Chapter 3

Trivially $\gamma$-endowed graphs

The aim of this chapter is to study a concept called trivially $\gamma$-endowed graphs. In any graph $G$ with $n$ vertices, any minimum dominating set and the whole set are always $\gamma$-endowed. It is interesting to study those graphs for which these are the only $\gamma$-endowed sets. A study of such graphs is made in this chapter.

3.1 Introduction

In a graph $G$, there may be dominating sets which do not contain minimum dominating sets. Obviously, the vertex set and any minimum dominating set contain $\gamma$-sets. It may happen that these are the only sets which are $\gamma$-endowed. This means that for any positive integer $k$ such that $\gamma(G) < k < n$, there exists a minimal dominating set of cardinality $k$. In this chapter, a study of these graphs called trivially $\gamma$-endowed graphs is made.
3.2 Trivially $\gamma$-endowed graphs

**Definition 3.2.1** $G$ is said to be trivially $\gamma$-endowed if $G$ is $k \gamma$-endowed only for $k = \gamma$ and $k = n$.

**Example 3.2.2**

\[ \gamma(G) = 2. \] $G$ is not $k \gamma$-endowed for any $t$, $3 \leq t \leq 5$.

**Example 3.2.3**

1. $P_{3n}$ is trivially $\gamma$ endowed.

2. $C_{3n}$ ($n \geq 2$) is trivially $\gamma$ endowed.

**Theorem 3.2.4** Let $G$ be a graph with support vertex $u$ and let $u$ support two pendant vertices. Then $G$ is trivially $\gamma$-endowed.
Proof:

Clearly, $u$ is a $\gamma$-fixed vertex of $G$. Let $D$ be a $\gamma$-set of $G$.

Then $(D - \{u\} \cup \{v_1, v_2\})$ where $v_1, v_2$ are pendants at $u$ is a dominating set of $G$ not containing any $\gamma$-set of $G$. Therefore, $G$ is not $k \gamma$-endowed where $k = \gamma + 1$.

Let $\gamma < k < n$. Choose $k - (\gamma + 1)$ vertices from $(V - D) - \{v_1, v_2\}$ (it is possible since $|V - D - \{v_1, v_2\}| = n - \gamma - 2 > k - \gamma - 2 \geq k - (\gamma + 1)$) and add them with $(D - \{u\}) \cup \{v_1, v_2\}$. The resulting set is a dominating set of $G$ of cardinality $k$ and it does not contain $u$. Hence, $G$ is not $k \gamma$-endowed for any $\gamma < k < n$. Therefore, $G$ is trivially $\gamma$-endowed.

Theorem 3.2.5 Let $G$ be a graph without isolates. Let $G$ have a unique minimum dominating set. Let $D$ be the unique minimum dominating set of $G$. Suppose there exists a dominating set of cardinality $\gamma + 1$ not containing $D$. Then, $G$ is trivially $\gamma$-endowed.

Proof:

Let $D = \{u_1, u_2, \cdots u_\gamma\}$. Let $S$ be a dominating set of cardinality $\gamma + 1$ not containing $D$. Let $u_i \in D - S$.

Case(i): Let $V - S = \{u_i\}$ . Therefore, $n - \gamma - 1 = 1$. Therefore, $\gamma = n - 2$. Since there exists a dominating set of cardinality $\gamma + 1 = n - 1$ not containing $D$, $G$ is trivially $\gamma$-endowed.
Case(ii): Let $|V - S| \geq 2$. Let $A$ be a subset of $V - S$ not containing $u_i$. Then $1 \leq |A| \leq n - \gamma - 2$. $S \cup A$ is a dominating set of $G$ not containing $D$. By adding $k_1$ vertices from $V - (S \cup \{u_i\})$ with $S$, $1 \leq k_1 \leq n - \gamma - 2$, we get dominating sets of cardinality $\gamma + 1 + k_1$ which do not contain $D$. Therefore, $G$ is not $k\gamma$-endowed for all $k$, $\gamma + 1 \leq k \leq \gamma + 1 + n - \gamma - 2 = n - 1$. Therefore $G$ is trivially $\gamma$-endowed.

**Corollary 3.2.6** Let $G$ be a graph without isolates. Let $D = \{u_1, u_2, u_3 \cdots u_\gamma\}$ be a unique minimum dominating set of $G$. If for some $i$, $1 \leq i \leq \gamma$, $\gamma(<pn(u_i, D)>)=2$, then there exists a dominating set of cardinality $\gamma + 1$ not containing $D$ and hence $G$ is trivially $\gamma$-endowed.

**Proof:**

Let $\gamma(<pn(u_i, D)>)=2$. Let $D_1$ be a minimum dominating set of $<pn(u_i, D)>$. Let $D_1 = \{x_1, x_2\}$. Then $S - (D - \{u_i\}) \cup \{x_1, x_2\}$ is a dominating set of $G$ of cardinality $\gamma + 1$ and $S$ does not contain $D$. Therefore, there exists a dominating set of cardinality $\gamma + 1$ not containing the minimum dominating set $D$. By the above theorem $G$ is trivially $\gamma$-endowed.

**Remark 3.2.7** The condition that $G$ has a unique dominating set can not be dropped: For example in $P_5$, there are three minimum dominating set of $G$. There exists a dominating set of $G$ of cardinality $(\gamma + 1)$ not containing any minimum dominating set and every four element dominating set contains a $\gamma$-set of $G$. 

132
Remark 3.2.8 If $G$ satisfies the hypothesis of the theorem

then $\gamma(G) \leq n - 2$.

Remark 3.2.9 There exists a graph $G$ without isolates having a unique minimum dominating set say $D = \{u_1, u_2 \ldots u_\gamma\}$ with $\gamma(<pn(u_i, D)) \geq 3$ for every $i$, $1 \leq i \leq \gamma$ and there exists a dominating set of cardinality $\gamma + 1$ not containing $D$. Let $G =$

\[
\begin{align*}
D = \{v_4, v_5\} \text{ is the unique minimum dominating set of } G, \\
pn(v_4, D) = \{v_1, v_2, v_3\} \text{ and } \gamma < pn(v_4, D) \geq 3, \\
pn(v_5, D) = \{v_6, v_7, v_8, v_9, v_{10}\} \\
\text{and } \gamma < pn(5, D) \geq 5. \ S = \{v_5, v_{10}, v_{11}\} \text{ is a dominating set of } G \text{ of cardinality } \gamma + 1 = 3 \text{ and } S \text{ does not contain } D. \text{ Clearly } G \text{ is trivially } \\
\gamma\text{-endowed.}
\end{align*}
\]
**Theorem 3.2.10** Let $G$ be a graph. Then $G$ satisfies

(i) the minimum dominating set of $G$ is unique. (ii) $\gamma(G) = n - 3$ and

(iii) There exists a $(\gamma(G) + 1)$ dominating set not containing $D$

if and only if $G = H \cup tK_1$ where $t \geq 0, H =$ \[ \text{Diagram} \]

**Proof:**

**Case(i):** Suppose $G$ has no isolates. Then $\gamma(G) \leq n/2$. That is,

$n - 3 \leq n/2$. Therefore, $n \leq 6$. Since $\gamma(G) = n - 3 \geq 1, n \geq 4$.

By inspecting

all the graphs of order 4, 5, 6. We get that $G =$ \[ \text{Diagram} \]

**Case(ii)**

Suppose $G$ has $t$ isolates say $u_1, u_2, \cdots u_t$. Let $G_1$ be the graph induced by $V(G) = \{u_1, u_2, \cdots u_t\}$. Then $G_1$ has no isolates

$\gamma(G_1) = \gamma(G) - t = n - 3 - t$. $|V(G_1)| = n - t$. Therefore,

$\gamma(G_1) = |V(G_1)| - 3$. Since $G$ has a unique minimum dominating set say $D$, $D_1 = D - \{u_1, u_2, \cdots u_t\}$ is the unique minimum dominating set of $G_1$. By hypothesis there exists a $(\gamma(G) + 1)$ dominating set $S$ in $G$ not containing $D$. Therefore, $S_1 = S - \{u_1, u_2, \cdots u_t\}$ is a dominating set in $G_1$ not containing $D$. 

134
Therefore, \( G_1 \) satisfies the hypothesis of case (i). Therefore,

\[
G_1 =
\]

Therefore,

\[
G =
\]

**Corollary 3.2.11** Let \( G \) be a graph. Then \( G \) satisfies

(i) the minimum dominating set of \( G \) is unique. (ii) \( \gamma(G) = n - 3 \) and

(iii) \( G \) is trivially \( \gamma \)-endowed

if and only if \( G = H \cup tk_1 \) Where \( t \geq 0 \),

\[
H =
\]
Proof:

If \( G = H \cup tK_1 \) where \( H = t \geq 0 \),
then \( G \) satisfies the three conditions. Conversely if \( G \) satisfies the three conditions, then there exists a \( (\gamma(G) + 1) \) dominating set of \( G \) not containing \( D \).

Therefore, by the above theorem, \( G = H \cup tK_1 \) where \( H = t \geq 0 \).

\[ \square \]

\textbf{Theorem 3.2.12} Let \( G \) be a graph. Then \( G \) satisfies

(i) the minimum dominating set of \( G \) is unique
(ii) \( \gamma(G) = n - 2 \) and
(iii) There exists a \( (\gamma(G) + 1) \) dominating set not containing \( D \)
if and only if \( G = P_3 \cup tk_1 \) where \( t \geq 0 \)

Proof:

Case(i): Suppose \( G \) has no isolates. Then \( \gamma(G) \leq n/2 \). That is
\[ n - 2 \leq n/2. \] Therefore, \( n \leq 4. \) Since \( \gamma(G) = n - 2 \geq 1, n \geq 3. \) Therefore, \( 3 \leq n \leq 4. \) By inspecting all the graphs of order 3 and 4, we get that \( G = P_3. \)

Case (ii)

Suppose \( G \) has \( t \) isolates say \( u_1, u_2, \cdots u_t. \) Let \( G_1 \) be the graph induced by \( V(G) = \{u_1, u_2, \cdots u_t\}. \) Then \( G_1 \) has no isolates \( \gamma(G_1) = \gamma(G) - t = n - 2 - t. \) \( |V(G_1)| = n - t. \) Therefore \( \gamma(G_1) = |V(G)| - 2. \) Since \( G \) has a unique minimum dominating set say \( D, D_1 = D - \{u_1, u_2, \cdots u_t\} \) is a unique minimum dominating set of \( G_1. \) By hypothesis there exists a \((\gamma(G) + 1)\) dominating set \( S \) in \( G \) not containing \( D. \) Therefore, \( S_1 = S - \{u_1, u_2, \cdots u_t\} \) is a dominating set in \( G_1 \) not containing \( D_1. \) Therefore \( G_1 \) satisfies the hypothesis of case (i). Therefore, \( G_1 = P_3. \) Therefore \( G = P_3 \cup tK_1. \)

**Corollary 3.2.13** Let \( G \) be a graph. Then \( G \) satisfies

(i) the minimum dominating set of \( G \) is unique

(ii) \( \gamma(G) = n - 2 \) and

(iii) \( G \) is trivially \( \gamma \)-endowed

if and only if \( G = P_3 \cup tK_1 \) where \( t \geq 0 \)

**Proof:**

If \( G = P_3 \cup tK_1 \) where \( t \geq 0 \), then \( G \) satisfies the three conditions. Conversely, if \( G \) satisfies the three conditions, then there exists a \( (\gamma(G) + 1) \)
dominating set of $G$ not containing $D$. Therefore, by the above theorem $G = P_3 \cup tK_1$.

\section*{Theorem 3.2.14} Let $G$ be a graph. Then $G$ satisfies

(i) the minimum dominating set of $G$ is unique

(ii) $\gamma(G) = n - 4$ and

(iii) There exists a $(\gamma(G) + 1)$ dominating set not containing $D$

if and only if $G = P_6 \cup tk_1, D_{2,2} \cup tk_1, H_1 \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1$ Where $t \geq 0$.

\textbf{Proof:}

\textbf{Case(i):} Suppose $G$ has no isolates. Then $\gamma(G) \leq n/2$. That is $n - 4 \leq n/2$.

Therefore, $n \leq 8$. Since $\gamma(G) = n - 4 \geq 1$, $n \geq 5$. Therefore $5 \leq n \leq 8$.

By inspecting

all the graphs of order 5, 6, 7 and 8, we get that $G = P_6 \cup tk_1, D_{2,2} \cup$

$tk_1, H_1 \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1$ where

\begin{itemize}
  \item $H_1 = \text{\begin{figure}}$
  \item $H_2 = \text{\begin{figure}}$
  \item $H_3 = \text{\begin{figure}}$
  \item $H_4 = \text{\begin{figure}}$
  \item $H_5 = \text{\begin{figure}}$
  \item $H_6 = \text{\begin{figure}}$
  \item $H_7 = \text{\begin{figure}}$
\end{itemize}
Case (ii)

Suppose \( G \) has \( t \) isolates say \( u_1, u_2, \ldots, u_t \). Let \( G_1 \) be the graph induced by \( V(G) - \{u_1, u_2, \ldots, u_t\} \). Then \( G_1 \) has no isolates \( \gamma(G_1) = \gamma(G) - t = n - 4 - t \). Therefore \( \gamma(G_1) = |V(G_1)| - 4 \). Since \( G \) has a unique minimum dominating set say \( D \), \( D_1 = D - \{u_1, u_2, \ldots, u_t\} \) is a unique minimum dominating set of \( G_1 \). By hypothesis there exists a \((\gamma(G) + 1)\) dominating set \( S \) in \( G \) not containing \( D \). \( S_1 = S - \{u_1, u_2, \ldots, u_t\} \) is a dominating set in \( G_1 \) not containing \( D_1 \). Therefore \( G_1 \) satisfies the hypothesis of case (i).

Therefore \( G_1 = P_6 \cup tk_1, D_{2,2} \cup tk_1, H_1 \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1 \). Therefore, \( G = P_6 \cup tk_1, D_{2,2} \cup tk_1, H \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1 \).  

**Corollary 3.2.15** Let \( G \) be a graph. Then \( G \) satisfies

(i) the minimum dominating set of \( G \) is unique

(ii) \( \gamma(G) = n - 4 \) and (iii) \( G \) is trivially \( \gamma \)-endowed

if and only if \( G = P_6 \cup tk_1, D_{2,2} \cup tk_1, H \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1 \) where \( t \geq 0 \)

**Proof:**

If \( G = P_6 \cup tk_1, D_{2,2} \cup tk_1, H_1 \cup tk_1, H_2 \cup tk_1 \cdots H_7 \cup tk_1 \) where \( t \geq 0 \), then \( G \) satisfies the three conditions. Conversely, if \( G \) satisfies three conditions, then there exists a \((\gamma(G) + 1)\) dominating set of \( G \) not
Corollary 3.2.16 Let \( G \) be a graph with a unique minimum dominating set. If \( G \) is not \((\gamma + 2)\) -endowed then \( G \) is not \( k \gamma \) -endowed for all \( k \) except possibly \( \gamma + 1 \).

Theorem 3.2.17 Let \( G \) be a graph with \( \gamma \) - fixed vertex \( u \). Suppose there exists a dominating set of cardinality \( \gamma + 1 \) not containing \( u \). Then \( G \) is trivially \( \gamma \) -endowed.

Proof:

Let \( D \) be a dominating set of \( G \) of cardinality \( \gamma + 1 \) and let \( u \notin D \).
Let \( S = D \cup T \) where \( u \notin T \subset V - D \). Then \( S \) is a dominating set of \( G \) of cardinality \( \gamma + 1 + |T| \) and \( 0 \leq |T| \leq n - (\gamma + 2) \). Therefore \( S \) is a dominating set of \( G \) with \( \gamma + 1 \leq |S| \leq n - 1 \). Therefore, \( G \) is trivially \( \gamma \) -endowed.

Corollary 3.2.18 Let \( G \) be a graph with \( \gamma \) - fixed vertices \( u_1, u_2 \cdots u_r \). Let the minimum cardinality of a dominating set not containing \( u_i \) for some \( i \), \( 1 \leq i \leq r \) be \( \gamma + t \), \( 1 \leq t \leq n - \gamma - 1 \). Then \( G \) is not \( k \gamma \) -endowed for all \( k \), \( \gamma + t \leq k \leq n - 1 \).

Theorem 3.2.19 Let \( G \) be a graph with a \( \gamma \) - fixed vertex \( u \). Then \( G \) is
trivially $\gamma$-endowed if for every positive integer $k$, $\gamma + 1 \leq k \leq n - 1$, there exists a dominating set of $G$ in $G - \{u\}$.

Proof:

Trivial. ■

**Theorem 3.2.20** Let $D$ be a unique minimum dominating set of $G$. Let $x_1, x_2, \cdots x_{r+1} \in V - D$ and let $N[x_1, x_2, \cdots x_{r+1}] \supseteq pn[u_i, D], 1 \leq i \leq r$ where $u_i \in D, 1 \leq i \leq r$. Then $G$ is trivially $\gamma$-endowed.
Proof:

Let \( S = (D - \{u_1, u_2, \ldots, u_r\} \cup \{x_1, x_2, \ldots, x_{r+1}\}) \). Then \( S \) is a dominating set of \( G \) not containing \( D \) and \( |S| = \gamma(G) + 1 \). Therefore, \( G \) is trivially \( \gamma \)-endowed.

\[ \square \]

**Remark 3.2.21** Let \( G \) be a graph without isolates. Let \( D = \{u_1, u_2, \ldots, u_r\} \) be a unique minimum dominating set of \( G \). Suppose there exist \( x, y \in V - D \) such that \( N(\{x, y\} \supseteq \text{pn}(u_i, D) \cup \{u_i\}) \). Then \( S = (D - \{u_i\}) \cup \{x, y\} \) is a dominating set of \( G \) of cardinality \( \gamma + 1 \). \( S \) does not contain \( D \). Therefore, \( G \) is trivially \( \gamma \)-endowed.

\[ \square \]

**Illustration 3.2.22**

\[
\begin{align*}
\gamma(G) &= 2 \quad \text{and} \quad \{v_2, v_4\} \text{ is a unique minimum dominating set of } G. \\
\{v_1, v_2, v_3\} \text{ is a dominating set of } G \text{ not containing the unique minimum dominating set of } G. \text{ The reason is that } N(\{v_1, v_3\} \supseteq \text{pn}(v_4, \{v_2, v_4\}) \cup \{v_4\}).
\end{align*}
\]
Remark 3.2.23 Let $G$ be a graph without isolates. Let $D = \{u_1, u_2, \cdots u_\gamma\}$ be a unique minimum dominating set of $G$. If $V-D$ contains a dominating set of $G$ of cardinality $\gamma+1$, then $G$ is trivially $\gamma$-endowed.

Illustration 3.2.24

$D = \{v_2, v_4\}$ is the unique minimum dominating set of $G$. $\{v_1, v_3, v_{11}\}$ is a dominating set of $G$ contained in $V-D$. Therefore, $G$ is trivially $\gamma$-endowed.

Remark 3.2.25 Let $D$ be a unique minimum dominating set of $G$. There exists a dominating set of $G$ of cardinality $\gamma+1$ not containing $D$. Then $G$ is trivially $\gamma$-endowed.
Remark 3.2.26 Given any positive integer \( k \) there exist a connected graph \( G \) with \( \gamma(G) = k \), \( \beta_0(G) = k + 1 \) and \( G \) is trivially \( \gamma \)-endowed. Let \( G \) be obtained from \( P_{k+2} \) by attaching pendant vertices one each at \( v_2, v_3 \cdots v_k \) where \( V(P_{k+2}) = \{v_1, v_2, v_3 \cdots v_{k+2}\} \). Then \( \gamma(G) = k \) and \( \beta_0(G) = k + 1 \) and \( G \) is trivially \( \gamma \)-endowed.

Problem: If \( \beta_0(G) = \gamma(G) + 1 \), then find necessary and sufficient condition such that \( G \) is trivially \( \gamma \)-endowed.

Theorem 3.2.27 Let \( G \) be a graph with \( \beta_0(G) = \gamma(G) + 1 \) , \( G \) is trivially \( \gamma \)-endowed if there exist a \( \beta_0 \)-set \( D = \{u_1, u_2, \cdots u_{\beta_0}\} \) with
\[
|pm[u_i, D]| = t_i, t_i \geq 2 \text{ for } i = i_1, i_2, \cdots i_s, s \geq 1 \text{ and } \beta_0 + \sum_{i \in \{i_1, i_2, \cdots i_s\}} t_i = n - 1.
\]

Proof:
Let \( D = \{u_1, u_2, \cdots u_{\beta_0}\} \) be a maximum independent set of \( G \). Suppose
\[
|pm[u_i, D]| = t_i, 1 \leq i \leq \beta_0.
\]
and let \( \beta_0 + \sum_{i \in \{i_1, i_2, \cdots i_s\}} t_i = n - 1 \) . Then \( G \) is not \( k \) \( \gamma \)-endowed for
\[
k = \beta_0 + \beta_0+1 + \cdots : \beta_0 + \sum_{i \in \{i_1, i_2, \cdots i_s\}} t_i \leq n - 1 .
\]
As \( \beta_0 = \gamma(G) + 1 \) and as \( \beta_0 + \Sigma_{i \in \{i_1, i_2, \cdots i_s\}} t_i = n - 1 \) . \( G \) is not \( k \) \( \gamma \)-endowed for \( k = \gamma(G) + 1, \gamma(G) + 2, \cdots (n - 1) \). That is \( G \) is trivially \( \gamma \)-endowed.

Theorem 3.2.28 Let \( G \) be a graph with a \( \gamma \)-fixed vertex and \( \beta_0(G) = \gamma(G) + 1 \) . Then \( G \) is trivially \( \gamma \)-endowed.
Proof:-

Let $D$ be a maximum independent set of $G$. Let $u$ be a $\gamma$ fixed vertex of $G$. Therefore, $G$ is not $(\gamma+1)$ $\gamma$ endowed. Let $D_1 = D \cup \{v\}$, $v \neq u$. Then $D_1$ is a dominating set of $G$ and $D_1$ does not contain a $\gamma$ set of $G$ since $u \notin D_1$. For any $\{v_1, v_2, \ldots, v_k\} \subset (V - \{u\}) - (D \cup \{v\})$, $D \cup \{v_1, v_2, \ldots, v_k\}$ is a dominating set of $G$ not containing a $\gamma$ set of $G$. Therefore, $G$ is trivially $\gamma$-endowed.

Remark 3.2.29 The converse of above theorem is not true since in $P_9$, $\beta_0 \neq \gamma + 1$, $P_9$ has $\gamma$ fixed vertices and $P_9$ is trivially $\gamma$ endowed.

Theorem 3.2.30 Suppose $G$ has a unique minimum dominating set.

i) If $\beta_0 = \gamma(G) + 1$, then $G$ is trivially $\gamma$-endowed.

ii) If $i(G) = \gamma(G) + 1$ then $G$ is trivially $\gamma$-endowed.

Proof:

If $\beta_0 = \gamma(G) + 1$ or $i(G) = \gamma(G) + 1$ then $G$ contains a minimal dominating set of cardinality $\gamma(G) + 1$. Therefore $G$ is trivially $\gamma$-endowed.

Corollary 3.2.31

(i) Let $i(G) = \gamma(G) + 1$ and $G$ has a $\gamma$-fixed vertex. Then $G$ is trivially $\gamma$-endowed.

(ii) Let $G$ have a $\gamma$ fixed vertex say $u$. Let $S$ be a dominating set of $G$ of
Remark 3.2.32 If $S$ is a dominating set of $G$ of cardinality $\gamma(G) + 1$ such that $S$ does not contain any minimum dominating set of $G$ then $S$ is minimal.

Theorem 3.2.33 Let $G$ be a graph with a $\gamma$ fixed vertex and let $G$ be not $k$ $\gamma$-endowed for $k = \gamma(G) + 1$. Then $G$ is trivially $\gamma$-endowed.

Proof:

Let $D$ be a dominating set of $G$ cardinality $\gamma(G) + 1$ such that $D$ does not contain a minimum dominating set of $G$. Then $D$ is a minimal dominating set of $G$. Let $u$ be a $\gamma$-fixed vertex of $G$. Clearly $u \notin D$. Let $\gamma(G) + 1 < l < n - 1$. Let $D_1 = D \cup \{v_{k+1}, v_{k+2}, \ldots v_l\}$ where $k = \gamma(G) + 1$ and $u \notin \{v_{k+1}, v_{k+2}, \ldots v_l\}$. Then $D_1$ is a dominating set of $G$ which does not contain a minimum dominating set of $G$. Hence the theorem.

Corollary 3.2.34 Let $G$ be a graph with a $\gamma$-fixed vertex and let $\beta_0(G) = \gamma(G) + 1$ or $i(G) = \gamma(G) + 1$ or $\Gamma(G) = \gamma(G) + 1$. Then $G$ is trivially $\gamma$-endowed.

Theorem 3.2.35 If a graph $G$ of cardinality of order $n$ has a minimal dominating set of cardinality $n - 1$ then
(i) \( \gamma(G) = n - 1 \) if \( G \) has exactly \( (n - 2) \) isolates

(ii) \( \gamma(G) = n - t \) if \( G \) has exactly \( (n - t - 1) \) isolates and the remaining vertices form a star.

Proof:

Let \( D \) be a minimal dominating set of cardinality \( n - 1 \). Let \( V(G) = \{u_1, u_2 \cdots u_n\} \) and let \( D = \{u_1, u_2 \cdots u_{n-1}\} \). \( u_n \) is adjacent to some point of \( D \) say \( u_i, 1 \leq i \leq n - 1 \).

Case(i) : \( D \) is independent. Then \( G \) is \( K_{1,t} \cup (n - t - 1)K_1 \) where \( 1 \leq t \leq n - 1 \). \( \gamma(G) = 1 + n - t - 1 = n - t \).

Case(ii) : \( D \) is not independent. Let without loss of generality \( u_1 \) and \( u_2 \) be adjacent. Then \( u_n \) is adjacent with exactly one of \( u_1, u_2 \) in \( D \) (since \( D \) is minimal). Therefore \( G = P_3 \cup (n - 3)K_1 \). \( \gamma(G) = n - 2 = n - t \) where \( t = 2 \) and \( G = K_{1,t} \cup (n - t - 1)K_1 \).

**Corollary 3.2.36** Let \( G \) be a graph order \( n \). \( G \) has a minimal dominating set of cardinality \( n - 1 \) iff \( G \) is \( K_{1,t} \cup (n - t - 1)K_1 \) where \( 1 \leq t \leq n - 1 \)

**Corollary 3.2.37** Let \( G \) be a graph of order \( n \) and let \( G \) have a minimal dominating set of cardinality \( n - 1 \). Then \( G \) is trivially \( \gamma \)-endowed iff \( G = K_{1,t} \cup (n - t - 1)K_1 \) where \( t = 1 \) or \( 2 \). Equivalently if \( G \) is a graph of order \( n \) with \( \Gamma(G) = n - 1 \) then \( G \) is trivially \( \gamma \)-endowed iff \( G = K_2 \cup (n - 2)K_1 \) or \( K_{1,2} \cup K_1 \).
Theorem 3.2.38 Suppose $\Gamma(G) = n - 2$. Then $G$ is $K_{1,t} \cup K_{1,s}$ with $t + s + 2 = n$ or $D_{r,s}$, $r + s + 2 = n$ or $G$ is obtained from $K_{1,t}$ by adding a vertex $v$ and making it adjacent with some or all the vertices of $K_{1,t}$ and possibly adjacent with newly added vertices which are independent. or $G$ is $P_4 \cup (n - 4)K_1$

Proof:

Let $G$ be a $\Gamma(G)$ set of $G$. Let $D = \{u_1, u_2, \cdots u_{n-2}\}$

Case(i): $D$ is independent. Then $u_{n-1}$ and $u_n$ are adjacent with at least one vertex of $D$. $u_{n-1}$ and $u_n$ may be or may not be adjacent. Let $N(u_{n-1}) \cap D = \{u_{i_1}, u_{i_2} \cdots u_{i_r}\}$ and $N(u_n) \cap D = \{v_{j_1}, v_{j_2} \cdots v_{j_s}\}$. There may be common vertices in $N(u_{n-1}) \cup D$ and $N(u_n) \cap D$. Then $G$ is $K_{1,t} \cup K_{1,s}$ with $t + s + 2 = n$ or $G$ is $D_{r,s}$, $r + s + 2 = n$ or $G$ is obtained from $K_{1,t}$ by adding a vertex $v$ and making it adjacent with some or all the vertices of $K_{1,t}$ and possibly adjacent with newly added vertices which are independent.

Case(ii): $D$ is not independent. Suppose $u_1$ and $u_2$ are adjacent, Then $u_1$ and $u_2$ must have private neighbour in $V - D$. Since $V - D = \{u_{n-1}, u_n\}$, $u_{n-1}$ is adjacent with exactly one of $u_1, u_2$ and $u_n$ is adjacent with the other. Therefore, $G$ is $P_4 \cup (n - 4)K_1$. ■

Corollary 3.2.39 Suppose $\Gamma(G) = n - 2$. Then $G$ is trivially $\gamma$-endowed if and only if $G$ is $K_{1,t} \cup K_{1,s}$, $t + s + 2 = n$ and at least one of $t$, $s$ is 2 and
\[ 2 \leq t, s \text{ or } D_{r,s}, r + s + 2 = n \text{ and at least one of } t, s \text{ is } 2 \text{ or } G \text{ is obtained from } K_{1,t} \text{ by adding a vertex } t \text{ and making it adjacent with some or all the vertices of } K_{1,t} \text{ and possibly adjacent with newly added say } s \text{ vertices which are independent and at least one of } t, s \text{ is less or equal to } 2 \text{ and the centre of } K_{1,t} \text{ or } v \text{ has a pendant vertex which is not a common vertex and if } v \text{ is adjacent with the centre of } K_{1,t} \text{ then the centre and } v \text{ should have at least one pendant vertex each which is not a common vertex.}

\[ \blacksquare \]

**Illustration 3.2.40**

\[ u_1 \quad u_2 \quad u_3 \quad u_t \]

\[ v_1 \quad v_2 \quad v_3 \]

where \( t \geq 2, \gamma(G) = 2, \{u_1, v_1\} \text{ is a minimum dominating set. } \Gamma(G) = n - 2 = t + 2. \]

\[ \blacksquare \]
\( G = \)

\[ \gamma(G) = 2, \Gamma(G) = n - 2. \]
Illustration 3.2.43

\[ G = \]

\[ \gamma(G) = 2, \Gamma(G) = n - 2. \]

Remark 3.2.44

There are graph \( G \) in which \( \gamma(G) = \beta_0(G) \) and \( \Gamma(G) = \beta_0(G) + 1 \) and \( G \) is \( k \gamma \)-endowed for all \( k \), \( \gamma(G) \leq k \leq n \) except when \( k = \Gamma(G) \).
For: Let

\[ v_1 v_2 G = v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} \]

\[ \gamma(G) = 4, \beta_0 = 4, \Gamma(G) = 5, \{v_3, v_4, v_6, v_7, v_{10}\} \text{ is a } \Gamma(G) = \text{set of } G. \] 

is \( k \) \( \gamma \)-endowed for all \( k \geq 6 \).

**Definition 3.2.45** A graph \( G \) is almost trivially \( \gamma \)-endowed if \( G \) is not \( k \) \( \gamma \)-endowed for all \( k, \gamma + 1 \leq k \leq n - 2 \).

**Example 3.2.46** \( K_{1,n} \) is almost trivially \( \gamma \)-endowed.

**Theorem 3.2.47** Let \( u, v \in V(G) \) be such that any minimum dominating set of \( G \) either contains \( u \) or \( v \). Suppose there exists a dominating set of cardinality \( \gamma + 1 \) not containing \( u \) and \( v \). Then \( G \) is almost trivially \( \gamma \)-endowed that is \( G \) is not \( k \) \( \gamma \)-endowed for all \( k, \gamma + 1 \leq k \leq n - 2 \).

**Proof:**

Let \( D \) be a dominating set of cardinality \( \gamma + 1 \) not containing \( u \) and \( v \). For any \( w \) not equal to \( u, v \), \( D \cup \{w\} \) is a dominating set of cardinality
\[ \gamma + 2 \text{ and it does not contain any minimum dominating set of } G \]. In general if \( v_1, v_2, \ldots, v_{n-(\gamma+3)} \notin D \cup \{u, v\}, (D \cup \{v_1, v_2, \ldots, v_{n-(\gamma+3)}\}) \) is a dominating set of cardinality \( n - 2 \) and this does not contain any minimum dominating set. Thus \( G \) is not \( \gamma \)-endowed for all \( k \), \( \gamma + 1 \leq k \leq n - 2 \). \[ \square \]

**Corollary 3.2.48** If \( u, v \in V(G) \) such that any minimum dominating set of \( G \) either contain \( u \) or contain \( v \) and there exists a dominating set of cardinality \( \gamma + 1 \) not containing \( u \) and \( v \), and also if \( G \) is not \( (n - 1) \gamma \)-endowed then \( G \) is trivially \( \gamma \)-endowed.