Chapter 2

BEGOE($k$): Spinless bosons and their spectral analysis

2.1 Introduction

The theory of random matrix explains the statistical behaviour of spectra of a large variety of complex systems such as atoms, nuclei, quantum dots etc. It was initially used by Wigner to study the spacing distribution of neutron resonances in heavy nuclei. As the interaction was not well known in this domain, Wigner assumed that the Hamiltonian here can be simulated in terms of ensembles of random matrices. The spacing distribution produced by such an ensemble agreed with that of nuclear resonance and the Eigen value distribution was found to be semicircular in character. Thereby it was manifested that spacing between the lines in the spectrum of heavy body should resemble the spacing between Eigen values of a random matrix. Such an ensemble of symmetric random matrices, with each matrix element being a Gaussian random variate with zero mean and unit variance (for diagonal matrix elements variance = 2), is called Gaussian Orthogonal Ensemble (GOE). The GOE for $m$-particles in the space defined by $N$ single particle states, in fact describes simultaneous interactions between all the particles because of the statistical independence of the matrix elements. But realistic interactions are however 2-body in nature. Previously, the
2-body interactions for the dilute limit of fermionic systems were studied by Mon, French [19] and Benet et al [20]. These 2-body interactions could be propagated to \( m \)-body space. This work concluded that the ensemble-averaged state density for GOE is semicircle, while the density for EGOE(2) with \( m \gg 2 \) is close to Gaussian. Further, the transition point at which the semicircle nature of the curve changes to gaussian shape was calculated and it was found to be near \( m = 2k \), as per the work done by Benet et al. Also the shape transition is numerically demonstrated [19, 20, 21, 22]. This work was purely done for the fermionic systems, but it actually stimulated the idea to expect a similar kind of nature for the bosonic systems. But for the case of bosons, we have the dense limit \( (m \to \infty, N \to \infty \text{ and } m/N \to \infty) \) which is a novel feature that is unseen for fermions. Considering this dense limit for \( m \) bosons in \( N \)-single particle states, it was previously established that the state density for BEGOE(2) approaches a Gaussian form [15, 16, 23]. Moreover, the nature of state density for a system of \( m \)-bosons with \( m \gg 2 \) comes out to be a Gaussian for BEGOE(2) [17]. However the transition of the state density from Gaussian to semicircle, as the rank of interaction \( k \) increases from 2 to \( m \) has never been explored. In this chapter, we focus on the shape change of the density curve for a finite system of bosons and also calculated is the transition point. Moreover, the motivation to work on the \( k \)-body interaction also comes from the recent researches in this field like - the study of quantum efficiency in the transfer of an initial excitation in disordered finite networks, modeled by the \( k \)-body embedded Gaussian ensembles of random matrices [24]. Also work is done by Asaga et al [25] on the shape of the spectrum and the spectral fluctuations for BEGOE\((k)\) and BEGUE\((k)\) model using \( m \) bosons \( (m \to \infty) \) for fixed sp states \( N \) and fixed body rank \( k \), which proves the NonErgodicity of BEGOE\((k)\) and BEGUE\((k)\). Thus, one can approach towards the exploration of other spectral properties and behavior of \( k \)-body Embedded Gaussian Ensembles for different cases.

For the present chapter, we have finite number of spinless bosons where \( k \)-body interaction is considered. Here, random interactions imply that the Hamiltonian
matrix in \( k \)-particle space is represented by a classical Gaussian Orthogonal random matrix Ensemble. Being restricted to GOE, one can address the system of bosons as Bosonic Embedded Gaussian Orthogonal Ensemble \( \text{BEGOE}(k) \) where \( k \) denotes the body rank. These ensembles basically depend on three parameters \((N, m, k)\).

The upcoming section of this chapter gives information about the dimensionality of the system. Stepping ahead to next section, the method of construction for the \( \text{BEGOE}(k) \) Hamiltonian is given. After that, a spectral analysis is made which covers the behavior of the eigenvalue density and the Nearest Neighbor Spacing Distribution (NNSD) for different body ranks \((k)\). Lastly, all the results are compiled in the conclusive section.

### 2.2 Construction of \( \text{BEGOE}(k) \)

We begin with a system of \( m \) spinless bosons distributed in \( N \) single particle states. The total number of states (dimensionality) for such a spinless boson system is 
\[
d(N, m) = \binom{N+m-1}{m}
\]
due to completely symmetric nature of the wavefunction under particle interchange [For fermionic system of course, the number of states is 
\[
d(N, m) = \binom{N}{m}
\]]. An individual state \(|\alpha\rangle\) of \( m \)-bosons is described by the distribution of \( m \)-particles in \( N \) single particle boson states as \((m_1, m_2, \ldots, m_N)\). It is convenient to use boson creation operators \( b_i^\dagger \) and boson creation operators \( b_i \) for the description of states, where \( i = 1, 2, \ldots, N \). These boson operators obey the following standard commutation rules:
\[
[b_i, b_j^\dagger] = \delta_{ij}, [b_i, b_j] = 0 = [b_i^\dagger, b_j^\dagger]
\]
(2.2.1)
The state \( m_1, m_2, \ldots, m_N \) particles in respective single particle states is,
\[
|m_1, m_2, \ldots, m_N\rangle = C(b_1^\dagger)^{m_1}(b_2^\dagger)^{m_2} \cdots (b_N^\dagger)^{m_N} = C \prod_{i=1}^{N}(b_i^\dagger)^{m_i}
\]
(2.2.2)
where \( C \) is the Normalization constant and equals \( \prod_{i=1}^{N} (m_i)!^{-1/2} \).

The Hamiltonian matrix is of \( d(N, m) \times d(N, m) \) and if the time reversal symmetry is obeyed, the matrix is symmetric. The number of independent matrix elements
(IME) would be,
\[ d(N, m) \cdot (d(N, m) + 1) \middle/ 2 \] (2.2.3)

Table (2.1) explicitly calculates the dimensionality and IME for different configuration of particles ranging from \( m = 2 \) to \( m = 10 \), for fixed \( N = 4 \).

Table 2.1: Dimensionality and number of Independent Matrix Elements for \( d(N, m) = d(4, 10) \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( d(4, m) )</th>
<th>No. of ime</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>55</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>210</td>
</tr>
<tr>
<td>4</td>
<td>35</td>
<td>630</td>
</tr>
<tr>
<td>5</td>
<td>56</td>
<td>1596</td>
</tr>
<tr>
<td>6</td>
<td>84</td>
<td>3570</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>7260</td>
</tr>
<tr>
<td>8</td>
<td>165</td>
<td>13695</td>
</tr>
<tr>
<td>9</td>
<td>220</td>
<td>24310</td>
</tr>
<tr>
<td>10</td>
<td>286</td>
<td>41041</td>
</tr>
</tbody>
</table>

Such a Hamiltonian, however corresponds to all the \( m \)-particles simultaneously interacting and hence involves \( m \)-body interaction. It is however known that for real systems the interaction is usually one-body or two-body in character. For generalization, if the interaction is \( k \)-body in character, it will be completely defined by matrix elements in \( k \)-particle space. For \( m > k \), the \( m \)-particle Hamiltonian matrix elements are linear combination of various \( k \)-particle Hamiltonian matrix elements. Thus, \( k \)-body Hamiltonian is embedded in \( m \)-particle space.

For our study we consider the matrix elements of \( k \)-body Hamiltonian in \( k \)-body space as gaussian random variates and when large number of such \( k \)-body Hamiltonians is considered for \( m \)-particle system, then the system is called Bosonic Embedded Gaussian Orthogonal Ensemble [BEGOE]. Hamiltonian for one member of such an ensemble is given as :

\[ H_k = \sum_{\alpha=1, \beta=1}^{d(N,k)} H_{\alpha\beta} |\alpha\rangle \langle \beta| \] (2.2.4)
where $|\alpha\rangle$ is $k$-boson state $|m_1(\alpha), m_2(\alpha), \ldots, m_N(\alpha)\rangle$, and $H_{\alpha\beta}$ is a Gaussian random variate. For the purpose of satisfying the symmetry properties, $H_{\alpha\beta}$ is zero centred Gaussian variate with $\sigma^2 = 1$ for $\alpha \neq \beta$ and $\sigma^2 = 2$ for $\alpha = \beta$.

The basic line of demarcation between the bosonic and fermionic systems is the Pauli’s exclusion principle which tells us about the number of particles that can accumulate on say any given state. Say for $m$ number of particles distributed in $N$ single particle states, the dimensionality for fermions is different from that of bosons. Also, the spin part has a contribution in the dimensionality term but for the present investigation we restrict ourselves with finite interacting bosons without spin.

We choose $m = 10$ bosons and $N = 4$, then the dimensionality of such a system is given as $d(4,10) = 286$. Similarly if one varies the value of $m$ and $N$, then the dimensionality also changes accordingly e.g. $d(5,10) = 1001$, $d(5,11) = 1365$, $d(10,20) = 10015005$ and so on. From this, one can calculate the number of Independent Matrix Elements (IME) in the Hamiltonian matrix as under:

$$d(N,m) \cdot (d(N,m) + 1) \over 2$$

Table explicitly calculates the dimensionality and IME for different configuration of particles ranging from $m = 2$ to $m = 10$, for fixed $N = 4$.

Now we can start with the basic construction of the BEGOE($k$) Hamiltonian matrix.

2.3 Spectral Analysis

2.3.1 Eigenvalue Density

For any system under investigation it is very important to know how the energy levels are arranged therein because this creates a platform that helps us in realization of various spectral properties of the system. For the given Hamiltonian matrix we calculate the energy eigenvalues and a normalized eigenvalue spectrum is generated. This helps in understanding the arrangement of energy levels in the given energy
spectrum. First of all we shall check how densely the energy levels are arranged and what kind of distribution it follows.

We have considered a system of \( m = 10 \) spinless bosons distributed in \( N = 4 \) single particle states and the body rank of the interaction going from \( k = 2, 3, 4, \ldots 10 \). Analysis is made for total 50 members. And during these calculations, the eigenvalue spectrum for each member of the ensemble is first zero centered and scaled to unit width. The ensemble-averaged state density can be expressed as a superposition of the two limiting forms by introducing an interpolating parameter \( \mu \) as

\[
\rho(E, \mu) = \mu \rho_G(E) + (1 - \mu) \rho_{SC}(E) \tag{2.3.1}
\]

where, \( \rho_G \) and \( \rho_{SC} \) are the densities for Gaussian and Semicircle distribution respectively and they are given as follows:

\[
\rho_G(E) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{(E - E_c)^2}{2\sigma^2}\right) \tag{2.3.2}
\]

\[
\rho_{SC}(E) = \frac{1}{2\pi} \left(4 - (E - E_c/\sigma)^2\right)^{1/2} \tag{2.3.3}
\]

Using the computational techniques and fortran codes, the eigenvalues were generated from the Hamiltonian matrix and complete data analysis is made as shown in Fig.(2.2). The ensemble averaged state density here is plotted as a function of normalized energy \((E - E_c/\sigma)\), where \( E_c \) is the centroid and \( \sigma \) being the standard deviation (or ‘variance’ if squared). One observes that when the rank of body interaction \((k)\) is low, the ensemble averaged state density is close to the Gaussian form whereas it is nearly semicircular in shape when the value of \((k)\) matches with the body rank \((m)\). The histograms (in black) in the figure represent the calculated values of the density, the red and blue curves are respectively for Gaussian and semicircle distribution which are compared with our numerical results. Thus, a clear Gaussian to semicircular transition for the state density is observed as we increase the body rank from \( k = 2 \) to \( k = m \) (here \( m = 10 \)).

Now, to determine the transition point for such a system, we calculated the values of interpolating parameter \( \mu \) with respect to the body rank \((k)\). This was done by
using the method of least square fitting. To verify this, a system with a different configuration \((N = 5 \text{ and } m = 10)\) was chosen which certainly gave us the same result. The values of \(\mu\) for different \(k\) are given in table (2.2). We have plotted these values in Fig. (2.3) where the red line is for \(d(N.m) \equiv (4,10)\) whereas the green line is for \(d(N.m) \equiv (5,10)\). In both the cases it is clearly seen that taking the value of parameter \(\mu \sim 0.7\), the transition in state density occurs at \(k = m/2\).

Table 2.2: Values of interpolating parameter \(\mu\) for different values of \(k\)-body interaction rank ; calculated for two different systems

<table>
<thead>
<tr>
<th>(k)-body rank</th>
<th>(\mu(4,10))</th>
<th>(\mu(5,10))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.894</td>
<td>0.905</td>
</tr>
<tr>
<td>3</td>
<td>0.836</td>
<td>0.824</td>
</tr>
<tr>
<td>4</td>
<td>0.649</td>
<td>0.59</td>
</tr>
<tr>
<td>5</td>
<td>0.376</td>
<td>0.304</td>
</tr>
<tr>
<td>6</td>
<td>0.164</td>
<td>0.099</td>
</tr>
<tr>
<td>7</td>
<td>0.067</td>
<td>0.028</td>
</tr>
<tr>
<td>8</td>
<td>0.031</td>
<td>0.015</td>
</tr>
<tr>
<td>9</td>
<td>0.016</td>
<td>0.011</td>
</tr>
<tr>
<td>10</td>
<td>0.009</td>
<td>0.0082</td>
</tr>
</tbody>
</table>

2.3.2 Nearest Neighbor Spacing Distribution (NNSD)

Whenever we deal with a physical system, it is important for us to know the chaotic limit of the system or the parameters which induce chaos in the system and leave behind the integrability. Also, recently the transition points for various finite quantum systems are also calculated at which the system makes a transition from integrability to chaos. One of the method that deals with this kind of study is the famous Nearest Neighbor Spacing Distribution (NNSD) which is a short range fluctuation measure that gives us exact knowledge about the present domain of the system whether regular or chaotic. It is a well known fact that whenever all the quantum numbers for a system
are defined or known, then the system is called integrable or regular, else it is non-integrable (approaching to chaos). Random matrix theory in fact is very successful in explaining the spacing distribution and many more statistical properties, and Wigner was probably the first one amongst the pioneers in this area who gave very significant results regarding the shape of the NNSD, and hence it is called as the Wigner surmise [26]. The expression for the Wigner surmise (GOE) can be seen below where $\beta$ is the Dyson parameter with value equal to 1 for GOE:

$$P(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4}s^2)$$  \hspace{1cm} (2.3.4)

Here $P(s)$ is the probability of finding a nearest neighbour spacing $S$ equal to $ss_0$ in a system of energy levels identified by one particular set of values of exact and good quantum numbers. $s_0$ is the local average spacing between the nearest neighbor levels [27]. For a successful analysis of NNSD, it is needed that there should be at least some minimum spacing between energy levels so the question of zero spacing amidst any two levels vanishes away [28]. The actual reason for inter-level spacing is the repulsion which occurs due to the same symmetry between them or the same quantum numbers. But, if we have a mixed symmetry sequence then in such case the two levels lie very close to each other and there is no repulsion. The nearest neighbour spacing distribution for such levels of mixed symmetries is then given by a Poisson distribution:

$$P(s) = \exp(-s)$$  \hspace{1cm} (2.3.5)

Thus one should be clear that if a system is in the integrable domain then a regular behaviour is expected which is expressed by Eq. (2.3.5), but if we observe a chaotic nature then Eq. (2.3.4) perfectly represents the situation. Nevertheless, if one is interested to observe a transition from regularity to chaos then also the method of NNSD is fairly applicable. In classical systems there are cases where such a transition is seen for e.g. period doubling route, Poincare maps etc. Also, hydrodynamics a branch in which this theory is supported by many experiments and well in quantum physics NNSD is a main tool to study this crossover form integrability to chaos.
Adding to this, recently a new method has been proposed by Oganesyan and Huse in 2007 called the ratio of consecutive level spacings where one deals with the ratio of the spacing between two levels. This method will be discussed in Chapter 6 and based on this studied is the Poisson to GOE transition for various systems.

For the present case, we have $(N, m) = (4, 10)$. NNSD is constructed for a 50 member ensemble of BEGOE($k$) for different body ranks. Unlike the case for eigenvalue density, here even if we increase the rank of body interaction from $k = 2$ to $k = m$ (here $m = 10$) we see no effect on the curve of NNSD which is close to the Wigner distribution as stated in Eq. (2.3.4). The histograms in black in Fig.(2.4) demonstrate the computational results for the ensemble-averaged NNSD and the colored curves are theoretical plots for Poisson and Wigner (GOE). Thus in the plot of $P(s)$ versus $s$ one clearly observes that the histograms match with the Wigner curve irrespective of the rank of body interaction.

2.4 Conclusion

Introduced here is a BEGOE($k$) ensemble of finite interacting spinless bosons which are to be distributed in $N$ number of single particle states. After discussing about the construction of the system Hamiltonian, we obtain the complete energy eigenvalue spectrum. From this, we analyze the nature of the eigenvalue density for different body interaction ranks ($k$) and a transition is observed from Gaussian to semicircle in state density as $k$ increases from $k = 2$ to $k = m$. Also, we study the behavior of the nearest neighbor spacing distribution (NNSD) varying body rank $k$ and it is seen that the distribution is consistently of GOE type independent of the body rank.
$m = 8 \ ; \ N = 4$

Figure 2.1: Figure shows few configurations or basis states for $m = 8$ spinless bosons in $N = 4$ sp states. The situation is analogous to a condition where $m$ particles are distributed in $N$ boxes with the conditions that the occupancy of each box lies between zero and $m$ and the maximum number of occupied boxes equals $m$. In the figure, the first panel corresponds to the basis state $|(j_1)^5(j_2)^2(j_3)^1\rangle$, the second panel corresponds to the basis state $|(j_1)^3(j_2)^2(j_3)^1(j_4)^2\rangle$, and the last one corresponds to the basis state $|(j_1)^2(j_4)^6\rangle$.
Figure 2.2: Ensemble averaged state density for different interaction ranks $k$ for a 50 member BEGOE($k$) ensemble with $N = 4$ and $m = 10$. The histogram (black) denotes the calculated result, which has a Gaussian nature (red curve) initially for lower value of $k$ but then makes a transition to Semicircular form (blue curve) as the value of $k$ approaches to $k = 10$. 
Figure 2.3: The interpolating parameter $\mu$ is plotted against the body rank $k$ for $(m, N) = (10, 4)$ (red line) and also for $(10, 5)$ (green line). One can clearly observe the point of transition between Gaussian density and semicircular density marked by the dotted line.
Figure 2.4: NNSD for a 50 member BEGOE($k$) ensemble with $N=4$ for different body ranks ($k$). The numerical results are compared with Poisson and Wigner (GOE) form which show that the results are consistently in accordance with the Wigner surmise irrespective of the body rank $k$. 