CHAPTER – VI
BATCH ARRIVAL QUEUEING SYSTEM UNDER WORKING VACATION POLICY

INTRODUCTION

Vacation queueing models had been the subject of interest in recent years because of their applicability and theoretical structures in real life congestion situations. In classical vacation queueing models (discussed in earlier chapters), the server completely stops the primary service and utilizes the vacation time to perform other supplementary tasks. Servi and Finn (2002) introduced a class of semi-vacation policy under which, the server works at a lower rate rather than completely stopping service during vacation. Such a vacation policy is called Working Vacation (WV) Policy.

In working vacation queues, a part of the service ability keeps the system operating at a lower speed during vacations. If the service speed degenerates to zero in a working vacation, then the working vacation queues become classical vacation queueing models. Therefore, the working vacation queue is the generalization of the classical vacation queue and the analysis of this kind is more complicated than the previous works.

The M/M/1 queueing system with working vacations has been analysed by Servi and Finn (2002), Liu et al. (2007), Zhang and Xu (2008) and Tian et al. (2008b). Later, Kim et al. (2003), Wu and Takagi (2006) and Li et al. (2009) generalized the model of Servi and Finn (2002) to an M/G/1 queue with general or exponential working vacations. Baba (2005) and Banik et al. (2007) studied GI/M/1 queue with working vacations for an infinite and finite queue respectively by using the matrix analytic method. The available research on working vacation mainly concentrated on single arrival and single service queueing systems.

Recently Xu et al. (2009a) and Julia Rose Mary and Afthab Begum (2010) analysed the $M^X / M / 1$ queue under working vacation and
studied the PGF of the stationary system size and the stochastic decomposition property of the model.

The aim of the present chapter is to analyse the non-Markovian batch arrival $M^X / G / 1$ queueing system under multiple and single working vacation using the supplementary variable technique and deduce the results of Li et al. (2009) for $M / G / 1$ queue and the results of Xu et al. (2009 a) and Julia Rose Mary and Afthab Begum (2010) for $M^X / M / 1$ queueing system with working vacation. It is found that, deriving the PGF of the system size at arbitrary epochs and at departure epochs using supplementary variable technique is much simpler than that of the other methods analysed in the literature.

SECTION 6.1

$M^X / G / 1$ QUEUEING SYSTEM WITH MULTIPLE WORKING VACATIONS (MWV)

6.1.1 Mathematical Analysis of MWV

6.1.1.1 Model Description

Arrival Pattern

Consider a batch arrival $M^X / G / 1$ queue with multiple vacations and exhaustive service such that the server works with different service rates rather than completely stops service during a vacation period. The arrival stream forms a Poisson process with group arrival rate $\lambda$ and the actual number of customers in any arriving module is a random variable $X$, which may take on any positive integral value $k (\leq \infty)$ with probability distribution $Pr(X = k) = g_k$.

Regular Busy Period

The server serves the customers one at a time and the normal service time $S_b$ during the regular busy period follows a general distribution. The normal service continues until the system becomes empty.

Working Vacation Period

When the system becomes empty at a service completion instant, the server starts a working vacation. The vacation time is assumed to follow an
exponential distribution with parameter $\eta$. During working vacation, an arriving customer is served at a lower rate and the service time $S_v$ follows a different general distribution.

The general distributions of service times during regular busy period and working vacation period are respectively denoted by $S_b(x) = \Pr(S_b < x)$ and $S_v(x) = \Pr(S_v < x)$. The LST and $k^{th}$ moment about the origin of the distributions are given by

$$S_b^*(\theta) = \int_0^\infty e^{-\theta t} dS_b(t) \quad S_v^*(\theta) = \int_0^\infty e^{-\theta t} dS_v(t)$$

$$E(S_b^k) = \int_0^\infty t^k dS_b(t) \quad E(S_v^k) = \int_0^\infty t^k dS_v(t), \quad k = 1, 2$$

At a vacation completion instant, if there are customers in the system, then the server will start a new regular busy period. Otherwise the server takes another working vacation. The service interrupted at the end of a vacation is lost and it is restarted with normal service rate $E(S_b)$. The interarrival times, service times and working vacation times are mutually independent. The service discipline is assumed to follow FCFS. This model is denoted by $M^X / G / 1 / MWV$.

The steady-state system size equations under the steady-state condition are analysed using supplementary variable technique. The remaining service times are introduced as supplementary variables and the following notations are used to discuss the model.

$\lambda$ : Group arrival rate
$X$ : Group size random variable
$g_k$ : $\Pr(X = k), k = 1, 2, 3, \ldots$
$X(z)$ : Probability Generating Function of $X$
$N(t)$ : The system size at time $t$.

Let $S_v^o(t)$ and $S_b^o(t)$ denote the remaining service time during working vacation period and regular busy period respectively at time $t$. 
Further the server states are denoted by $Y(t)$ at time $t$.

Let

$$Y(t) = \begin{cases} 
0, & \text{if the server is in idle state in vacation} \\
1, & \text{if the server is in working vacation state} \\
2, & \text{if the server is in regular busy state}
\end{cases}$$

At an arbitrary time $t$, the state of the system can be described by the Markov process $\{(N(t), Y(t), S^0_b(t), S^0_v(t)), t \geq 0\}$.

Now the system state probabilities at time $t$ are defined as follows:

Let

$$Q_0(t) = \Pr(N(t) = 0, Y(t) = 0)$$

$$Q_n(x, t) = \Pr(N(t) = n, x \leq S^0_v(t) \leq x + dt, Y(t) = 1), \quad n \geq 1$$

$$P_n(x, t) = \Pr(N(t) = n, x \leq S^0_b(t) \leq x + dt, Y(t) = 2), \quad n \geq 1$$

Thus, $Q_n(x, t)$ ($P_n(x, t)$) denotes the probability that there are $n$ ($\geq 1$) customers in the system at arbitrary epoch with the remaining working vacation time (regular service time) lies between $x$ and $x + \Delta t$ and $Q_0(t)$ denotes the probability that the server is idle in vacation at time $t$.

Further, $Q_n(0)$ ($P_n(0)$) represents the probability that there are $n$ customers in the system at the departure epoch during working vacation period (service completion instant during regular service).

### 6.1.1.2 The System Size Distribution

Assuming the steady-state probabilities $Q_n(x) = \lim_{t \to \infty} Q_n(x, t)$ and $P_n(x) = \lim_{t \to \infty} P_n(x, t)$ exist and independent of time $t$, the following steady-state equations are obtained for the queueing system using supplementary variable technique.

**Idle State**

$$\lambda \cdot Q_0 = P_1(0) + Q_1(0)$$

**Working Vacation State**

$$-\frac{d}{dx} Q_1(x) = -(\lambda + \eta) Q_1(x) + Q_2(0) s_v(x) + \lambda Q_0 g_1 s_v(x)$$
\[-\frac{d}{dx} Q_n(x) = - (\lambda + \eta) Q_n(x) + Q_{n+1}(0) s_v(x) + \lambda Q_0 g_n s_v(x) + \lambda \sum_{k=1}^{n-1} Q_{n-k}(x) g_k, \quad n \geq 2\]

**Regular Busy State**

\[-\frac{d}{dx} P_1(x) = - \lambda P_1(x) + P_2(0) s_b(x) + \int_0^\infty Q_1(y) dy \eta s_b(x)\]

\[-\frac{d}{dx} P_n(x) = - \lambda P_n(x) + P_{n+1}(0) s_b(x) + \int_0^\infty Q_n(y) dy \eta s_b(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k, \quad n \geq 2\]

The LST of the above equations are obtained as

\[\lambda Q_0 = P_1(0) + Q_1(0) \quad (6.1.1)\]

\[\theta Q_1^*(\theta) - Q_1(0) = (\lambda + \eta) Q_1^*(\theta) - Q_2(0) S_v^*(\theta) - \lambda Q_0 g_1 S_v^*(\theta) \quad (6.1.2)\]

\[\theta Q_n^*(\theta) - Q_n(0) = (\lambda + \eta) Q_n^*(\theta) - Q_{n+1}(0) S_v^*(\theta) - \lambda Q_0 g_n S_v^*(\theta) - \lambda \sum_{k=1}^{n-1} Q_{n-k}^*(\theta) g_k, \quad n \geq 2 \quad (6.1.3)\]

\[\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0) S_b^*(\theta) - S_b^*(\theta) \int_0^\infty Q_1(y) dy \eta \quad (6.1.4)\]

\[\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0) S_b^*(\theta) - S_b^*(\theta) \int_0^\infty Q_n(y) dy \eta\]

\[-\lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k, \quad n \geq 2 \quad (6.1.5)\]

where \(P_n^*(\theta) = \int_0^\infty e^{-\theta t} P_n(t) dt\) and \(Q_n^*(\theta) = \int_0^\infty e^{-\theta t} Q_n(t) dt\)

**Steady-State Solutions**

The following probability generating functions are defined for \(|z| \leq 1\) to derive the system size distribution of the model.

\[Q^*(z, \theta) = \sum_{n=1}^{\infty} Q_n^*(\theta) z^n, \quad Q(z, 0) = \sum_{n=1}^{\infty} Q_n(0) z^n\]
\[ P^*(z, \theta) = \sum_{n=1}^{\infty} P_n^*(0) z^n, \quad P(z, 0) = \sum_{n=1}^{\infty} P_n(0) z^n \]

Multiplying the equations (6.1.2) and (6.1.3) by \( z^n \), \( n \geq 1 \) and summing up over \( n = 1 \) to \( \infty \), it is found after some manipulation that,

\[(\theta - h_X(z)) Q^*(z, \theta) = Q(z, 0) \left( \frac{z - S^*_v(\theta)}{z} \right) - S^*_v(\theta) \ (\lambda X(z) Q_0 - Q_1(0)) \] (6.1.6)

At \( \theta = h_X(z) = \eta + \lambda (1 - X(z)) \)

\[ Q(z, 0) \left( \frac{z - S^*_v(h_X(z))}{z} \right) = S^*_v(h_X(z)) \ (\lambda X(z) Q_0 - Q_1(0)) \] (6.1.7)

Since \( \phi(z) = S^*_v(h_X(z)) \) satisfies the following conditions:

\[ 0 < \phi(0) = S^*_v(\eta + \lambda) < \phi(1) = S^*_v(\eta) < 1 \]

and for \( 0 < z < 1 \),

\[ \phi'(z) = \lambda X'(z) \int_0^{\infty} t e^{-(\theta + \lambda (1 - X(z)))t} d S_v(t) > 0 \]

\[ \phi''(z) = \lambda^2 X'(z) \int_0^{\infty} t^2 e^{-(\theta + \lambda (1 - X(z)))t} d S_v(t) > 0 \]

The equation \( z = \phi(z) \) has a unique root \( z_1 \) in the range \( 0 < z < 1 \) [Li et al. (2009)]. Hence the equations (6.1.7) and (6.1.1) at \( z = z_1 \) imply,

\[ Q_1(0) = \lambda X(z_1) Q_0 \] (6.1.8)

\[ P_1(0) = \lambda Q_0 (1 - X(z_1)) \] (6.1.9)

Substituting the value of \( Q_1(0) \) in equation (6.1.7),

\[ Q(z, 0) = \frac{z S^*_v(h_X(z)) \lambda Q_0 (X(z) - X(z_1))}{z - S^*_v(h_X(z))} \] (6.1.10)

Thus equation (6.1.6) can be simplified as,

\[ Q^*(z, \theta) = \frac{z \lambda Q_0 (X(z) - X(z_1)) (S^*_v(h_X(z)) - S^*_v(\theta))}{(z - S^*_v(h_X(z))) (\theta - h_X(z))} \]
At $\theta = 0$,
\[
Q^*(z, 0) = \frac{\lambda z Q_0(X(z) - X(z_1)) (1 - S^*_v(h_X(z)))}{h_X(z) (z - S^*_v(h_X(z)))} \tag{6.1.11}
\]

To obtain the partial PGFs $P^*(z, \theta)$ and $P(z, 0)$, equations (6.1.4) and (6.1.5) are used.

Multiplying equations (6.1.4) and (6.1.5) by suitable powers of $z$ and summing over $n = 1$ to $\infty$, we find
\[
(\theta - w_X(z)) P^*(z, \theta) = P(z, 0) - \frac{S^*_b(\theta)}{z} (P(z, 0) - P_1(0) z)
- \eta S^*_b(\theta) \sum_{n=1}^{\infty} \left( \int_0^\infty Q_n(y) dy \right) z^n \tag{6.1.12}
\]

Since,
\[
\sum_{n=1}^{\infty} z^n \left( \int_0^\infty Q_n(y) dy \right) = \sum_{n=1}^{\infty} z^n \left( \int_0^\infty e^{-\theta y} Q_n(y) dy \right) \bigg|_{\theta=0}
= \sum_{n=1}^{\infty} z^n Q^*_n(0) = Q^*(z, 0) \tag{6.1.13}
\]
equation (6.1.12) can be rewritten as
\[
(\theta - w_X(z)) P^*(z, \theta) = P(z, 0) \left( \frac{z - S^*_b(\theta)}{z} \right) - S^*_b(\theta) (\eta Q^*(z, 0) - P_1(0)) \tag{6.1.14}
\]

At $\theta = w_X(z)$,
\[
P(z, 0) = \frac{z S^*_b(w_X(z))}{z - S^*_b(w_X(z))}(\eta Q^*(z, 0) - P_1(0)) \tag{6.1.15}
\]

Substituting for $P(z, 0)$ in equation (6.1.14), we have
\[
P^*(z, \theta) = \frac{z (S^*_b(w_X(z)) - S^*_b(\theta)) (\eta Q^*(z, 0) - P_1(0))}{(\theta - w_X(z)) (z - S^*_b(w_X(z)))} \tag{6.1.16}
\]

Using the equations (6.1.9) and (6.1.11) in (6.1.15) and (6.1.16), the expressions for $P(z, 0)$ and $P^*(z, \theta)$ are given by
\[ P(z, 0) = \frac{\lambda Q_0 z S_b^*(w_X(z))}{(z - S_b^*(w_X(z)))} \left[ \frac{\eta z (X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - (1 - X(z_1)) \right] \]  

(6.1.17)

\[ P^*(z, 0) = \frac{\lambda Q_0 z (S_b^*(w_X(z)) - S_b^*(0))}{(\theta - w_X(z))(z - S_b^*(w_X(z)))} \left[ \frac{\eta z (X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - (1 - X(z_1)) \right] \]  

(6.1.18)

Therefore, the total PGF \( P_{MWV}(z) \) of the system size probabilities of the model is given by

\[ P_{MWV}(z) = P^*(z, 0) + Q^*(z, 0) + Q_0 \]

\[ = \frac{(z - 1) Q_0 S_b^*(w_X(z))}{(z - S_b^*(w_X(z)))} \psi_X(z) \], where

(6.1.19)

\[ \psi_X(z) = 1 + \frac{\lambda z(X(z) - X(z_1))}{z - S_v^*(h_X(z))} \left[ \frac{1 - S_v^*(h_X(z))}{h_X(z)} - \frac{S_v^*(h_X(z))(1 - S_b^*(w_X(z)))}{w_X(z) S_b^*(w_X(z))} \right] \]

(6.1.20)

Using the normalizing condition \( P_{MWV}(1) = 1 \), it is found that

\[ Q_0 = \frac{(1 - \rho_b)}{d_{MWV}} \]

(6.1.21)

where \( d_{MWV} = \psi_X(1) = \frac{h_X(z_1)}{\eta} - \frac{\rho_b(1 - X(z_1)) S_v^*(\eta)}{E(X)(1 - S_v^*(\eta))} \)

(6.1.22)

Thus the PGF \( P_{MWV}(z) \) can be written as

\[ P_{MWV}(z) = \frac{(1 - \rho_b)(z - 1) S_b^*(w_X(z))}{(z - S_b^*(w_X(z)))} \frac{\psi_X(z)}{\psi_X(1)} \]

(6.1.23)

\[ = P_{M^X/G/1}(z) \frac{\psi_X(z)}{\psi_X(1)} \]

And this confirms the Stochastic decomposition property.

### 6.1.2 Decomposition Property

If \( \rho_b = \lambda E(X) E(S_b) < 1 \) and \( \mu_v < \mu_b \), then PGF of the system size of \( M^X / G / 1 \) multiple working vacation model is decomposed into the product of the PGF of two random variables of which one is the system size PGF of the
classical $M^X / G / 1$ queueing model $P_{M^X/G^1}(z)$ and the other is the PGF $(\psi_X(z)/\psi_X(1))$ of the additional queue length.

### 6.1.3 Performance Measures

In this section, the steady-state system size probabilities at various states are derived.

(i) The probability that the server is on vacation ($P_v$) is given by

$$P_v = \lim_{z \to 1} (Q^*(z, 0) + Q_0) = \frac{h_x(z)}{\eta} Q_0$$

(ii) The probability that the server is on regular busy period ($P_{busy}$) is given by

$$P_{busy} = \lim_{z \to 1} P^*(z, 0) = \frac{Q_0 \rho_b}{1 - \rho_b} \left[ \frac{h_x(z)}{\eta} - \frac{(1 - X(z)) S_v^*(\eta)}{E(X)(1 - S_v^*(\eta))} \right]$$

(iii) The expected system size when the server is on vacation ($L_v$) is

$$L_v = \left[ \frac{d}{dz} (Q^*(z, 0) + Q_0) \right]_{z=1} = \lambda Q_0 \left[ \frac{E(X) h_x(z)}{\eta} - \frac{(1 - X(z)) S_v^*(\eta)}{\eta(1 - S_v^*(\eta))} \right]$$

(iv) The expected system size of the model is given by

$$L_{MWV} = \frac{d}{dz} (P_{MWV}(z))_{z=1}$$

$$= L_{M^X/G^1} + \left( \frac{\lambda E(X)}{\eta} - \frac{S_v^*(\eta)}{1 - S_v^*(\eta)} \right) + \frac{\rho_b w_x(z) S_v^{*(1)}(\eta)}{d_{MWV} (1 - S_v^*(\eta))^2}$$

$$+ \frac{S_v^*(\eta)}{d_{MWV} (1 - S_v^*(\eta))} [1 - \rho_b + w_x(z) (\frac{\rho_b}{\eta} + \lambda E(x) (E(S_b^2) - \frac{E(S_b^2)}{2}))]$$

(6.1.24)

where $L_{M^X/G^1} = \rho_b + \frac{(\lambda E(X))^2 E(S_b^2) + \lambda E(X(X-1)) E(S_b)}{2}$ denotes the mean system size for the classical $M^X / G / 1$ queueing model, $S_v^{*(1)}(\eta) = \frac{d}{dz} (S_v^*(z))_{z=\eta}$ and $d_{MWV}$ is given by equation (6.1.22).
6.1.4 Queue Size Distribution at Departure Epoch

By following the argument of Lee et al. (1995) that a departing customer will see \( j \) customers in the system just after the departure if and only if there were \( (j + 1) \) customers in the system just before the departure, the probability that there are \( j \) customers in the system at departure epoch \((\Pi^+_j)\) is given by

\[
\Pi^+_j = D(Q_{j+1}(0) + P_{j+1}(0)), \quad j \geq 0
\]

where \( D \) is the normalizing constant.

Let \( \Pi^+(z) \) be the PGF of \( \{\Pi^+_j, j \geq 0\} \).

Then \( \Pi^+(z) = \sum_{j=0}^{\infty} \Pi^+_j z^j = \frac{D}{z} (Q(z, 0) + P(z, 0)) \)

Substituting for \( Q(z, 0) \) and \( P(z, 0) \) from equations (6.1.10) and (6.1.17) and evaluating \( D \) using the normalizing condition, it is found that

\[
\Pi^+(z) = \frac{(1-X(z))}{E(X)(1-z)} P_{MWV}(z)
\]

(6.1.25)

And this gives the relation between the PGF of the steady state system size probabilities at arbitrary epochs and departure epochs for \( M^X / G / 1 / MWV \) model.

SECTION 6.2

\( M^X / G / 1 \) QUEUEING SYSTEM WITH SINGLE WORKING VACATION (SWV)

6.2.1 Mathematical Analysis of SWV

6.2.1.1 Model Description

In this section, we consider \( M^X / G / 1 \) queue with single working vacation. As in section 6.1, customers arrive in batches with group arrival rate \( \lambda \) and the regular service time follows an arbitrary distribution \( S_{ub}(t) \).

The server begins a working vacation of random length \( V \) at the instant when the system becomes empty and the vacation duration follows an exponential distribution with parameter \( \eta \). During a WV the customers are
served at a lower service rate $E(S_v)$ and the service time follows an arbitrary distribution $S_v(t)$.

When the vacation ends, if there are customers in the queue, the server changes service rate from $E(S_v)$ to $E(S_b)$ and a regular busy period starts. Otherwise, the server enters an idle period and a new regular busy period starts when a batch of customers arrive. This model is denoted by $M^X / G / 1 / SWV$.

### 6.2.1.2 The System Size Distribution

To derive the steady-state system size equations, the following notations and probabilities are defined.

The notations except at idle state are the same as before.

Let $N(t)$ denote the system size including the one in service at time $t$.

$$ Y(t) = \begin{cases} 
0, & \text{if the server is idle in vacation at time } t, \\
1, & \text{if the server is idle in the system at time } t, \\
2, & \text{if the server is busy in vacation with lower service rate at time } t, \\
3, & \text{if the server is busy with regular service rate at time } t 
\end{cases} $$

$$ Q_0(t) = Pr(N(t) = 0, Y(t) = 0) $$

$$ P_0(t) = Pr(N(t) = 0, Y(t) = 1) $$

$$ Q_n(x, t) \, dt = Pr(N(t) = n, x \leq S_v^v(t) \leq x + dt, Y(t) = 2), \quad n \geq 1 $$

$$ P_n(x, t) \, dt = Pr(N(t) = n, x \leq S_b^b(t) \leq x + dt, Y(t) = 3), \quad n \geq 1 $$

Thus $P_0(t)$ gives the probability that the server is idle in the system at time $t$ and the interpretation of the other probabilities are as in section 6.1. The steady-state equations for this queueing system using supplementary variable technique by introducing remaining service times as supplementary variables are:

**Idle State**

$$ \lambda \cdot P_0 = \eta \cdot Q_0 $$

$$ (\lambda + \eta) \cdot Q_0 = P_1(0) + Q_1(0) $$
Working Vacation State

\[-\frac{d}{dx} Q_1(x) = - (\lambda + \eta) Q_1(x) + Q_2(0) s_v(x) + \lambda g_1 Q_0 s_v(x)\]

\[-\frac{d}{dx} Q_n(x) = - (\lambda + \eta) Q_n(x) + Q_{n+1}(0) s_v(x) + \lambda g_n Q_0 s_v(x) + \lambda \sum_{k=1}^{n-1} Q_{n-k}(x) g_k, \quad n \geq 2\]

Regular Busy State

\[-\frac{d}{dx} P_1(x) = - \lambda P_1(x) + P_2(0) s_b(x) + \lambda g_1 P_0 s_b(x) + \int_0^\infty Q_1(y) dy \eta s_b(x)\]

\[-\frac{d}{dx} P_n(x) = - \lambda P_n(x) + P_{n+1}(0) s_b(x) + \lambda g_n P_0 s_b(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k, \quad n \geq 2\]

The LST of the above equations are given by

\[\lambda P_0 = \eta Q_0\]

(6.2.1)

\[(\lambda + \eta) Q_0 = P_1(0) + Q_1(0)\]

(6.2.2)

\[\theta Q_1^*(\theta) - Q_1(0) = (\lambda + \eta) Q_1^*(\theta) - Q_2(0) S_v^*(\theta) - \lambda g_1 Q_0 S_v^*(\theta)\]

(6.2.3)

\[\theta Q_n^*(\theta) - Q_n(0) = (\lambda + \eta) Q_n^*(\theta) - Q_{n+1}(0) S_v^*(\theta) - \lambda g_n Q_0 S_v^*(\theta)
\]

\[-\lambda \sum_{k=1}^{n-1} Q_{n-k}^*(\theta) g_k, \quad n \geq 2\]

(6.2.4)

\[\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0) S_b^*(\theta) - \lambda g_1 P_0 S_b^*(\theta)
\]

\[-\int_0^\infty Q_1(y) dy \eta S_b^*(\theta)\]

(6.2.5)

\[\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0) S_b^*(\theta) - \lambda g_n P_0 S_b^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k
\]

\[-\int_0^\infty Q_n(y) dy \eta S_b^*(\theta), \quad n \geq 2\]

(6.2.6)
Steady-State Solutions

In order to derive the distribution of the system size probabilities, the following probability generating functions are derived.

\[
Q^*(z, \theta) = \sum_{n=1}^{\infty} Q_n^*(\theta) z^n, \quad Q(z, 0) = \sum_{n=1}^{\infty} Q_n(0) z^n
\]

\[
P^*(z, \theta) = \sum_{n=1}^{\infty} P_n^*(\theta) z^n, \quad P(z, 0) = \sum_{n=1}^{\infty} P_n(0) z^n
\]

The steady-state equations corresponding to the working vacation state are exactly the same as in section 6.1. Thus the PGFs at departure epoch and at arbitrary epoch, when the server is busy on vacation are given by

\[
Q(z, 0) = \frac{\lambda z Q_0(X(z)-X(z_1)) S_v^*(h_X(z))}{z - S_v^*(h_X(z))}
\]

\[
Q^*(z, 0) = \frac{\lambda z Q_0(X(z)-X(z_1))(1-S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))}
\]

The PGF of the system at the service termination epochs and at arbitrary epochs, when the server is in regular busy state are obtained using equations (6.2.5) and (6.2.6) as,

\[
P(z, 0) = \frac{z S_b^*(w_X(z)) Q_0}{z - S_b^*(w_X(z))} \left[ \frac{\eta \lambda z (X(z)-X(z_1))(1-S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - \eta(1-X(z)) - \lambda(1-X(z_1)) \right]
\]

\[
P^*(z, 0) = \frac{z Q_0(1-S_b^*(w_X(z)))}{w_X(z)(z - S_b^*(w_X(z)))} \left[ \frac{\eta \lambda z (X(z)-X(z_1))(1-S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - \eta(1-X(z)) - \lambda(1-X(z_1)) \right]
\]

Thus the total PGF \( P_{SWV}^S(z) \) for the SWV is given by

\[
P_{SWV}(z) = P^*(z, 0) + Q^*(z, 0) + Q_0 + P_0 = \frac{(z-1)Q_0 S_b^*(w_X(z)) \psi_X^S(z)}{z - S_b^*(w_X(z))}
\]
where $\psi^S_X(z) = \psi_X(z) + \frac{\eta}{\lambda} \psi_X(z)$ as in equation (6.1.20)

and $\psi^S_X(1) = d_{SWV} = \left[ \frac{h_X(z_1)}{\eta} + \frac{\eta}{\lambda} - \frac{\rho_b (1 - X(z_1)) S^*_V(\eta)}{E(X) (1 - S^*_V(\eta))} \right]$ (6.2.11a)

Using the normalizing condition $P_{SMV}(1) = 1$, $Q_0$ is obtained as $Q_0 = \frac{1 - \rho_b}{\psi^S_X(1)}$.

Substituting for $Q_0$ in equation (6.2.11),

$$P_{SMW}(z) = \left( \frac{\rho_b (z - 1) S^*_b(w_X(z))}{z - S^*_b(w_X(z))} \right) \left( \psi^S_X(z) \right) \psi^S_X(1)$$

$$= P_{M^X/G/1}(z) \frac{\psi^S_X(z)}{\psi^S_X(1)}$$ (6.2.12)

### 6.2.2 Decomposition Property

Thus the equation (6.2.12), confirms the stochastic decomposition property that the PGF of the $M^X / G / 1$ single working vacation model is decomposed into the product of PGF of two random variables one is the PGF of the classical $M^X / G / 1$ queueing model (without vacation) and the other is a PGF of the additional queue length.

### 6.2.3 Performance Measures

(i) The probability that the server is on vacation state ($P_v$) is

$$P_v = \lim_{z \to 1} (Q^*(z, 0) + Q_0) = \frac{h_X(z_1)}{\eta} Q_0$$

(ii) The probability that the server is on regular busy state ($P_{busy}$) is

$$P_{busy} = \lim_{z \to 1} P^*(z, 0) = \frac{Q_0 \rho_b}{1 - \rho_b} \left[ \frac{h_X(z_1)}{\eta} - \frac{(1 - X(z_1)) S^*_V(\eta)}{E(X)} \right]$$

(iii) The probability that the server is in idle state ($P_i$) is given by

$$P_i = P_0 = \frac{\eta Q_0}{\lambda}$$

(iv) The expected system size when the server is on vacation state ($L_v$) is

$$L_v = \left[ \frac{d}{dz} (Q^*(z, 0) + Q_0) \right]_{z=1}$$
The expected system size of the model is given by

\[ L_{SWV} = \frac{d}{dz} \bigg|_{z=1} (P_{SWV}(z)) \]

\[ = L_{M^X/G/1} + \left( \frac{\lambda E(X)}{\eta} - \frac{S_V^*(\eta)}{1 - S_V^*(\eta)} \right) + \frac{\rho_b w_x(z_1) S_V^{(1)}(\eta)}{d_{SWV} (1 - S_V^*(\eta))^2} - \frac{E(X)}{d_{SWV}} \]

\[ + \frac{S_V^*(\eta)}{d_{SWV} (1 - S_V^*(\eta))} \left[ 1 - \rho_b + \frac{\eta}{\lambda} + w_x(z_1) \left( \frac{\rho_b}{\eta} + \lambda E(X) \left( (E(S_b))^2 - \frac{E(S_b^2)}{2} \right) \right) \right] \]

where \( d_{SWV} \) is given by equation (6.2.11a).

6.2.4 Queue Size Distribution at Departure Epoch

If \( \Pi_j^+ \) denotes the probability that there are \( j \) customers in the system at departure epoch, then its PGF \( \Pi^+(z) \) for the single working vacation model is given by \( \Pi^+(z) = (1 - X(z)) \frac{1}{E(X)(1 - z)} P_{SWV}(z) \).

6.3 PARTICULAR CASES

In this section, the steady-state results of some vacation queueing models published previously in literature are derived as special cases.

I. The \( M^X/G/1 \) queue with classical vacations

If the server does not serve the customers during the vacation period, then \( S_V^*(h_X(z)) \equiv 0 \) and hence \( z_1 = 0 \). Thus the PGF (P(z)) and the expected system size (L) for the classical vacation models are given by:

(a) For Multiple Vacation

\[ P(z) = P_{M^X/G/1}(z) \frac{\eta}{\eta + \lambda (1 - X(z))} \]

\[ L = L_{M^X/G/1} + \frac{\lambda E(X)}{\eta} \] (from equations (6.1.23) and (6.1.24))
(b) For Single Vacation

\[
P(z) = \left( \frac{1}{h_X(z)} + \frac{\eta}{\lambda (\eta + \lambda)} \right) \left( \frac{1}{\eta} + \frac{\eta}{\lambda (\eta + \lambda)} \right)
\]

\[
L = L_{M^x/G/1} + \frac{\lambda E(X)}{\eta^2} \left( \frac{1}{\eta} + \frac{\eta}{(\lambda + \eta) \lambda} \right)
\]  

(from equations (6.2.12) and (6.2.13)).

II. The M/G/1 queue with working vacations (Li et al. (2009))

If the arrival is single and follows Poisson process then \( X(z) = z \) and \( E(X) = 1 \). Thus the equations (6.1.23) and (6.2.11) lead to

(a) \( P(z) = \Pi^+(z) = P_{M/G/1} \frac{\psi(z)}{\psi(1)} \), for multiple working vacation model.

(b) \( P(z) = \Pi^+(z) = P_{M/G/1} \frac{\psi^s(z)}{\psi^s(1)} \), for single working vacation model

where \( \psi(z) = 1 + \frac{\lambda (z - 1) z}{z - S^*_V (h(z))} \left( \frac{1 - S^*_V (h(z))}{h(z)} - \frac{S^*_V (h(z)) (1 - S^*_V (w(z)))}{w(z) S^*_b (w(z))} \right) \)

\[
\psi^s(z) = \psi(z) + \eta / \lambda, w(z) = \lambda (1 - z) \text{ and } h(z) = \eta + w(z).
\]

It is verified that \( P(z) \) for M/G/1/MWV exactly coincides with the corresponding result of Li et al. (2009). Also for M/G/1 working vacation models, the PGF of the stationary system size at departure epoch \( \Pi^+(z) \) and at arbitrary epoch \( P(z) \) coincide.

III. The M^x/M/1 working vacation queueing model (Julia Rose Mary and Afthab Begum (2009))

If the service times follow exponential distribution, then \( S^*_b (w_X(z)) = \frac{\mu_b}{\mu_b + w_X(z)} \), \( S^*_V (h_X(z)) = \frac{\mu_v}{\mu_v + h_X(z)} \) and \( z_1 \) is the root of characteristic equation \( \lambda z (1 - X(z)) - \mu_v (1 - z) + \eta z = 0 \). Then by using equations (6.1.11), (6.1.18), (6.1.23), (6.2.8), (6.2.10) to (6.2.13) the PGFs for the Markovian M^x / M / 1 / MV models are obtained :
(a) For Multiple Working Vacation

\[ Q^*(z,0) + Q_0 = \frac{\mu_v (z-z_1) Q_0}{z_1 (\mu_v (z-1) + \eta z + \lambda z (1-X(z)))} \]  \hspace{1cm} (6.2.14)

\[ P^*(z,0) = \frac{\lambda z \mu_v Q_0 [(z-1) z_1 (X(z) - 1) - (z_1 - 1) z (X(z) - 1)]}{z_1 (\lambda z (1-X(z)) + \eta z + \mu_v (z-1)) (\lambda z (1-X(z)) + \mu_b (z-1))} \]  \hspace{1cm} (6.2.15)

The total PGF \( P(z) = P^*(z,0) + Q^*(z,0) + Q_0 \)

\[ P(z) = P_{M^x/M/1}(z) \frac{\mu_b (z-z_1) + \lambda z z_1 (X(z) - X(z))}{\lambda z (1-X(z)) + \mu_v (z-1) + \eta z} + \frac{\eta}{\mu_b (1-z_1) + \lambda z_1 (X(z) - 1)} \]  \hspace{1cm} (6.2.17)

(b) For Single Working Vacation

\[ Q^*(z,0) + Q_0 = \frac{\mu_v (z-z_1) Q_0}{z_1 (\mu_v (z-1) + \eta z + \lambda z (1-X(z)))} \]  \hspace{1cm} (6.2.16)

\[ P^*(0,0) + P_0 = \frac{z Q_0}{\lambda z (1-X(z)) + \mu_b (z-1)} \]

\[ \left[ \frac{\lambda \mu_v}{z_1} \left( \frac{z_1 (1-z) (X(z) - 1) - z (1-z_1) (X(z) - 1)}{\lambda z (1-X(z)) + \eta z + \mu_v (z-1)} \right) - \eta (1-X(z)) \right] \]  \hspace{1cm} (6.2.17)

The total PGF and the mean system size are given by,

\[ P(z) = P_{M^x/M/1}(z) \left[ \frac{\mu_v}{z_1} \left( \frac{\mu_b (z-z_1) + \lambda z z_1 (X(z) - X(z))}{\lambda z (1-X(z)) + \eta z + \mu_v (z-1)} \right) + \frac{\eta \mu_b}{\lambda} \right] + \frac{\mu_v}{z_1} \left( \frac{\mu_b (1-z_1) + \lambda z_1 (X(z) - 1)}{\eta} + \frac{\eta \mu_b}{\lambda} \right) \]

\[ L = L_{M^x/M/1} + \frac{\lambda E(X) - \mu_v}{\eta} + \frac{\mu_b z_1 (1-\rho_b + \eta \frac{1-\lambda E(X)}{\mu_v})}{\mu_b (1-z_1) + \lambda z_1 (X(z) - 1) + \frac{\eta^2 \mu_b z_1}{\lambda \mu_v}} \]

IV. The M/M/1 working vacation queueing model

If the interarrival times, the service times \( S_b \) and \( S_v \) follow the exponential distributions with parameters \( \lambda_b \), \( \mu_b \) and \( \mu_v \), then the characteristic equation \( z = S^*_v (\eta + w_x(z)) \) leads to \( \lambda (1-z) + \mu_v (z-1) + \eta z = -\lambda (z-z_1)(z-\frac{1}{r_v}) \),
where \( z_1 < 1 \) and \( z_2 = \frac{\mu_v}{\lambda z_1} \geq 1 \) are the roots of the characteristic equation. Thus equations (6.2.14) and (6.2.15) give the PGF of the system size, when the server is in working vacation state \((Q(z))\) and regular busy state \((P(z))\) for MWV model.

**a) For Multiple Working Vacation**

The generating functions in working vacation state and regular busy state respectively give,

\[
Q(z) = \frac{Q_0}{1-r_v z} = \sum_{n=0}^{\infty} (r_v)^n z^n Q_0
\]

\[
P(z) = \frac{\rho_b (1-z_1) z Q_0}{(1-\rho_b z)(1-r_v z)} = \frac{\eta_r Q_0}{\mu_b (\rho_b - r_v)} \left( \frac{1}{1-r_v z} - \frac{1}{1-z_1 z} \right)
\]

where \( Q_0 = \left[ (1-\rho_b) + \frac{\eta r_v}{\mu_b (1-r_v)} \right]^{-1} (1-\rho_b) (1-r_v) \)

\[
L = \frac{\rho_b}{1-\rho_b} + \left( \frac{1-r_v \mu_v}{\mu_b} \right)^{-1} \left( 1-\frac{\mu_v}{\mu_b} \right) \left( \frac{r_v}{1-r_v} \right) \text{where } \rho_b = \frac{\lambda}{\mu_b}.
\]

**b) For Single Working Vacation**

Similarly equations (6.2.16) and (6.2.17) give the corresponding results for M/M/1 single working vacation model.

\[
Q(z) = \sum_{n=0}^{\infty} r_v^n z^n Q_0
\]

\[
P(z) = \left[ \frac{\eta r_v}{\mu (\rho_b - r_v)} \sum_{n=0}^{\infty} (\rho_b^n - r_v^n) z^n + \frac{\eta z}{\mu_b} \sum_{n=0}^{\infty} (\rho_b z)^n \right] Q_0
\]

\[
P_0 = \frac{\eta}{\lambda} Q_0
\]

\[
L = \frac{\rho_b}{1-\rho_b} + \left( \frac{1-\mu_v}{\mu_b} \right) \left( \frac{r_v}{1-r_v} \right) \left[ \frac{1-r_v \mu_v}{\mu_b} \right] \frac{\eta}{\lambda} (1-r_v)^{-1}
\]

It is verified that the results thus obtained coincide with the corresponding results of M / M / 1 MWV model (Servi and Finn (2002)) and M / M / 1 / SWV model (Tian et al. (2008b) and Liu et al. (2007)).
6.4 NUMERICAL ANALYSIS

In this section numerical results are obtained to study the effects of the parameters namely (i) mean vacation time \(1/\eta\), (ii) service rate during working vacation \(\mu_v\), (iii) regular service rate \(\mu_b\) and (iv) the mean batch size \(E(X)\) on the expected system size of the batch arrival multiple working vacation model \(L_{MWV}\) and single working vacation \(L_{SWV}\) queueing model. The effects of the parameters on the system size probabilities \(P_v, P_{\text{busy}}, P_{\text{idle}}\) when the server is in different states are also analysed. To present the numerical examples for the \(M^X / G (= (S_b, S_v)) / 1\) working vacation model, different service time distributions such as deterministic (D), exponential (M) and Erlang \((E_k)\) are considered for regular service time \(S_b\) and vacation service time \(S_v\).

Thus, the first and second moments

\[
(E(S_b), E(S_b^2)) = \left(\frac{1}{\mu_b}, \frac{(k+1)}{k \mu_b^2}\right) \text{ for Erlang-}k \text{ distribution and}
\]

\[
\left(\frac{1}{\mu_b}, \frac{1}{\mu_b^2}\right) \text{ for Deterministic distribution.}
\]

The characteristic equation \(z = S_v^\eta (\eta + \lambda (1 - X(z)))\) corresponds to

\[z = \left(\frac{k \mu_v}{k \mu_v + \eta + \lambda (1 - X(z))}\right)^k\] and \(z = e^{-\frac{\eta + \lambda (1 - X(z))}{\mu_v}}\) according as \(S_v\) follows Erlang-\(k\) distribution and deterministic distribution respectively. \(z_1\) denotes the root of the characteristic equation which lies in \((0, 1)\). The batch size is assumed to follow Geometric distribution \((\text{Geo}(p))\) in all the tables except Table 6.3.

In classical vacation models, since the service is stopped completely during vacation, the system size increases notably as the mean vacation time increases. But in working vacation models, since the service is done with a smaller rate \(\mu_v (< \mu_b)\) during vacation, the vacation parameter \(\eta\) has less effect on the system size.
The values of Table 6.1 and Figures 6.1 (a), 6.1 (b), 6.1 (c) and 6.1 (d) show that as the vacation service rate ($\mu_v$) or vacation parameter ($\eta$) increases, the mean system size decreases. The table values also show that the effect of $\eta$ on expected system size is very much significant for smaller values of $\eta$. It is interesting to note that (first row) as $\mu_v$ approaches 0, the system size of the working vacation models approaches the system size of the corresponding classical vacation models. Further the mean system size of working vacation models and classical non-vacation models coincide (last row) when $\mu_v = \mu_b$.

**Figure 6.1a** $M^X / (D, D) / MWV$

**Figure 6.1b** $M^X / (D, D) / 1 / SWV$

**Figure 6.1c** $M^X / (D, E_3) / 1 / MWV$

**Figure 6.1d** $M^X / (D, E_3) / 1 / SWV$
In Table 6.2, the values of the expected system size for both the models (multiple and single) are presented for different arrival rates $\lambda$ and vacation service rates $\mu_v$ for two different regular service rates ($\mu_b$). It is noted from Table 6.2 that expected system size increases with $\lambda$ and decreases as $\mu_v$ increases for $M^X/(D, E_3)/1/WV$.

Table 6.2 Expected system size with respect to $\lambda$ and $\mu_v$

$$(\eta, p) = (0.1, 0.75)$$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu_v$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$\mu_b = 1.5$</td>
<td>11.587</td>
<td>10.795</td>
<td>10.036</td>
<td>9.320</td>
<td>8.650</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 4$</td>
<td>8.419</td>
<td>7.743</td>
<td>7.100</td>
<td>6.502</td>
<td>5.952</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7.322</td>
<td>6.690</td>
<td>6.092</td>
<td>5.539</td>
<td>5.036</td>
</tr>
<tr>
<td>0.25</td>
<td>$\mu_b = 1.5$</td>
<td>16.583</td>
<td>15.713</td>
<td>14.867</td>
<td>14.049</td>
<td>13.263</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 4$</td>
<td>10.686</td>
<td>9.946</td>
<td>9.231</td>
<td>8.551</td>
<td>7.911</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.694</td>
<td>8.982</td>
<td>8.294</td>
<td>7.642</td>
<td>7.038</td>
</tr>
<tr>
<td>0.3</td>
<td>$\mu_b = 1.5$</td>
<td>25.589</td>
<td>24.661</td>
<td>23.748</td>
<td>22.853</td>
<td>21.976</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 4$</td>
<td>13.005</td>
<td>12.217</td>
<td>11.448</td>
<td>10.706</td>
<td>9.995</td>
</tr>
</tbody>
</table>

Multiple working vacation  Single working vacation

The influence of arrival rate $\lambda$ and slow service rate $\mu_v$ on the expected system size of $M^X/(D, E_3)/1MWV$ and $SWV$, when batch arrival follows binomial distribution $N(5, 0.75)$ are tabulated in Table 6.3 and depicted in Figures 6.3a and 6.3b.

Table 6.3 Expected system size with respect to $\lambda$ and $\mu_v$

$$(N, \eta, \mu_b) = (5, 0.1, 0.9)$$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho_b$</th>
<th>$\mu_v$</th>
<th>0.005</th>
<th>0.05</th>
<th>0.07</th>
<th>0.1</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.42</td>
<td></td>
<td>5.387</td>
<td>5.235</td>
<td>5.111</td>
<td>4.918</td>
<td>2.919</td>
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<td></td>
<td></td>
<td></td>
<td>4.136</td>
<td>4.027</td>
<td>3.935</td>
<td>3.788</td>
<td>2.352</td>
</tr>
<tr>
<td>0.12</td>
<td>0.5</td>
<td></td>
<td>6.750</td>
<td>6.575</td>
<td>6.436</td>
<td>6.221</td>
<td>3.854</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5.513</td>
<td>5.377</td>
<td>5.267</td>
<td>5.093</td>
<td>3.241</td>
</tr>
<tr>
<td>0.14</td>
<td>0.58</td>
<td></td>
<td>8.342</td>
<td>8.145</td>
<td>7.993</td>
<td>7.758</td>
<td>5.053</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>7.137</td>
<td>6.976</td>
<td>6.848</td>
<td>6.649</td>
<td>4.403</td>
</tr>
<tr>
<td>0.16</td>
<td>0.67</td>
<td></td>
<td>10.333</td>
<td>10.116</td>
<td>9.951</td>
<td>9.699</td>
<td>6.691</td>
</tr>
<tr>
<td>0.18</td>
<td>0.75</td>
<td></td>
<td>13.125</td>
<td>12.887</td>
<td>12.711</td>
<td>12.444</td>
<td>9.165</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>12.007</td>
<td>11.798</td>
<td>11.640</td>
<td>11.399</td>
<td>8.465</td>
</tr>
</tbody>
</table>

Multiple working vacation  Single working vacation
The Table 6.4 shows that, the batch size significantly affects the system size for different parametric values of \( \mu_v \) and \( \eta \). Two different distributions of \( X \) are considered.

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Binomial batch size</th>
<th>Geometric batch size</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_v = 0.1 )</td>
<td>( \mu_v = 0.3 )</td>
<td>( \mu_v = 0.1 )</td>
</tr>
<tr>
<td>( E(X) = 2.5 )</td>
<td>( E(X) = 3.75 )</td>
<td>( E(X) = 2.5 )</td>
</tr>
</tbody>
</table>

- Multiple working vacation
- Single working vacation
The system size probabilities $P_v$, $P_{\text{busy}}$, $P_1$ and the expected system size $L_v$ and $L_{\text{busy}}$, when the server is on vacation, busy and idle are presented for both multiple and single working vacation models in Table 6.5.

**Table 6.5 System measures of $M^X/(D, E_3)/1$ MWV and SWV with respect to system parameters**

$$(\lambda, p, \mu_b, \mu_v, \eta) = (0.2, 0.75, 1.5, 0.5, 0.1)$$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$P_v$</th>
<th>$P_1$</th>
<th>$P_{\text{busy}}$</th>
<th>$L_v$</th>
<th>$L_{\text{busy}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.666</td>
<td>-</td>
<td>0.334</td>
<td>23.371</td>
<td>14.280</td>
</tr>
<tr>
<td></td>
<td>0.662</td>
<td>0.003</td>
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<td>23.213</td>
<td>14.212</td>
</tr>
<tr>
<td>0.05</td>
<td>0.605</td>
<td>-</td>
<td>0.395</td>
<td>5.536</td>
<td>6.691</td>
</tr>
<tr>
<td></td>
<td>0.552</td>
<td>0.040</td>
<td>0.407</td>
<td>5.057</td>
<td>6.481</td>
</tr>
<tr>
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<td>0.570</td>
<td>-</td>
<td>0.430</td>
<td>3.012</td>
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</tr>
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<td></td>
<td>0.453</td>
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