the analytic signal in multiple dimensions preserves the frequency components on one side of the half-space. A drawback of the directional formulation is that it is not isotropic. The quadrature filters from multiple orientations are combined to address the anisotropy of the half-space formulation. Four filter of directions \( \{0^\circ, 45^\circ, 90^\circ \text{ and } 135^\circ\} \) are generated with kernel size of 15 x 15 using lognormal radial functions with bandwidths of 4 octaves and center frequencies of \( 5 \pi / 7 \) (Soleimani et al 2013). The filter kernel is applied in four uniformly distributed directions respectively. Thus, the local phase is a characteristic along particular direction. A rotation invariant phase map is generated by summing the filter responses for all directions.

The local phase information which present observable magnitudes of structural variations at different scales are integrated to generate the global phase map. This is obtained by weighted sum over all scales, to get high strength response and is expressed as

\[
q = \frac{\sum_{i=1}^{N}|q_i|^\beta q_i}{\sum_{i=1}^{N}|q_i|^\beta}
\]
where \( N \) is the number of scales, \( q_i \) is the combined response for each scale and \( \beta \) is a weight parameter. The output magnitude is normalized by

\[
a(\sigma) = \frac{1}{(1+(\sigma/a)^2)}
\]  

where \( a = |q| \), \( q \) is the global phase map and \( \sigma \) is a data dependent threshold parameter. The scaling issues associated with different inputs are removed by performing normalization (Ali et al 2007). As both positive and negative values are obtained, thresholding is employed to retrieve only the positive values. The statistical texture features are extracted from the global phase map of breast tissues.

### 3.4.4 Statistical Analysis

The statistical features such as mean, kurtosis, skewness, coarseness, contrast and directionality are derived from transformed coefficients of breast tissues. The transformed coefficients are obtained using Wavelet, Radon, QHT, Riesz, steerable Riesz and MQF analyses.

Mean of transformed coefficients is given by

\[
Mean: \quad \mu = \sum_{i=0}^{G-1} ip(i)
\]  

(3.74)

where \( p(i) \) is the probability density of occurrence of the \( i^{th} \) magnitude value.

The skewness is a measure of the inequality of distribution about mean. The value may be positive or negative of the skewness. The negative value represents that the tail of the block pixel values on the left side of the probability distribution is skewed or longer than the right side and the bulk of
the values lie to the right of the mean. A positive value represents that the tail on the right side is longer than the left side and the bulk of the values lie to the left of the mean. A zero value of skewness means that the values on both sides of the mean are relatively equally distributed. The skewness can be calculated as

\[
\text{Skewness : } \mu_3 = \sigma^{-3} \sum_{i=0}^{G-1} (i - \mu)^3 p(i)
\] (3.75)

Where \( \mu \) is the mean and \( \sigma \) is the standard deviation of the values of a transformed image. The fourth moment kurtosis is a measure of the peakedness of the probability distribution of pixel values of transformed image (Myint et al 2002). For high value of kurtosis, the distribution has a sharper peak and longer, fatter tails, while for low value, distribution gives more rounded peak and shorter thinner tails. Kurtosis can be defined as

\[
\text{Kurtosis : } \mu_4 = \sigma^{-4} \sum_{i=0}^{G-1} (i - \mu)^4 p(i)
\] (3.76)

Coarseness measures the granularity measurement of texture. The moving average \( A_z(s, t) \) over the neighborhood of size \( 2^z \times 2^z \) (\( z=0, 1, 2, 3, 4, 5 \)) at the point \( (s, t) \) is computed as

\[
A_z(s, t) = \frac{\sum_{j=s+2^{z-1}}^{s-2^{z-1}} \sum_{k=t+2^{z-1}}^{t-2^{z-1}} x(i, j)}{2^{2z}}
\] (3.77)

where \( x(i, j) \) is the transformed coefficient at \( (i, j) \). The differences between pairs of non-overlapping moving averages in the horizontal and vertical directions for each coefficients are computed as

\[
E_{zh}(s, t) = | A_z(s + 2^{z-1}, t) - A_z(s - 2^{z-1}, t)|
\] (3.78)
\[ E_z(s, t) = | A_z(s, t + 2^{z-1}) - (s, t - 2^{z-1}) | \]  \hspace{1cm} (3.79)

The best neighborhood size that maximizes \( E \) in either direction is determined as \( S_{\text{best}}(s, t) = 2^z \). The global coarseness is calculated by averaging \( S_{\text{best}} \) over the transformed image

\[
\text{Coarseness} = \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} S_{\text{best}}(j, k)
\]  \hspace{1cm} (3.80)

Contrast is derived as mean of local contrast value of image. The local contrast (\( l_{\text{con}} \)) over the neighborhood of size \( W \times W \) is computed as

\[
l_{\text{con}}(i, j) = \frac{\max_{x \in W} x - \min_{x}}{\max_{x \in W} x + \min_{x}}
\]  \hspace{1cm} (3.81)

where \( x \in W \times W \) and \( \min_{x} \) represent the maximum and minimum magnitude of transformed coefficients. The global contrast (\( \text{Con} \)) is derived using

\[
\text{Con} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} l_{\text{con}}(i, j).
\]  \hspace{1cm} (3.82)

Directionality measures the frequency distribution of orientation of local edges against their directional angle. The edge strength \( e(x, y) \) and the directional angle \( a(x, y) \) are computed using

\[
e(x, y) = 0.5(|\Delta_x(x, y)| + |\Delta_y(x, y)|)
\]  \hspace{1cm} (3.83)

\[
a(x, y) = \tan^{-1}(\Delta_y(x, y) / \Delta_x(x, y))
\]  \hspace{1cm} (3.84)

where \( \Delta_x(x, y) \) and \( \Delta_y(x, y) \) are the horizontal and vertical grey level differences between the neighboring pixels, respectively (Muwei & Lei 2009).

A histogram of quantized direction values \( H_{\text{dir}}(a) \) is constructed by counting numbers of the edge pixels of transformed image with the
corresponding directional angles and edge strength greater than a predefined threshold. The histogram is relatively uniform for images without strong orientation and exhibits peaks for highly directional images. The degree of directionality is given by

$$F_{\text{dir}} = 1 - n_p \sum_{p=1}^{n_p} \sum_{a \in w_p} (a - a_p)^2 H_{\text{dir}}(a)$$

(3.85)

where $n_p$ is the number of peaks, $a$ is the quantized directional angle (cyclically in modulo $180^\circ$), $a_p$ is the position of the $p^{\text{th}}$ peak, $w_p$ is the range of the angles attributed to the $p^{\text{th}}$ peak (that is, the range between valleys around the peak) and $r$ denotes a normalizing factor related to quantizing levels of the angles $a$. 