Chapter 5

Weakly well covered graphs

5.1 Introduction

Well covered graphs are those in which every maximal independent set is a maximum independent set. A graph is weakly well covered if every non-maximal independent set is contained in a maximum independent set. Weakly well covered graphs are studied in this chapter.

5.2 Weakly well covered graphs.

Definition 5.2.1

A graph $G$ is said to be well covered if every maximal independent set of $G$ is a maximum independent set of $G$. Equivalently, $G$ is completely extendable.
Example 5.2.2

(i). $C_4$ is well covered.

(ii). $K_{m,n}$ is well covered.

Definition 5.2.3

A graph $G$ is said to be weakly well covered if every non-maximal independent set of $G$ is contained in a maximum independent set of $G$.

Example 5.2.4

Let $G = W_7$

Here $\beta_0(G) = 3$. Also, $\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}$ are maximum independent sets of $G$. Clearly any non-maximal independent set of $G$ is contained in $\beta_0$-set of $G$. Therefore $G$ is weakly well covered graph.

Remark 5.2.5

Every well covered graph is a weakly well covered but the converse is not true. The example given above is an example of a graph which is weakly well covered but not well covered. Some more examples are given below
Example 5.2.6

Let $G = C_6$

Here $\beta_0(G) = 3$. Also, $\{u_1, u_3, u_5\}$, $\{u_2, u_4, u_6\}$ are maximum independent sets of $G$. Clearly any non-maximal independent set of $G$ is contained in $\beta_0$-set of $G$. Hence $G$ is weakly well covered graph. Also, $S = \{u_1, u_4\}$ form a maximal independent set of $G$ which is not contained in any $\beta_0$-set of $G$. Therefore $G$ is not well covered.

Example 5.2.7
Let $G = K_{1,n}$

In this graph $G = K_{1,n}$, the pendent vertices form a unique maximum independent sets of $K_{1,n}$. Clearly any non-maximal independent set of $K_{1,n}$ is contained in a unique $\beta_0$-set of $K_{1,n}$. Hence $K_{1,n}$ is weakly well covered graph. Also, the full degree vertex form a maximal independent set of $K_{1,n}$ which is not contained in a unique $\beta_0$-set of $K_{1,n}$. Therefore $K_{1,n}$ is not well covered.

**Remark 5.2.8**

*Every weakly well covered graph is weakly k-extendable for every* $k$, $1 \leq k \leq \beta_0(G)$ *and conversely.*

**Definition 5.2.9**

*A graph $G$ is said to be trivially weakly well covered if for any positive integer $k$, $1 \leq k \leq (\beta_0 - 2)$, there exist a non-maximal independent set of cardinality $k$ not contained in a maximum independent set of $G$.*

**Example 5.2.10**

*The graph in example 4.2.8 (iii) above is trivially weakly well covered.*
Definition 5.2.11

A graph $G$ is trivially well covered if there exist maximal independent sets of cardinality $k$, $1 \leq k \leq (\beta_0(G) - 1)$.

Observation 5.2.12

Let $G$ be a weakly well covered graph. Let $S$ be a maximal independent set of $G$ containing at least two vertices. Let $u \in S$. Then $u$ is $\beta_0$-good.

Proof:

Let $S$ be a maximal independent set of $G$ containing at least two vertices. Let $u \in S$. $\{u\}$ is a non-maximal independent set contained in $S$. Therefore $\{u\}$ is contained in a maximum independent set of $G$. Therefore $u$ is $\beta_0$-good.

Theorem 5.2.13

Let $G$ be a graph $G \neq K_n$ and without a full degree vertex having a unique $\beta_0$-set say $S$. Then $G$ is not weakly well covered.

Proof:

Let $G$ be a graph without a full degree vertex having a unique $\beta_0$-set say $S$. Since $G \neq K_n$, $V - S \neq \phi$. Let $u \notin S$. $\{u\}$ is non-maximal, since $G$ has no full degree vertex. Since $u \notin S$, $\{u\}$ is not contained in the unique maximum independent set $S$. Therefore $G$ is not weakly well covered.

Observation 5.2.14

Let $G$ be a graph without full degree vertex. If $G$ is a weakly well covered graph
containing two disjoint maximal independent sets whose union is $V(G)$, then $G$ is $\beta_0$-excellent.

**Proof:**

Let $u \in V(G)$. Let $S_1, S_2$ be maximal independent sets of $G$ whose union is $V(G)$. Then $u \in S_1$ (or) $S_2$. Suppose $u \in S_1$. Since $u$ is not a full degree vertex, $S_1$ is not a singleton. Therefore $\{u\}$ is not maximal. Therefore $u$ is contained in a maximum independent set of $G$. Hence every vertex in $S_1$ is $\beta_0$-good. Similarly every vertex in $S_2$ is $\beta_0$-good. Therefore $G$ is $\beta_0$-excellent.

**Example 5.2.15**

Let $G = P_6$

\begin{center}
\begin{tikzpicture}
\node [vertex] (1) at (0,0) {$u_1$};
\node [vertex] (2) at (1,0) {$u_2$};
\node [vertex] (3) at (2,0) {$u_3$};
\node [vertex] (4) at (3,0) {$u_4$};
\node [vertex] (5) at (4,0) {$u_5$};
\node [vertex] (6) at (5,0) {$u_6$};
\end{tikzpicture}
\end{center}

Here $\beta_0(G) = 3$. Also, $\{u_1, u_3, u_5\}$, $\{u_2, u_4, u_6\}$, $\{u_1, u_3, u_6\}$, $\{u_1, u_4, u_6\}$ are
maximum independent sets of $G$. Further $S_1 = \{u_2, u_5\}$, $S_2 = \{u_1, u_3, u_6\}$ form two disjoint maximal independent sets of $G$, whose union is $V(G)$ and $G$ has no full degree vertex. Clearly any non-maximal independent sets of $G$ is contained in $\beta_0$-set of $G$. Hence $G$ is weakly well covered graph. That is $G$ is $\beta_0$-excellent.

**Observation 5.2.16**

There are non $\beta_0$-excellent graphs which are weakly well covered.

*Example,* consider $G = W_5$

![Graph](image)

Here $V(G) = \{u_1, \ldots, u_5\}$ and $\beta_0(G) = 2$. Further $\{u_2, u_4\}$, $\{u_3, u_5\}$ are maximum independent sets of $G$. Clearly any non-maximal independent sets of $G$ is contained in $\beta_0$-set of $G$. Hence $G$ is weakly well covered graph. But $\{u_1\}$ is a maximal independent set of $G$ and which is not contained in $\beta_0$-set of $G$. That is $u_1 \notin S$. Therefore $G$ is not $\beta_0$-excellent.

**Proposition 5.2.17**

1. $K_n$ is well covered.

2. $\overline{K_n}$ is weakly well covered (also well covered).
3. \( C_n \) is not well covered. Moreover \( C_{2n} \) is not weakly well covered and 
\( C_{2n+1} \) is weakly well covered.

4. \( P_n \) is neither well covered nor weakly well covered.

5. \( W_n \) is not well covered but weakly well covered.

6. \( K_{1,n} \) is weakly well covered but not well covered.

7. \( D_{r,s} \) is neither well covered nor weakly well covered.

8. (i). \( K_{m,n} \) is well covered and weakly well covered if \( m = n \).

(ii). \( K_{m,n} \) is neither well covered nor weakly well covered if \( m \neq n \).

9. Let \( G \) be a non-complete graph with a full degree vertex and \( |V(G)| \geq 3 \). 
Then \( G \) is not well covered.

10. There exist a graph \( G \) with a full degree vertex which is not weakly well 
covered. For example, let \( G = F_n \), where \( n \) is even, \( (n \geq 4) \). Then \( G \) 
is not weakly well covered and \( G \) has a full degree vertex.

11. \( G^+ \) is well covered.

**Definition 5.2.18**

Let \( G \) be a weakly well covered graph. Let \( u \in V(G) \). \( u \) is said to be weakly 
well covered critical if \( G - u \) is not weakly well covered. Otherwise \( u \) is said 
to be non-critical.
Example 5.2.19

Let $G = W_7$

Here $\beta_0(G) = 3$. Also, $\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}$ are maximum independent sets of $G$. Clearly any non-maximal independent sets of $G$ is contained in $\beta_0$-set of $G$. Therefore $G$ is weakly well covered graph. Let $u = u_1$. Then $G - u_1$ is the fan $F_6$ which is shown below

Here $\beta_0(G - u_1) = 3$. Also, $\{u_2, u_4, u_6\}$ is the unique maximum independent sets of $G - u_1$. Clearly, $\{u_3\}, \{u_5\}$ are non-maximal independent sets of $G - u_1$ of cardinality 1, which is not contained in the unique maximum independent sets of $G - u_1$. Therefore $G - u_1$ is not weakly well covered. Therefore $u_1$ is a weakly well covered critical.

Example 5.2.20

$P_4$ is weakly well covered and all vertices are weakly well covered non-critical.
Example 5.2.21

$K_{1,n}$ is weakly well covered and every pendent vertex is non-critical.

Theorem 5.2.22

Let $G$ be a weakly well covered. Suppose $G - v$ is not weakly well covered for some $v \in V(G)$. Then $\beta_0(G - v) = \beta_0(G)$.

Proof:

Let $v \in V(G)$ be such that $G - v$ is not weakly well covered. Then there exists a non-maximal independent set say $D$ in $G - v$ such that $D$ is not contained in a maximum independent set of $G - v$. Clearly $D$ is a non-maximal independent set of $G$. Since $G$ is weakly well covered, there exists a maximum independent set say $S$ of $G$ such that $D \subset S$. If $v \notin S$, then $S$ is a maximum independent set of $G - v$ containing $D$, a contradiction. Therefore $v \in S$. $S - v$ is an independent set of $G - v$ containing $D$. Clearly, $S - v$ is not a maximum independent set of $G - v$. Therefore $|S - v| < \ldots$
\[ \beta_0(G - v) \leq \beta_0(G) = |S|. \] That is \(|S| - 1 < \beta_0(G - v) \leq \beta_0(G) = |S|\).

Therefore \(\beta_0(G - v) = \beta_0(G)\). \[\square\]

**Corollary 5.2.23**

If \(G\) is weakly well covered and \(\beta_0(G - v) \neq \beta_0(G)\) for some \(v \in V(G)\), then \(G - v\) is weakly well covered.

**Example 5.2.24**

Let \(G = W_7\). Let \(v\) be a vertex of \(G\) which is not the central vertex. \(G\) is weakly well covered. \(G - v\) is a fan \(F_6\) and \(G - v\) is not weakly well covered \(\beta_0(G - v) = 3 = \beta_0(G)\).

**Remark 5.2.25**

There exists graph \(G\) such that for some vertex \(v\) in \(V(G)\), \(G\) and \(G - v\) are weakly well covered and \(\beta_0(G - v) = \beta_0(G)\).

**Example:** Let \(G = P_4\). Let \(v\) be a pendent vertex of \(P_4\). Then \(G - v\) in \(P_3\).

Both \(P_4\) and \(P_3\) are weakly well covered and \(\beta_0(G - v) = 2 = \beta_0(G)\).

**Remark 5.2.26**

There exists graph \(G\) such that \(G\) is not weakly well covered and there exist vertices \(u_1, u_2\) such that \(G - u_i\) is weakly well covered \(1 \leq i \leq 2\) and \(\beta_0(G - u_1) = \beta_0(G), \beta_0(G - u_2) < \beta_0(G)\).

For example, let \(G\) :
Here $\beta_0(G) = 4$. Also $\{u_1, u_2, u_3, u_5\}, \{u_1, u_2, u_3, u_6\}$ are $\beta_0$-set of $G$.

$G - u_4$:

Here $\beta_0(G - u_4) = 4$. Also $\{u_1, u_2, u_3, u_5\}, \{u_1, u_2, u_3, u_6\}$ are $\beta_0$-set of $(G - u_4)$. Therefore $\beta_0(G - u_1) = 4 = \beta_0(G)$.

$G - u_2$:

Here $\beta_0(G - u_2) = 2$. Also $\{u_1, u_3\}, \{u_1, u_6\}, \{u_4, u_6\}, \{u_3, u_5\}, \{u_3, u_6\}$ are $\beta_0$-set of $(G - u_2)$. Therefore $\beta_0(G - u_2) < \beta_0(G)$.

**Observation 5.2.27** There are graphs satisfying the following property

1. $G$ is weakly well covered, $G - v$ is weakly well covered for some $v \in V(G)$.

Example: Let $G = P_4$. 
2. $G$ is weakly well covered, $G - v$ is not weakly well covered for some $v \in V(G)$.

Example: Let $G = P_6$ or $P_8$.

3. $G$ is not weakly well covered, $G - v$ is weakly well covered for some $v \in V(G)$.

Example: Let $G$:

![Graph](image)

4. $G$ is not weakly well covered, $G - v$ is not weakly well covered for some $v \in V(G)$.

Example: Let $G = D_{3,2}$.

**Theorem 5.2.28**

Let $G$ be a simple graph which is not well covered. Suppose $G$ is $k$-extendable for all $k \leq \beta_0(G) - 2$. Then $\beta_0(G) = i(G) + 1$.

**Proof:**

Let $G$ be extendable for all $k \leq i(G) - 2$. If $\beta_0(G) = i(G)$, then $G$ is
(β₀(G) − 1)-extendable.

For: Let S be an independent set of cardinality (i(G) − 1). S is not a maximal independent set, since |S| < i(G). Therefore S is contained in a maximal independent set of cardinality > |S|. Since β₀(G) = i(G), any super set of S which is independent is of cardinality i(G), S is contained in a maximum independent set. Therefore G is well covered, a contradiction. Hence β₀(G) > i(G). That is i(G) ≤ β₀(G) − 1. By hypothesis, G is k-extendable for all k ≤ (β₀(G) − 2). If i(G) ≤ (β₀(G) − 2), then G is i(G)-extendable, a contradiction, since i(G) < β₀(G). Therefore i(G) > (β₀(G) − 2). That is i(G) > (β₀(G) − 1). But i(G) ≤ (β₀(G) − 1). Therefore i(G) = β₀(G) − 1. Hence β₀(G) = i(G) + 1.

Example 5.2.29

Let G = C₉

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Let \( i(G) = 3 \) and \( \beta_0(G) = 4 \). \( \{u_1, u_4, u_7\} \) is the \( i(G) \)-set.

\( \{u_1, u_3, u_5, u_7\}, \{u_2, u_4, u_6, u_8\}, \{u_1, u_3, u_5, u_8\}, \{u_2, u_4, u_6, u_9\}, \{u_1, u_4, u_6, u_8\}, \{u_2, u_5, u_7, u_9\}, \{u_1, u_3, u_6, u_8\}, \{u_3, u_5, u_7, u_9\}, \{u_2, u_4, u_7, u_9\} \) are maximum independent sets. Clearly, \( G \) is 1-extendable and 2-extendable, but not 3-extendable, since \( \{u_1, u_4, u_7\} \) is not contained in any maximum independent sets of \( C_9 \). Therefore \( G \) is \( k \)-extendable for \( 1 \leq k \leq 2 \). That is \( G \) is \( k \)-extendable for \( 1 \leq k \leq (\beta_0(G) - 2) \). That is \( \beta_0(G) = i(G) + 1 \).

**Theorem 5.2.30**

*Let \( G \) be a graph which is not well covered. Let \( G \) be \( k \)-extendable for \( k = i(G) - r, r \geq 1 \). Then \( G \) is \( k \)-extendable for all \( k \leq (i(G) - r) \).*

**Proof:**

Let \( D \) be an independent set of \( G \) of cardinality \( k \leq (i(G) - r) \). Let \( S \) be a maximal independent set of \( G \) containing \( D \). Then \( |S| \geq i(G) \).

Hence \( |S - D| \geq i(G) - (i(G) - r) = r \). Let \( u_1, u_2, \ldots, u_t \) belong to \( S - D \)
such that $|S - \{u_1, u_2, \ldots, u_t\}| = i(G) - r$ where $t \geq r$. Therefore $S - \{u_1, u_2, \ldots, u_t\}$ is extendable. Therefore $D \subset S - \{u_1, u_2, \ldots, u_t\}$ is extendable. Hence the theorem.

**Corollary 5.2.31**

Let $G$ be a graph which is not well covered. Let $G$ be $k$-extendable for $k = (i(G) - 1)$. Then $G$ is $k$-extendable for all $k \leq (i(G) - 1)$.

**Theorem 5.2.32**

Let $G$ be a simple graph. $G$ is $k$-extendable for all $k$ except $k = (\beta_0(G) - 2)$ if and only if $G$ is $(\beta_0(G) - 1)$ and $(\beta_0(G) - 3)$ extendable and $i(G) = (\beta_0(G) - 2)$.

**Proof:**

Suppose $G$ is $k$-extendable for all $k$ except $k = (\beta_0(G) - 2)$. Clearly, $G$ is $k$-extendable for $k = (\beta_0(G) - 1)$ and $(\beta_0(G) - 3)$. Further $i(G) = (\beta_0(G) - 2)$, since $G$ is $k$-extendable for all $k$ except $(\beta_0(G) - 2)$.

Conversely, suppose $G$ is $(\beta_0(G) - 1)$ and $(\beta_0(G) - 3)$ extendable and $i(G) = (\beta_0(G) - 2)$. By theorem 5.2.30, $G$ is $k$-extendable for all $k \leq (\beta_0(G) - 3)$. Therefore $G$ is $k$-extendable for all $k$ except $k = (\beta_0(G) - 2)$.

**Example 5.2.33**

Let $G$ :
$i(G) = 2$ and $\beta_0(G) = 4$. $\{v_2, v_5\}$ is the unique $i(G)$-set.

$\{v_1, v_6, v_4, v_7\}$, $\{v_1, v_6, v_3, v_8\}$, $\{v_1, v_6, v_4, v_8\}$, $\{v_2, v_4, v_6, v_7\}$, $\{v_1, v_3, v_5, v_8\}$ are maximum independent sets of $G$. Clearly, $G$ is 1-extendable, 3-extendable, 4-extendable but not 2-extendable. That is $G$ is $k$-extendable for all $k$ except $k = (\beta_0(G) - 2)$ if and only if $G$ is $\beta_0(G) - 1)$ and $\beta_0(G) - 3)$ extendable and $i(G) = (\beta_0(G) - 2)$.

**Theorem 5.2.34**

Let $G$ be a simple graph. $G$ is $k$-extendable for all $k$ except $k = (\beta_0(G) - 3)$ if and only if $G$ is $\beta_0(G) - 1)$, $\beta_0(G) - 2)$ and $\beta_0(G) - 4)$ extendable and $i(G) = (\beta_0(G) - 3)$.

**Proof:**

Suppose $G$ is $k$-extendable for all $k$, except $k = (\beta_0(G) - 3)$. Clearly, $i(G) \leq \beta_0(G) - 3$ and as $G$ is extendable for all $k$ except $\beta_0(G) - 3)$, $i(G) = (\beta_0(G) - 3)$. Therefore $G$ is $(\beta_0(G) - 1)$, $(\beta_0(G) - 2)$ and $(\beta_0(G) - 4)$ extendable.

Conversely, let $i(G) = (\beta_0(G) - 3)$ and let $G$ be $(\beta_0(G) - 1)$, $(\beta_0(G) - 2)$ and $(\beta_0(G) - 4)$ extendable. Since $G$ is $(i(G) - 1)$-extendable, by corollary 5.2.31, $G$ is $k$-extendable for all $k \leq (i(G) - 1) = (\beta_0(G) - 4)$. By hypothesis, $G$ is $k$-extendable for $k = (\beta_0(G) - 1)$, $(\beta_0(G) - 2)$. Therefore $G$ is $k$-extendable for all $k$ except $(\beta_0(G) - 3)$.

**Example 5.2.35**

Let $G$:

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$S_1 = \{u_1, u_6, u_7, u_8, u_9\}$ and $S_2 = \{u_2, u_3, u_4, u_5, u_{10}\}$ are maximum independent sets of $G$ and $\beta_0(G) = 5$. $\{u_1, u_{10}\}$ is the unique $i(G)$-set in $G$ and $i(G) = 2$. Here $i(G) = \beta_0(G) - 3 = 2$. Also $\{u_1, u_{10}\}$ is an independent set of cardinality 2 which is not in any of the maximum independent sets of $G$.

$G$ is 1-extendable, 3-extendable and 4-extendable. That is $G$ is $k$-extendable for all $k$, except at $k = (\beta_0(G) - 3)$ if and only if $G$ is $(\beta_0(G) - 1)$, $(\beta_0(G) - 2)$ and $(\beta_0(G) - 4)$-extendable and $i(G) = \beta_0(G) - 3$.

### 5.3 Graph operations on Weakly well covered graphs

**Proposition 5.3.1**

Let $G$ and $H$ be weakly well covered. Then $G \cup H$ need not be weakly well covered.

For example, let $G_1 = C_{10}$, $G_2 = C_{12}$. Both $G_1$ and $G_2$ are weakly well covered. Let $V(G_1) = \{u_1, u_2, \ldots, u_{10}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{12}\}$. Let $S_1 = \{u_1, u_3, u_5, u_7, u_9\}$ and $S_2 = \{v_1, v_4, v_7, v_{10}\}$. $S_2$ is not contained in any maximum independent set in $G_2$. Therefore $S_1 \cup S_2$ is a non-maximal
independent set in $G_1 \cup G_2$, which is not contained in a maximum independent set. Therefore $C_{10} \cup C_{12}$ is not weakly well covered.

**Remark 5.3.2**

Let $G$ and $H$ be weakly well covered. Let $S$ be a non-maximal independent set in $G \cup H$. Let $S \cap V(G) = S_1$ and $S \cap V(H) = S_2$. Let $S_1$ and $S_2$ be non-maximal. Then $S_1 \cup S_2$ is contained in a maximum independent set of $G \cup H$.

**Proof:**

Let $S_1$ and $S_2$ be non-maximal. By hypothesis, there exist maximum independent sets $T_1$ and $T_2$ in $G$ and $H$ respectively such that $S_1 \subset T_1$ and $S_2 \subset T_2$. Clearly $T_1 \cup T_2$ is a maximum independent set in $G \cup H$ and $S_1 \cup S_2 = S \subset (T_1 \cup T_2)$. Therefore $G \cup H$ is weakly well covered.

**Remark 5.3.3**

Let $G$ be weakly well covered and $H$ be well covered. Let $S$ be a non-maximal independent set in $G \cup H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. Let $S_1$ be non-maximal. Then $S$ is contained in a maximum independent set of $G \cup H$.

**Proof:**

Let $S_1$ be non-maximal and $S_2$ be a maximal independent set. Since $G$ is weakly well covered, there exists a maximum independent set $T_1$ in $G$ such that $S_1 \subset T_1$. Since $H$ is well covered, $S_2$ is a maximum independent set of
Therefore $S \subset (T_1 \cup S_2)$ and $T_1 \cup S_2$ is a maximum independent set in $G \cup H$. Therefore $G \cup H$ is weakly well covered.

**Remark 5.3.4**

*If $G$ and $H$ are well covered, then $G \cup H$ is well covered and hence weakly well covered.*

**Theorem 5.3.5**

*Let $G$ and $H$ be weakly well covered graphs and $\beta_0(G) = \beta_0(H)$. Then $G + H$ is weakly well covered.*

**Proof:**

Let $S$ be a non-maximal independent set in $G + H$. Then $S$ is either completely contained in $V(G)$ (or) completely contained in $V(H)$. Since $S$ is non-maximal, either $S$ is a non-maximal independent set in $G$ (or) non-maximal independent set in $H$. Since $G$ and $H$ are weakly well covered and $\beta_0(G) = \beta_0(H)$. $S$ is contained in a maximum independent set of $G + H$. Therefore $G + H$ is weakly well covered.

**Example 5.3.6**

*Let $G = W_7$ and $H = P_6$. Let $V(G) = \{u_1, u_2, \ldots, u_7\}$ and $V(H) = \{v_1, v_2, \ldots, v_6\}$. Let $S = \{u_1, u_3\}$. Then $S$ is a non-maximal independent set in $G + H$. $S$ is contained in a maximum independent set namely $T = \{u_1, u_3, u_5\}$ in $G$. $T$ is also a maximum independent set in $G + H$.***
Remark 5.3.7

If $G$ and $H$ are weakly well covered and $\beta_0(G) \neq \beta_0(H)$, then $G + H$ need not be weakly well covered.

For example, let $G = W_7$ and $H = P_8$. $\beta_0(G) = 3$ and $\beta_0(H) = 4$. $\beta_0(G + H) = 4$. Any non-maximal independent set in $G$ is contained in a maximum independent set of $G$ of cardinality 3. But $\beta_0(G + H) = 4$. Therefore $G + H$ is not weakly well covered.

5.4 1-$k$-extendable graphs

Definition 5.4.1

A graph $G$ is 1-$k$-extendable if for every $u \in V(G)$, $\beta_0(G - u) = \beta_0(G)$ and $G - u$ is $k$-extendable.

Example 5.4.2

Let $G$ be a component wise complete graph. Let $u \in V(G)$. Let $G = G_1 \cup G_2 \cup$
... ∪ \( G_r \), where each \( G_i \) is complete and \(|V(G_i)| \geq 2\) for every \( i, 1 \leq i \leq r \).
\( \beta_0(G) = r \). \( \beta_0(G - u) = r \) for any \( u \) in \( V(G) \). \( G \) is 1-extendable. Therefore \( G \) is 1-1-extendable. \( G \) is also \( k \)-extendable, \( 1 \leq k \leq (r - 1) \). Therefore \( G \) is \( 1 - k \)-extendable for \( 1 \leq k \leq (\beta_0(G) - 1) \).

**Example 5.4.3**

Let \( G = C_n \). Then \( \beta_0(G - u) = \beta_0(G) \) for every \( u \) in \( V(G) \). If \( n \) is odd, \( P_{n-1} \) is 1-extendable. Therefore \( C_n \) is 1-1-extendable.

**Definition 5.4.4**

Let \( n, k \) be the positive integers. Let \( S \) be a set of \( n \) elements. Let \( T_1, T_2, \ldots, T_{nk} \) be the set of all subsets of \( S \) of cardinality \( k \). Let \( G \) be the graph whose vertices are \( T_1, T_2, \ldots, T_{nk} \). Two vertices \( T_i, T_j \) are adjacent in \( G \) if \( T_i \cap T_j = \emptyset \), \( 1 \leq i, j \leq nk, i \neq j \). \( G \) is called a kneser graph denoted by \( K(n,k) \).

**Remark 5.4.5**

Kneser graphs are regular. The degree of any vertex of \( K(n,k) \) is \((n-k)_{C_k}\).

**Remark 5.4.6**

When \( n \) is \( 2k + 1 \) the kneser graph is called an odd graph.

**Remark 5.4.7**

\( K(n,2) \) is the compliment of the line graph of \( K_n \). That is \( K(n,k) = \overline{L(K_n)} \).

**Proposition 5.4.8**

When \( n = 2 \), \( K(n,2) = K_1 \).
When $n = 3$, $K(n, 2) = \overline{K_3}$.

When $n = 4$, $K(n, 2) = 3K_2$.

**Proposition 5.4.9**

When $n \geq 5$, $\beta_0(K(n, 2)) = (n - 1)$.

**Proposition 5.4.10**

1. When $n \geq 5$, $K(n, 2)$ is neither well covered nor weakly well covered.

2. When $n = 2$, $K(n, 2)$ is well covered.

3. When $n = 3$, $K(n, 2)$ is well covered.

4. When $n = 4$, $K(n, 2)$ is well covered and weakly well covered.

**5.5 An application**

Vulnerability of online transactions over the internet can be studied by making use of extensibility of independent sets in graphs.

In crucial online transactions over the internet, there occurs a most important type of attack by the intruders to find out users crucial information from the stored database. The attackers normally modify the queries passed to the database into a malformed query and retrieve the crucial information from the server database without the knowledge of the service requester and the service provider. For instance, during an online banking transaction, the user-id and password can be taken by the intruders and insert some attack
keywords into them to fetch all the details from the bank database.

To avoid these kinds of attacks, the industries are currently applying vulnerability analysis to find out such malformed queries and they tried to stop their execution. This identification can be done by formal manual analysis by going through each and every query and find out the possible attacks. But when we are trying to do it by means of textual analysis of each and every program statement and finding out the potential attack-prone queries, the task will be too time consuming and will be error-prone as we may not know the maximum possible attack types. Hence, this research work proposes to apply a graph based approach to represent the query statements and then evaluate the grammar of the queries to find any mismatch in syntax of queries.

The query grammar graph is constructed as follows. The keywords in the query statement are represented by the nodes. Further each node has specific attributes which are the malicious inputs for each query. Two nodes are joined if the queries do not have connection. The maximum possible extensibility level of an attack prone node is to be found through the study of extensibility of independent nodes. Thus the study of extensibility helps to devise an effective vulnerability analysis.
List of Publications


2. K. Angaleeswari, P. Sumathi and V. Swaminathan \textit{k-extendable graphs and Weakly k-extendable graphs}, Pre print.

3. K. Angaleeswari, P. Sumathi and V. Swaminathan \textit{k-extensibility in Graphs}, (Communicated).
